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ON SOME RESULTS OF $\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

ERDİNÇ DÜNDAR

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Abstract. In this work, we investigate some results of $\mathcal{I}_2$-convergence of double sequences of real valued functions and prove a decomposition theorem.

1. background and introduction

The concept of convergence of a real sequence was independently extended to statistical convergence by Fast [11] and Schoenberg [30]. This concept was extended to the double sequences by Mursaleen and Edely [21]. A lot of developments have been made in this area after the works of Šalát [29] and Fridy [13, 14]. Furthermore Gökhan et al. [16] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [11, 13, 14, 28]. Çakan and Altay [5] presented multidimensional analogues of the results presented by Fridy and Orhan [12].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [18] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Nuray and Ruckle [26] indepedently introduced the same concept with another name generalized statistical convergence. Kostyrko et al. [19] gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some of its properties. Also Das and Malik [7] introduced the concept of $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points and $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of double sequences. Balcerzak et al. [4] discussed various kinds of statistical convergence and $\mathcal{I}$-convergence of sequences of functions with values in $\mathbb{R}$ or in a metric space. Gezer and Karakuş [15] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and then they examined the relation between them. Dündar and Altay [8] studied the

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concepts of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy for double sequences in a linear metric space and investigated the relation between $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions defined between linear metric spaces. Also, some results on $\mathcal{I}$-convergence may be found in [2, 3, 9, 20, 22, 23, 24, 25, 31].

In this study, we investigate some results of $\mathcal{I}_2$-convergence of double sequences of real valued functions and prove a decomposition theorem for $\mathcal{I}_2$-convergent double sequences.

2. DEFINITIONS AND NOTATIONS

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively.

Now, we recall the concept of statistical and ideal convergence of the sequences (See [6, 8, 10, 11, 16, 18, 21, 27]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N(x,\varepsilon)$ such that

$$|f_{mn}(x) - f(x)| < \varepsilon,$$

for all $m, n > N$. In this case we write

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x) \ \mathit{or} \ f_{mn} \to f, \ \mathit{as} \ m,n \to \infty,$$

for each $x \in S$.

A double sequence of functions $\{f_{ij}\}$ is said to be pointwise statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j), i \leq m \ \mathit{and} \ j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$,

$$|f_{ij}(x) - f(x)| < \varepsilon, \ \ a.a.(i,j).$$

In this case we write

$$st - \lim_{i,j\to\infty} f_{ij}(x) = f(x) \ \mathit{or} \ f_{ij} \xrightarrow{st} f,$$

for each $x \in S$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset \in \mathcal{I}$, ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1. [18] *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})_{m,n \in \mathbb{N}}$ of elements of X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n \to \infty} x_{mn} = L,$$

for $(m, n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n \to \infty} x_{mn} = L.$$

Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each (fixed) $x \in S$. This can be written by the formula

$$(\forall x \in S)\ (\forall \varepsilon > 0)\ (\exists H \in \mathcal{I}_2)\ (\forall (m, n) \notin H)\ \ |f_{mn}(x) - f(x)| < \varepsilon.$$

This is written as

$$f_{mn} \xrightarrow{\mathcal{I}_2} f, \ \textit{as } m, n \to \infty.$$

The function f is called the double $\mathcal{I}_2$-limit (or Pringsheim $\mathcal{I}_2$-limit) function of the $\{f_{mn}\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise $\mathcal{I}_2^*$- convergent to f on $S \subset \mathbb{R}$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e. $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for $(m, n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \stackrel{\mathcal{I}_2^*}{\to} f, \text{ as } m, n \to \infty,$$

for each $x \in S$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Now we begin with quoting the lemmas due to Dndar and Altay [8, 10] which are needed throughout the paper.

Lemma 2.2 ([10],). *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ is a double sequence of functions and f be a function on $S \subset \mathbb{R}$. Then*

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$.

Lemma 2.3 ([8]). *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2), (X, d_x) and (Y, d_y) two linear metric spaces, $f_{mn} : X \to Y$ a double sequence of functions and $f : X \to Y$. If $\{f_{mn}\}$ double sequence of functions is $\mathcal{I}_2$-convergent, then it is $\mathcal{I}_2^*$-convergent.*

3. SOME RESULTS OF $\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

Throughout the paper we use convergence instead of pointwise convergence.

Theorem 3.1. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$. If $c \in \mathbb{R}$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$, then we have*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} cf_{mn}(x) = cf(x),$$

for each $x \in S$.

Proof. Let $c \in \mathbb{R}$ and

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$. If $c = 0$, there is nothing to prove, so we assume that $c \neq 0$.

Let $\varepsilon > 0$ be given. Then,

$$\Big\{(m, n) \in \mathbb{N} \times \mathbb{N} : |cf_{mn}(x) - cf(x)| \geq \varepsilon\Big\} \subseteq \Big\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{|c|}\Big\} \in \mathcal{I}_2.$$

Hence, $\mathcal{I}_2 - \lim_{m,n\to\infty} cf_{mn}(x) = cf(x)$ for each $x \in S$. □

Theorem 3.2. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ and $\{g_{mn}\}$ be two double sequences of functions, f and g be two functions on $S \subset \mathbb{R}$ and*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = g(x),$$

for each $x \in S$. Then, we have
(i) $\mathcal{I}_2 - \lim_{m,n\to\infty}(f_{mn} + g_{mn})(x) = f(x) + g(x)$,
(ii) $\mathcal{I}_2 - \lim_{m,n\to\infty}(f_{mn}g_{mn})(x) = f(x)g(x)$,
for each $x \in S$.

Proof. (i) Let $\varepsilon > 0$ be given. Since

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = g(x),$$

therefore

$$A\Big(\frac{\varepsilon}{2}\Big) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2}\} \in \mathcal{I}_2$$

and

$$B\Big(\frac{\varepsilon}{2}\Big) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |g_{mn}(x) - g(x)| \geq \frac{\varepsilon}{2}\} \in \mathcal{I}_2,$$

for each $x \in S$ and by definition of ideal we have $A(\frac{\varepsilon}{2}) \cup B(\frac{\varepsilon}{2}) \in \mathcal{I}_2$. Now define the set

$$C(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x))| \geq \varepsilon\}$$

and it is sufficient to prove that $C(\varepsilon) \subset A(\frac{\varepsilon}{2}) \cup B(\frac{\varepsilon}{2})$, for each $x \in S$. Let $(m,n) \in C(\varepsilon)$, then we have

$$\begin{aligned} \varepsilon &\leq |(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x))| \\ &\leq |f_{mn}(x) - f(x)| + |g_{mn}(x) - g(x)|, \end{aligned}$$

for each $x \in S$. As both of $\{|f_{mn}(x) - f(x)|, |g_{mn}(x) - g(x)|\}$ can not be (together) strictly less than $\frac{\varepsilon}{2}$, and therefore we have either

$$|f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2} \;\; or \;\; |g_{mn}(x) - g(x)| \geq \frac{\varepsilon}{2},$$

for each $x \in S$. This shows that

$$(m,n) \in A(\frac{\varepsilon}{2}) \;\; or \;\; (m,n) \in B(\frac{\varepsilon}{2})$$

and so we have

$$(m,n) \in A(\frac{\varepsilon}{2}) \cup B(\frac{\varepsilon}{2}).$$

Hence, $C(\varepsilon) \subset A(\frac{\varepsilon}{2}) \cup B(\frac{\varepsilon}{2})$.

(ii) Since $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$, therefore for $\varepsilon = 1 > 0$

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq 1\} \in \mathcal{I}_2$$

for each $x \in S$ and so

$$A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| < 1\} \in \mathcal{F}(\mathcal{I}_2)$$

for each $x \in S$. Also for any $(m,n) \in A$

$$|f_{mn}(x)| < 1 + f(x),$$

for each $x \in S$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$0 < 2\delta < \frac{\varepsilon}{|f| + |g| + 1}.$$

It follows from the assumption that

$$B = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| < \delta\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$C = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |g_{mn}(x) - g(x)| < \delta\} \in \mathcal{F}(\mathcal{I}_2),$$

for each $x \in S$. Since $\mathcal{F}(\mathcal{I}_2)$ is a filter, therefore $A \cap B \cap C \in \mathcal{F}(\mathcal{I}_2)$. Then for each $(m,n) \in A \cap B \cap C$ we have

$$\begin{aligned} |f_{mn}(x)g_{mn}(x) - f(x)g(x)| &= |f_{mn}(x)g_{mn}(x) - f_{mn}(x)g(x) + f_{mn}(x)g(x) - f(x)g(x)| \\ &\leq |f_{mn}(x)|.|g_{mn}(x) - g(x)| + |g(x)|.|f_{mn}(x) - f(x)| \\ &< (|f(x)| + 1)\delta + |g(x)|\delta = (|f(x)| + |g(x)| + 1)\delta < \varepsilon, \end{aligned}$$

for each $x \in S$. Hence we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x)g_{mn}(x) - f(x)g(x)| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in S$. This completes the proof of theorem. □

Now, we give the decomposition theorem for double sequences of functions.

Theorem 3.3. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2), $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$. Then the following conditions are equivalent:*

(i) $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$, for each $x \in S$,

(ii) There exist $\{g_{mn}\}$ and $\{h_{mn}\}$ be two double sequences of functions such that

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \quad \lim_{m,n\to\infty} g_{mn}(x) = f(x) \text{ and } supp\ h_{mn}(x) \in \mathcal{I}_2,$$

for each $x \in S$, where supp $h_{mn}(x) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\}$.

Proof. *(i)* $\Rightarrow$ *(ii)* : $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ for each $x \in S$. Then by Lemma 2.3 there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x),$$

for each $x \in S$. Let us define the double sequence $\{g_{mn}\}$ by

$$g_{mn}(x) = \begin{cases} f_{mn}(x) & , \quad (m,n) \in M \\ f(x) & , \quad (m,n) \in \mathbb{N} \times \mathbb{N} \backslash M. \end{cases} \tag{3.1}$$

It is clear that $\{g_{mn}\}$ is a double sequence of functions on S and

$$\lim_{m,n\to\infty} g_{mn}(x) = f(x),$$

for each $x \in S$. Also let

$$h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \quad m,n \in \mathbb{N}, \tag{3.2}$$

for each $x \in S$. Since

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\} \subset \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2,$$

for each $x \in S$, so we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2.$$

It follows that *supp* $h_{mn}(x) \in \mathcal{I}_2$ and by (3.1) and (3.2) we get $f_{mn}(x) = g_{mn}(x) + h_{mn}(x)$, for each $x \in S$.

$(ii) \Rightarrow (i)$: Suppose that there exist two sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ on S such that

$$(3.3)\ f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \lim_{m,n\to\infty} g_{mn}(x) = f(x), supp\ h_{mn}(x) \in \mathcal{I}_2,$$

for each $x \in S$, where $supp\ h_{mn}(x) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\}$. We will show that

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$. Let

$$(3.4) \qquad M = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) = 0\} = \mathbb{N}\times\mathbb{N}\backslash supp\ h_{mn}(x).$$

Since

$$supp\ \ h_{mn}(x) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2,$$

from (3.3) and (3.4) we have $M \in \mathcal{F}(\mathcal{I}_2)$, $f_{mn}(x) = g_{mn}(x)$ for $(m,n) \in M$ and

$$\mathcal{I}_2^* - \lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x),$$

for each $x \in S$. By Lemma 2.2 it follows that

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$. This completes the proof. □

Corollary 3.1. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2), $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$. Then*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$$

if and only if there exist two double sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ of functions on S such that

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \ \lim_{m,n\to\infty} g_{mn}(x) = f(x), \ and\ \mathcal{I}_2 - \lim_{m,n\to\infty} h_{mn}(x) = 0,$$

for each $x \in S$.

Proof. Let $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ and $\{g_{mn}\}$ is the sequence defined by (3.1). Consider the sequence

$$(3.5) \qquad h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \ \ m,n \in \mathbb{N},$$

for each $x \in S$. Then we have

$$\lim_{m,n\to\infty} g_{mn}(x) = f(x)$$

and since $\mathcal{I}_2$ is a strongly admissible ideal so

$$\mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = f(x),$$

for each $x \in S$. By Theorem 3.2 and by (3.5) we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} h_{mn}(x) = 0,$$

for each $x \in S$.

Now let $f_{mn}(x) = g_{mn}(x) + h_{mn}(x)$, where

$$\lim_{m,n\to\infty} g_{mn}(x) = f(x) \ \ and\ \mathcal{I}_2 - \lim_{m,n\to\infty} h_{mn}(x) = 0,$$

for each $x \in S$. Since $\mathcal{I}_2$ is a strongly admissible ideal so

$$\mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = f(x)$$

and by Theorem 3.2 we get

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$. □

Remark 3.1. *In Theorem 3.3, if (ii) is satisfied then the strongly admissble ideal $\mathcal{I}_2$ need not have the property $(AP2)$. Since*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |h_{mn}(x)| \geq \varepsilon\} \subset \{(m,n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2$$

for each $\varepsilon > 0$, then

$$\mathcal{I}_2 - \lim_{m,n\to\infty} h_{mn}(x) = 0.$$

Thus, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$.

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Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, Afyonkarahisar, Turkey
E-mail address: erdincdundar79@gmail.com, edundar@aku.edu.tr