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ON COMPLEX NUMBERS AND TAXICAB PLANE

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Abstract. In this work, we apply complex theoretic information to the taxicab plane geometry. Also, the formulations of the isometries of the taxicab plane are given in terms of complex numbers.

1. Introduction

Recall that one can think of the taxicab plane as the set of all pairs of real numbers (x, y) equipped with the taxicab metric

$$d_T(A(x_1, y_1), B(x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

instead of the well known Euclidean metric

$$d_E(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

The d_T taxicab distance between A and B is the length of a shortest path from A and B compose of line segments parallel to the coordinate axes. The taxicab geometry have been studied and improved by some authors [1-8].

We want to see the role played by complex numbers in the taxicab plane. We have come to think of the points of the taxicab plane as complex numbers. This allows us to apply complex numbers in to theoretic information to taxicab plane geometry.

Informally, a complex number is a number that can be put in the form $z = x+iy$, where x and y are real numbers and $i^2 = -1$. x and y are called the real part and the imaginary part of the complex number z, respectively.

One can pack coordinates (x, y) of a point in the taxicab plane, in a complex number $z = x + iy$. This way we get one-to-one correspondence between points of the taxicab plane and $\mathbb{C}$.

In this paper we give an interpretation of taxicab plane geometry using complex coordinates. By using elementary mathematics and an Euclidean approach, it is straightforward to formalize the taxicab plane trigonometry in the taxicab plane with the same coherence as the Euclidean trigonometry. We will explore complex

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numbers which are a slightly modified version of taxicab geometry instead of using complex numbers and we formulate the isometries of the taxicab plane given in terms of complex numbers.

2. Modulus of Complex Number in Taxicab Plane

It is well known that the conjugate of $z = x + iy$ is $x - iy$ and is denoted by $\overline{z}$. The absolute value or modulus of a complex number z is $|z| = \sqrt{z.\overline{z}}$. Now, we give the taxicab modulus of z, denoted by $|z|_T$, with the complex notation.

Definition 2.1. The taxicab modulus, or taxicab absolute value, of a complex number $z = x + yi$, denoted $|z|_T$, is the nonnegative real number $|z|_T = |x| + |y|$.

Geometrically, the modulus is the taxicab distance from the origin 0 to the point z. Some properties of the modulus are given in the following proposition:

Proposition 2.1. *The function $z \to |z|_T$ has the following properties: for all $z, w \in \mathbb{C}$,*

i. $|z|_T = 0 \Leftrightarrow z = 0$,

ii. $z\overline{z} = \dfrac{|z|_T}{|\frac{1}{z}|_T}$,

iii. $|z|_T = |\overline{z}|_T$,

iv. $|zw|_T \leq |z|_T |w|_T$,

v. $|z + w|_T \leq |z|_T + |w|_T$,
$|z + w|_T = |z|_T + |w|_T$ *with* $w \neq 0 \Leftrightarrow z = \lambda w$ *for some* $\lambda > 0$.

Proof. *i*) and *iii*) are obvious from the definition1.

ii) One can easily calculated $|\frac{1}{z}|_T = \left|\dfrac{1}{x+iy}\right| = \dfrac{|x|+|y|}{x^2+y^2} = \dfrac{|z|_T}{|z|^2}$ and $z\overline{z} = x^2 + y^2 = |z|^2$. From these equalities, $z\overline{z} = \dfrac{|z|_T}{|\frac{1}{z}|_T}$ is obtained.

iv) Let $z = x + iy$ and $w = u + iv$ be two complex numbers.

$$|zw|_T = |xu - yv| + |xv + yu| \leq |xu| + |xv| + |yv| + |yu| = |z|_T |w|_T$$

is obtained.

v) Let $z = x+iy$ and $w = u+iv$ be two complex numbers with $z \neq w \neq 0$. Then

$$|z + w|_T = |x + u| + |y + v| \leq |x| + |y| + |u| + |v| = |z|_T + |w|_T$$

is obtained. If $z = \lambda w$, one can get

$$|z + w|_T = |1 + \lambda||w|_T = |w|_T + \lambda|w|_T = |w|_T + |\lambda w|_T = |z|_T + |w|_T.$$

□

Definition 2.2. If z and $w \in \mathbb{C}$, the taxicab distance between z and w is the nonnegative real number $|z - w|_T$.

This is the taxicab distance function in the taxicab plane, and it may be expressed in coordinate form as well as in the complex form:
if $z = x + iy = (x, y)$ and $w = u + iv = (u, v)$, then

$$|z - w|_T = |(x - u) + i(y - v)|_T = |x - u| + |y - v|\,.$$

Proposition 2.2. *The taxicab distance function* $(z,w) \to |z-w|_T$ *satisfies the following properties:*

i) For all $z,\ w \in \mathbb{C}$, $|z-w|_T \geq 0$. *Equality holds* $\Leftrightarrow z=w$.

ii) For all $z,\ w \in \mathbb{C}$, $|z-w|_T = |w-z|_T$.

iii) For all $z,\ w,\ r \in \mathbb{C}$, $|z-r|_T \leq |z-w|_T + |r-w|_T$.

$|z-w|_T = |z-r|_T + |r-w|_T \Leftrightarrow r=(1-t)z+tw$ *for some* t *with* $0 \leq t \leq 1$.

Proof. *i)* Let $z=x+iy$ and $w=u+iv$ be two complex numbers with $z \neq w$. Since $|z-w|_T = |x-u|+|y-v|$ and $|x-u| \geq 0,\ |y-v| \geq 0$ then $|z-w|_T \geq 0$. In equality case,

$$\begin{aligned} |z-w|_T = 0 \quad &\Leftrightarrow \quad |x-u|=0,\ |y-v|=0 \\ &\Leftrightarrow \quad x=u,\ y=v \\ &\Leftrightarrow \quad z=w. \end{aligned}$$

ii) Let $z=x+iy$ and $w=u+iv$ be two complex numbers. One can easily get the following equality

$$\begin{aligned} |z-w|_T &= |(x-u)+i(y-v)|_T = |x-u|+|y-v| \\ &= |u-x|+|v-y| = |w-z|_T. \end{aligned}$$

iii) Let $z=x+iy$, $w=u+iv$ and $r=p+iq$ be three complex numbers. Using the triangular inequality in the following equality

$$|z-w|_T = |x-u|+|y-v|\,,$$

one can obtain $|z-w|_T \leq |z-r|_T + |r-w|_T$. Furthermore, equality holds if and only if $z-r=\lambda(r-w)$ for nonnegative real number λ from the Proposition 2.1 . Setting $t=\dfrac{\lambda}{1+\lambda}$, it is seen that $0 \leq t \leq 1$, while $1-t=\dfrac{1}{1+\lambda}$ so that $r=(1-t)z+tw$. □

3. Polar form in taxicab plane

Let $z=x+yi=(x,y)$ be a nonzero complex number. The standard argument of z is the angle from the positive x axis to the ray Oz , denoted by $arg(z)$ whose values lie in $[0,2\pi)$. It is well known in Euclidean that

$$|z|=\sqrt{x^2+y^2},\ \ arg(z)=arctan\frac{y}{x}$$

and if $r=|z|$ and $\theta=arg(z)$, then the polar form of z is

$$z=r(cos\theta+isin\theta)=re^{i\theta}.$$

Now, we give the polar form of z in the taxicab plane. Any point $z=x+yi=(x,y)\in\mathbb{R}^2_T$ can be written in polar coordinates (r_T,θ), $r_T \geq 0$, $\theta\in[0,2\pi)$ where $r_T=|x|+|y|$ and where, if $(x,y)\neq(0,0)$, is the angle, measured in radians and proceeding counterclockwise, from the positive x-axis to the ray joining $(0,0)$ to (x,y). Note that r_T is the modulus $|z|_T$ of the complex number z. The point $(0,0)$ $\in\mathbb{R}^2_T$ has non-unique polar coordinates $(0,\theta)$, any real number.

In [1], for the point $P=(x,y)$ on the taxicab unite circle

$$\begin{aligned} x=\cos_T\theta &= \frac{\cos\theta}{|\cos\theta|+|\sin\theta|},\ 0\leq\theta<2\pi \\ y=\sin_T\theta &= \frac{\sin\theta}{|\cos\theta|+|\sin\theta|},\ 0\leq\theta<2\pi \end{aligned}\ .$$

It is well known that Taxicab tangent function is the same as standard Euclidean tangent function. Hence, Euclidean argument is same the taxicab argument for a complex number. The Cartesian coordinates (x, y) of a point in taxicab plane can be recovered from the polar coordinates (r_T, θ) by $x = r_T \cos_T \theta$, $y = r_T \sin_T \theta$, [1,6] . Thus we have shown that $z = x + iy$ complex number can be expressed in the polar form or trigonometric form in the taxicab plane as

$$z = |z|_T (\cos_T \theta + i \sin_T \theta).$$

The modulus of the product of the complex numbers is product of their moduli, and the argument of the product is sum of their arguments according to Euclidean metric. Although the argument of the product is sum of their arguments, the modulus of the product of the complex numbers is not product of their moduli in taxicab plane. This is given by the following proposition.

Proposition 3.1. *Let z_1 and z_2 be two complex numbers with arguments θ_1 and θ_2, respectively. Then the modulus of the product of these complex numbers in terms of the moduli and arguments is*

$$|z_1.z_2|_T = |z_1|_T |z_2|_T \sqrt{\frac{(\cos_T^2 \theta_1 + \sin_T^2 \theta_1)(\cos_T^2 \theta_2 + \sin_T^2 \theta_2)}{\cos_T^2(\theta_1 + \theta_2) + \cos_T^2(\theta_1 + \theta_2)}}.$$

Proof. The polar forms of $z_1$and z_2 are $z_1 = |z_1|_T (\cos_T \theta_1 + i \sin_T \theta_1)$, $z_2 = (\cos_T \theta_2 + i \sin_T \theta_2)$ and $z_1.z_2 = |z_1.z_2|_T (\cos_T(\theta_1 + \theta_2) + i \sin_T(\theta_1 + \theta_2))$. The relations between the taxicab moduli and Euclidean moduli of $z_{1,}$ z_2 and $z_1.z_2$ complex numbers are

$$|z_1| = |z_1|_T \sqrt{(\cos_T^2 \theta_1 + \sin_T^2 \theta_1)},$$
$$|z_2| = |z_2|_T \sqrt{(\cos_T^2 \theta_2 + \sin_T^2 \theta_2)},$$
$$|z_1.z_2| = |z_1 z_2|_T \sqrt{\cos_T^2(\theta_1 + \theta_2) + \cos_T^2(\theta_1 + \theta_2)}.$$

Since the modulus of the product of the complex numbers is product of their moduli and the argument of the product is sum of their arguments according to Euclidean metric,

$$|z_1.z_2|_T = |z_1|_T |z_2|_T \sqrt{\frac{(\cos_T^2 \theta_1 + \sin_T^2 \theta_1)(\cos_T^2 \theta_2 + \sin_T^2 \theta_2)}{\cos_T^2(\theta_1 + \theta_2) + \sin_T^2(\theta_1 + \theta_2)}}$$

is obtained by using above relations. □

4. Isometries in terms of complex numbers in taxicab plane

It is well known that the group of the isometries of Euclidean plane with the usual metric is the semi–direct product of its the symmetry group of the unit circle and the group of all translations of the plane. Also the group of the isometries of Euclidean plane with respect to the taxicab metric is the semi–direct product of the symmetry group of the square and the group of all translations of the plane, [3], [4], [5], [8]. The goal of this section is to obtain formulations in terms of complex numbers for the taxicab isometries.

Lemma 4.1. *If c is any fixed complex number, then the function*

$$T_c(z) = z + c$$

is a translation of the taxicab plane.

Lemma 4.2. *For any angle θ, the function*

$$R_\theta(z) = \left(\frac{\cos_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} + i \frac{\sin_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} \right) z$$

is the rotation of the vector Oz with the origin center by the angle θ in taxicab plane.

Proof. Let $z = x + yi$ be complex number corresponding to the point (x, y) in taxicab plane. Let the polar form of z be $z = |z|_T (\cos_T \alpha + i \sin_T \alpha)$. From [7], the image of Oz under the rotation with an angle θ is

$$Oz' = (|z_T| \sqrt{\frac{(\cos_T^2 \alpha + \sin_T^2 \alpha)}{\cos_T^2(\theta+\alpha) + \sin_T^2(\theta+\alpha)}} \cos_T(\theta+\alpha), |z_T| \sqrt{\frac{(\cos_T^2 \alpha + \sin_T^2 \alpha)}{\cos_T^2(\theta+\alpha) + \sin_T^2(\theta+\alpha)}} \sin_T(\theta+\alpha)).$$

So,

$$z' = |z_T| \sqrt{\frac{(\cos_T^2 \alpha + \sin_T^2 \alpha)}{\cos_T^2(\theta+\alpha) + \sin_T^2(\theta+\alpha)}} (\cos_T(\theta+\alpha) + i \sin_T(\theta+\alpha)).$$

By Proposition 3.1 , the product of z and $\left(\frac{\cos_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} + i \frac{\sin_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} \right)$ is equal to $z' = R_\theta(z)$ and this proves the lemma. □

The geometric interpretation of the multiplication of complex numbers is given in the following corollary.

Corollary 4.1. *The multiplication of two complex numbers is the product of the image of first complex number under the rotation counter-clockwise by the other's argument and the Euclidean norm of the second complex number.*

Proof. Let the polar forms of $z_1$and z_2 be $z_1 = |z_1|_T (\cos_T \theta_1 + i \sin_T \theta_1)$, $z_2 = |z_2|_T (\cos_T \theta_2 + i \sin_T \theta_2)$. From Proposition 3.1 and Lemma 4.2, it is easily seen that

$$z_1.z_2 = |z_2| R_{\theta_2}(z_1) = |z_1| R_{\theta_1}(z_2).$$

□

The reflections of the taxicab plane are same the reflections of Euclidean plane. If $z = x + iy$ is any complex number, the function $\rho_x(z) = \overline{z} = x - iy = R_{-\theta}$ is the reflection in $x-$axis.

Lemma 4.3. *If m is the line through the origin with inclination θ to positive $x-$axis, then the reflection in m in terms of complex numbers can be written as*

$$\rho_x(z) = \left(\frac{\cos_T 2\theta}{\sqrt{\cos_T^2 2\theta + \sin_T^2 2\theta}} + i \frac{\sin_T 2\theta}{\sqrt{\cos_T^2 2\theta + \sin_T^2 2\theta}} \right) \overline{z}$$

Proof. Since the reflection in the line m is the composition $R_\theta \circ \rho_x \circ R_{-\theta}$, it is clear that

$$\rho_m(z) = (R_\theta \circ \rho_x \circ R_{-\theta})(z) = \left(\frac{\cos_T 2\theta}{\sqrt{\cos_T^2 2\theta + \sin_T^2 2\theta}} + i \frac{\sin_T 2\theta}{\sqrt{\cos_T^2 2\theta + \sin_T^2 2\theta}} \right) \overline{z}.$$

□

The isometries of taxicab plane in terms of complex numbers can be summarized in the following theorem by using the above lemmas.

Theorem 4.1. *The isometries of taxicab plane all have the form*

$$f(z) = \left(\frac{\cos_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} + i \frac{\sin_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} \right) z + c$$

or

$$f(z) = \left(\frac{\cos_T \dfrac{\theta}{2}}{\sqrt{\cos_T^2 \dfrac{\theta}{2} + \sin_T^2 \dfrac{\theta}{2}}} + i \frac{\sin_T \dfrac{\theta}{2}}{\sqrt{\cos_T^2 \dfrac{\theta}{2} + \sin_T^2 \dfrac{\theta}{2}}} \right) \overline{z} + c,$$

where $\theta \in \left\{ \frac{k\pi}{2} : k \in \mathbb{Z} \right\}$ *and* c *is an arbitrary complex number. Conversely, every function of either of these forms is an isometry of the taxicab plane.*

Proof. We know that the isometries of the taxicab plane are the compositions of translations, the rotations with the angles θ and the reflections in lines with inclination angles $\dfrac{\theta}{2}$, $\theta \in \left\{ \frac{k\pi}{2} : k \in \mathbb{Z} \right\}$. By using the above Lemma 4.1 and Lemma 4.2, the composition of the rotation with θ, $\theta \in \left\{ \frac{k\pi}{2} : k \in \mathbb{Z} \right\}$ and a translation in taxicab plane is obtained as

$$f(z) = \left(\frac{\cos_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} + i \frac{\sin_T \theta}{\sqrt{\cos_T^2 \theta + \sin_T^2 \theta}} \right) z + c.$$

Similarly, from the above Lemma 4.1 and Lemma 4.3, the composition of the reflection in the line with inclination angle $\frac{\theta}{2}$, $\theta \in \left\{ \frac{k\pi}{2} : k \in \mathbb{Z} \right\}$ and a translation in taxicab plane is obtained as

$$f(z) = \left(\frac{\cos_T \dfrac{\theta}{2}}{\sqrt{\cos_T^2 \dfrac{\theta}{2} + \sin_T^2 \dfrac{\theta}{2}}} + i \frac{\sin_T \dfrac{\theta}{2}}{\sqrt{\cos_T^2 \dfrac{\theta}{2} + \sin_T^2 \dfrac{\theta}{2}}} \right) \overline{z} + c.$$

Conversely, we know that every function of the first form is either a rotation with the angles θ or a translation, and every function of the second form is the composition of a translation with a reflection. □

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