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MANNHEIM PARTNER CURVES OF $AW(k)-$TYPE IN MINKOWSKI $3-$SPACE

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Abstract. In this study, firstly, we investigate curvature conditions of non-null $AW(k)-$type curves $(1 \leq k \leq 3)$ in E_1^3. Morever, we give a classification for $W-$curves of type $AW(k)$ in E_1^3. Secondly, according to types of non-null Mannheim partner curves in E_1^3, we obtain conditions to be $AW(k)-$type curve.

1. Introduction

The notion of $AW(k)-$type submanifols was defined by *Arslan* and *West* in [1]. After, many works related to $AW(k)-$type submanifolds had been studied by several authors, [2], [4] and [5]. Then, many studies on curves of $AW(k)-$type have been done by many mahematicians. For example, the authors gave curvature conditions and characterizations related to these curves in E^n [3, 6]. Furthermore, *Külahçı* et al. studied the curves of $AW(k)-$type in $3-$dimensional null cone and null curves of the $AW(k)-$type in Lorentzian space [7, 8]. *Ersoy* et al. studied Mannheim partner curves of $AW(k)-$type in E^3 [9]. Considering Mannheim curves, they investigated the necessary and sufficent conditions for Mannheim curve to be $AW(k)-$type in E^3. However, in the literature, there is no studies related with Mannheim partner curves of $AW(k)-$type in E_1^3. Therefore, it is necessary to research Mannheim partner curves of $AW(k)-$type in E_1^3.

The main purpose of this paper is to carry out some results which were given in [3] and [6] to non-null curves of $AW(k)-$type and to obtain conditions to be $AW(k)-$type for any types of non-null Mannheim partner curves, in E_1^3.

2. Preliminaries

The Minkowski 3-space E_1^3 is the real vector space E^3 provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. According to this metric, in E_1^3 an arbitrary vector $v = (v_1, v_2, v_3)$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (ligthlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively. The vector product of x and y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2)$$

for the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3 [14].

Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively. Depending on the causal character of the non-null curve α, we have the following Frenet formulae [12, 13]:

$$\begin{cases} T' = \kappa N, N' = -\kappa T + \tau B, B' = \tau N \\ \langle T, T\rangle = \langle N, N\rangle = 1, \langle B, B\rangle = -1, \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0 \end{cases}$$

if α is a spacelike curve with a spacelike principal normal N,

$$\begin{cases} T' = \kappa N, N' = \kappa T + \tau B, B' = \tau N \\ \langle T, T\rangle = \langle B, B\rangle = 1, \langle N, N\rangle = -1, \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0 \end{cases}$$

if α is a spacelike curve with a timelike principal normal N,

$$\begin{cases} T' = \kappa N, N' = \tau N, B' = -\kappa T - \tau B \\ \langle T, T\rangle = 1, \langle N, N\rangle = \langle B, B\rangle = 0, \langle T, N\rangle = \langle T, B\rangle = 0, \langle N, B\rangle = 1 \end{cases}$$

and finally

$$\begin{cases} T' = \kappa N, N' = \kappa T + \tau B, B' = -\tau N \\ \langle T, T\rangle = -1, \langle B, B\rangle = \langle N, N\rangle = 1, \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0 \end{cases}$$

if α is a timelike curve. The functions $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called the curvature and the torsion of α, respectively.

3. $AW(k)-$Type Curves in E_1^3

Let $\alpha : I \subset E \to E_1^3$ be a unit speed curve in E_1^3. The curve α is a Frenet curve of osculating order 3 when its higher order derivatives $\alpha'(s), \alpha''(s), \alpha'''(s)$ are linearly independent, and $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)$ are linearly dependent for all $s \in I$. To each Frenet curve of osculating order 3 one can associate an frame $\{T, N, B\}$ along α called the Frenet frame and curvature functions κ and τ.

A regular curve $\alpha : I \subset E \to E_1^3$ is called a $W-$curve of rank 3, if α is a Frenet curve of osculating order 3 and the Frenet curvatures κ and τ are non-zero constants.

In this section, we consider a non-null α curve of osculating order 3 and investigate conditions to be of $AW(k)-$type curve of α, in E_1^3. Then, we have the following cases:

Case 1. Let α be a spacelike curve with a spacelike principal normal.

Proposition 3.1. Let α be a spacelike curve with a spacelike principal normal of osculating order 3 in E_1^3. Thus, we have

$$\begin{aligned}
\alpha'(s) &= T(s),\\
\alpha''(s) &= D_T\alpha'(s) = \kappa(s)N(s),\\
\alpha'''(s) &= D_TD_T\alpha'(s) = -\kappa^2(s)T(s) + \kappa'(s)N(s) + \kappa(s)\tau(s)B(s),\\
\alpha''''(s) &= D_TD_TD_T\alpha'(s) = -3\kappa(s)\kappa'(s)T(s) + (\kappa''(s) - \kappa^3(s) + \kappa(s)\tau^2(s))N(s)\\
&\quad +(2\kappa'(s)\tau(s) + \kappa(s)\tau'(s))B(s).
\end{aligned}$$

Notation 1. Let us write

$$N_1(s) = \kappa(s)N(s), \tag{3.1}$$

$$N_2(s) = \kappa'(s)N(s) + \kappa(s)\tau(s)B(s), \tag{3.2}$$

$$\begin{aligned} N_3(s) &= (\kappa''(s) - \kappa^3(s) + \kappa(s)\tau^2(s))N(s)\\ &\quad +(2\kappa'(s)\tau(s) + \kappa(s)\tau'(s))B(s). \end{aligned} \tag{3.3}$$

Corollary 3.1. $\left\{\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)\right\}$ is linearly dependent if and only if $\{N_1(s), N_2(s), N_3(s)\}$ is linearly dependent.

As in Euclidean 3−space, we give the following definition for $AW(k)-$type curves in Minkowski 3−space.

Definition 3.1. [1]A regular curve of osculating order 3 in E_1^3 is

i) of type $AW(1)-$type if they satisfy $N_3(s) = 0$,
ii) of type $AW(2)$ if they satisfy $\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s)$,
iii) of type $AW(3)$ if they satisfy $\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s)$.

Theorem 3.1. *Let α be a spacelike curve with a spacelike principal normal of osculating order 3 in E_1^3. Then, α is $AW(1)-$type curve if and only if .*

$$\kappa''(s) - \kappa^3(s) + \kappa(s)\tau^2(s) = 0 \tag{3.4}$$

and

$$\tau(s) = \frac{c}{\kappa^2(s)} (c = constant) \tag{3.5}$$

Proof. Let α be an $AW(1)-$type curve. From Definition 3.1, $N_3(s) = 0$. Then, we have

$$(\kappa''(s) - \kappa^3(s) + \kappa(s)\tau^2(s))N(s) + (2\kappa'(s)\tau(s) + \kappa(s)\tau'(s))B(s) = 0.$$

Furthermore, since N and B are linearly independent, one can obtain (3.4) and (3.5).The converse statement is trivial. The proof is completed. □

Corollary 3.2. Spacelike $W-$curves with spacelike principal normal with the curvature $\kappa = |\tau|$ are $AW(1)-$type curves in E_1^3.

Theorem 3.2. *Let α be a spacelike curve with a spacelike principal normal of osculating order 3 in E_1^3. Then, α is $AW(2)-$type curve if and only if .*

$$2(\kappa'(s))^2\tau(s) + \kappa(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s) - \kappa^4(s)\tau(s) + \kappa^2(s)\tau^3(s). \tag{3.6}$$

Proof. Let α be an $AW(2)-$type curve. From Definition 3.1, $\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s)$. Then, we have

$$\|N_2(s)\|^2 N_3(s) = \left[\begin{array}{c}(\kappa'(s))^2\kappa''(s)-(\kappa'(s))^2\kappa^3(s)+(\kappa'(s))^2\kappa(s)\tau^2(s)\\ -\kappa^2(s)\kappa''(s)\tau^2(s)+\kappa^5(s)\tau^2(s)-\kappa^3(s)\tau^4(s)\end{array}\right]N(s)$$
$$(3.7)\qquad +\left[\begin{array}{c}2(\kappa'(s))^3\tau(s)+(\kappa'(s))^2\kappa(s)\tau'(s)\\ -2\kappa^2(s)\kappa'(s)\tau^3(s)-\kappa^3(s)\tau^2(s)\tau'(s)\end{array}\right]B(s)\,(\mathbf{1})$$

and

$$\langle N_3(s), N_2(s)\rangle N_2(s) = [(\kappa'(s))^2\kappa''(s)-(\kappa'(s))^2\kappa^3(s)$$
$$-(\kappa'(s))^2\kappa(s)\tau^2(s)-\kappa'(s)\kappa^2(s)\tau'(s)\tau(s)]N(s)$$
$$(3.8)\qquad +\left[\begin{array}{c}\kappa'(s)\kappa''(s)\kappa(s)\tau(s)-\kappa'(s)\kappa^4(s)\tau(s)\\ -\kappa'(s)\kappa^2(s)\tau^3(s)-\kappa^3(s)\tau^2(s)\tau'(s)\end{array}\right]B(s)\,.\,(\mathbf{2})$$

From (1), and (2), we get

$$(\kappa'(s))^2\kappa(s)\tau^2(s)-\kappa^2(s)\kappa''(s)\tau^2(s)+\kappa^5(s)\tau^2(s)-\kappa^3(s)\tau^4(s)$$
$$(3.9)\qquad = -(\kappa'(s))^2\kappa(s)\tau^2(s)-\kappa'(s)\kappa^2(s)\tau'(s)\tau(s)\,(\mathbf{3})$$

and

$$2(\kappa'(s))^3\tau(s)+(\kappa'(s))^2\kappa(s)\tau'(s)-2\kappa^2(s)\kappa'(s)\tau^3(s)$$
$$(3.10)\qquad = \kappa'(s)\kappa''(s)\kappa(s)\tau(s)-\kappa'(s)\kappa^4(s)\tau(s)-\kappa'(s)\kappa^2(s)\tau^3(s)\,.\,(\mathbf{4})$$

If we multiply by $\frac{1}{\kappa(s)\tau(s)}$ both sides of (3), we obtain (3.6). The contrary is clearly established. Thus, our theorem is proved. □

Corollary 3.3. Spacelike $W-$curves with spacelike principal normal with the curvature $\kappa = |\tau|$ are $AW(2)-$type curves in E_1^3.

Example 3.1. Let α be defined by $\alpha(s) = \left(-\frac{1}{6}s^3, -\frac{1}{6}s^3 + s, \frac{1}{2}s^2\right)$ in E_1^3. Then, α is a $AW(2)-$type curve with the curvatures $\kappa = \tau = 1$.

Corollary 3.4. From Corollary 3.2 and Corollary 3.3, every spacelike $W-$curves with spacelike principal normal of type $AW(2)$ with the curvature $\kappa = |\tau|$ are $AW(1)-$type curves in E_1^3.

Theorem 3.3. *Let α be a spacelike curve with a spacelike principal normal of osculating order 3 in E_1^3. Then, α is $AW(3)-$type curve if and only if*

$$(3.11)\qquad \tau(s) = \frac{c}{\kappa^2(s)}(c = constant)$$

Proof. Let α be an $AW(3)-$type curve. From Definition 3.1, $\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s)$. Then, we have

$$\|N_1(s)\|^2 N_3(s) = \left[\kappa^2(s)\kappa''(s)-\kappa^5(s)+\kappa^3(s)\tau^2(s)\right]N(s)$$
$$(3.12)\qquad +\left[2\kappa^2(s)\kappa'(s)\tau(s)+\kappa^3(s)\tau'(s)\right]B(s)\ (\mathbf{5})$$

and

$$\langle N_3(s), N_1(s)\rangle N_1(s) = [\kappa^2(s)\kappa''(s)-\kappa^5(s)$$
$$(3.13)\qquad +\kappa^3(s)\tau^2(s)]N(s)\,(\mathbf{6})$$

By virtue of (5) and (6), we get

$$2\kappa^2(s)\kappa'(s)\tau(s)+\kappa^3(s)\tau'(s) = 0.$$

Thus, we obtain (3.11). □

Corollary 3.5. All spacelike $W-$curves with spacelike principal normal are $AW(3)-$type curves in E_1^3.

Corollary 3.6. From Corollary 3.4 and Corollary 3.5, we get

$$AW(1) \subset AW(2) \subset AW(3)$$

for every spacelike $W-$curves with spacelike principal normal with the curvature $\kappa = |\tau|$.

For the other cases, the proof can be shown similarly.

Case 2. Let α be a spacelike curve with a timelike principal normal.

Proposition 3.2. Let α be a spacelike curve with a timelike principal normal of osculating order 3 in E_1^3. Thus, we have

$$\begin{aligned}
\alpha'(s) &= T(s), \\
\alpha''(s) &= \kappa(s)N(s), \\
\alpha'''(s) &= \kappa^2(s)T(s) + \kappa'(s)N(s) + \kappa(s)\tau(s)B(s), \\
\alpha''''(s) &= 3\kappa(s)\kappa'(s)T(s) + (\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s))N(s) \\
&\quad +(2\kappa'(s)\tau(s) + \kappa(s)\tau'(s))B(s).
\end{aligned}$$

Notation 2. Let us write

$$\begin{aligned}
N_1(s) &= \kappa(s)N(s), \\
N_2(s) &= \kappa'(s)N(s) + \kappa(s)\tau(s)B(s), \\
N_3(s) &= (\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s))N(s) + (2\kappa'(s)\tau(s) + \kappa(s)\tau'(s))B(s).
\end{aligned}$$

Corollary 3.7. $\left\{\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)\right\}$ is linearly dependent if and only if $\{N_1(s), N_2(s), N_3(s)\}$ is linearly dependent.

Theorem 3.4. *Let α be a spacelike curve with a timelike principal normal of osculating order* 3 *in E_1^3. Then, α is $AW(1)-$type curve if and only if .*

$$\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s) = 0$$

and

$$\tau(s) = \frac{c}{\kappa^2(s)}(c = constant).$$

Theorem 3.5. *Let α be a spacelike curve with a timelike principal normal of osculating order* 3 *in E_1^3. Then, α is $AW(2)-$type curve if and only if*

$$2(\kappa'(s))^2\tau(s) + \kappa(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s) + \kappa^4(s)\tau(s) + \kappa^2(s)\tau^3(s).$$

Corollary 3.8. As a result of Teorem 4 and Teorem 5, there are no $W-$curves of $AW(1)-$type and of $AW(2)-$type.

Theorem 3.6. *Let α be a spacelike curve with a timelike principal normal of osculating order* 3 *in E_1^3. Then, α is $AW(3)-$type curve if and only if*

$$\tau(s) = \frac{c}{\kappa^2(s)}(c = constant).$$

Corollary 3.9. All spacelike $W-$curves with timelike principal normal are $AW(3)-$type curves in E_1^3.

Example 3.2. Let α be defined by

$$\alpha(s)=\left(\frac{\sqrt{2}}{\sqrt{3}}s+\frac{1}{2\sqrt{5}}e^{\sqrt{5}s}+\frac{1}{3\sqrt{5}}e^{-\sqrt{5}s},\frac{\sqrt{2}}{\sqrt{3}}s+\frac{1}{2\sqrt{5}}e^{\sqrt{5}s},-\frac{\sqrt{2}}{\sqrt{3}}s-\frac{1}{3\sqrt{5}}e^{-\sqrt{5}s}\right)$$

in E_1^3. Then, α is a $AW(3)-$type curve with the curvatures $\kappa=1$, $\tau=2$.

Case 3. Let α be a timelike curve.

Proposition 3.3. Let α be a timelike curve of osculating order 3 in E_1^3. Thus, we have

$$\begin{aligned}\alpha'(s) &= T(s),\\ \alpha''(s) &= \kappa(s)N(s),\\ \alpha'''(s) &= \kappa^2(s)T(s)+\kappa'(s)N(s)+\kappa(s)\tau(s)B(s),\\ \alpha''''(s) &= 3\kappa(s)\kappa'(s)T(s)+(\kappa''(s)+\kappa^3(s)-\kappa(s)\tau^2(s))N(s)\\ &\quad +(2\kappa'(s)\tau(s)+\kappa(s)\tau'(s))B(s).\end{aligned}$$

Notation 3. Let us write

$$\begin{aligned}N_1(s) &= \kappa(s)N(s),\\ N_2(s) &= \kappa'(s)N(s)+\kappa(s)\tau(s)B(s),\\ N_3(s) &= (\kappa''(s)+\kappa^3(s)-\kappa(s)\tau^2(s))N(s)+(2\kappa'(s)\tau(s)+\kappa(s)\tau'(s))B(s).\end{aligned}$$

Corollary 3.10. $\left\{\alpha'(s),\alpha''(s),\alpha'''(s),\alpha''''(s)\right\}$ is linearly dependent if and only if $\{N_1(s),N_2(s),N_3(s)\}$ is linearly dependent.

Theorem 3.7. *Let α be a timelike curve of osculating order 3 in E_1^3. Then, α is $AW(1)-$type curve if and only if*

$$\kappa''(s)+\kappa^3(s)-\kappa(s)\tau^2(s)=0$$

and

$$\tau(s)=\frac{c}{\kappa^2(s)}(c=constant)$$

Corollary 3.11. $W-$ timelike curves with the curvature $\kappa=|\tau|$ are $AW(1)-$type curves in E_1^3.

Theorem 3.8. *Let α be a timelike curve of osculating order 3 in E_1^3. Then, α is $AW(2)-$type curve if and only if*

$$-2(\kappa'(s))^2\tau(s)-\kappa(s)\kappa'(s)\tau'(s)=-\kappa(s)\kappa''(s)\tau(s)-\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

Corollary 3.12. $W-$timelike curves with the curvature $\kappa=|\tau|$ are $AW(2)-$type curves in E_1^3.

Corollary 3.13. From Corollary 3.11, and Corollary 3.12, every $W-$timelike curves of type $AW(2)-$type with the curvature $\kappa=|\tau|$ are $AW(1)-$type curves in E_1^3.

Theorem 3.9. *Let α be a timelike curve of osculating order 3 in E_1^3. Then, α is $AW(3)-$type curve if and only if*

$$\tau(s)=\frac{c}{\kappa^2(s)}(c=constant).$$

Corollary 3.14. All $W-$timelike curves are $AW(3)-$type curves in E_1^3.

Example 3.3. Let α be defined by $\alpha(s) = \left(2\sinh s, 2\cosh s, \sqrt{3}s\right)$ in E_1^3. Then, α is a $AW(3)-$type curve with the curvatures $\kappa = 2$, $\tau = \sqrt{3}$.

Corollary 3.15. From Corollary 3.13 and Corollary 3.14, we get

$$AW(1) \subset AW(2) \subset AW(3)$$

for every $W-$timelike curves with the curvature $\kappa = |\tau|$ in E_1^3.

4. Mannheim Partner Curves of $AW(k)-$type in Minkowski 3−Space

Definition 4.1. [10]Let α and α^* be two curves in the Minkowski 3−space given by the parametrizations $\alpha(s)$ and $\alpha^*(s^*)$, respectively, and let them have at least four continous derivatives. If there exist a correspondence between the space curves α and α^* such that the principal normal lines of α coincide with the binormal lines of α^* at the corresponding points of curves, then α is called a Mannheim curve and α^* is called a Mannheim partner curve of α. The pair $\{\alpha, \alpha^*\}$ is said to be a Mannheim pair.

By considering the casual characters of the non-null curves, it is easily seen from Definition 4.1 that there are five different types of the Mannheim partner curves in the Minkowski 3−space. Let the pair $\{\alpha, \alpha^*\}$ be a Mannheim pair. Then, according to the characters of the curves α and α^* we have the following cases [11]:

Case 4. The curve α^* is timelike. Then, there are two cases.

- The curve α is a spacelike curve with a timelike principal normal. In this case, we say that the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of **type 1**.
- The curve α is a timelike curve. In this case, we say that the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of **type 2**.

Case 5. The curve α^* is spacelike. Then, there are three cases.

- The curve α^* is spacelike curve with a timelike binormal vector and the curve α is a spacelike curve with a timelike principal normal vector. In this case, we say that the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of **type 3**.
- The curve α^* is spacelike curve with a timelike binormal vector and the curve α is a timelike curve. In this case, we say that the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of **type 4**.
- The curve α^* is spacelike curve with a timelike principal vector and the curve α is a spacelike curve with a timelike binormal vector. In this case, we say that the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of **type 5**.

Theorem 4.1. [11]*Let α be a curve in E_1^3.*

i) • If α is a Mannheim curve of type 1,2 or 5, then the relationship between the curvature and torsion of the curve α is given as follows:

$$\mu\tau(s) + \lambda\kappa(s) = 1.$$

• If α is a Mannheim curve of type 3 or 4, then the relationship is given by

$$\mu\tau(s) - \lambda\kappa(s) = 1,$$

where λ and μ are nonzero real numbers.

ii) Let $\{\alpha, \alpha^*\}$ be a Mannheim pair. Then, we have

$$\alpha^* = \alpha - cN$$

for a nonzero constant c.

As a result of Theorem 4.1, we have the following corollaries:

Corollary 4.1. Let $\{\alpha, \alpha^*\}$ be Mannheim pair in E_1^3.

i) If the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of type 1,2,3 or 4, then we have

$$\alpha^{*\prime} = (1 - c\kappa)\, T + c\tau B$$

and

ii) If the pair $\{\alpha, \alpha^*\}$ is a Mannheim pair of type 5, then we have

$$\alpha^{*\prime} = (1 + c\kappa)\, T + c\tau B$$

where c is a nonzero real number.

Corollary 4.2. Let α be a curve of osculating order 3 in E_1^3. Then,

i) α is a Mannheim curve of type 1,2 or 5 if and only if there is a nonzero real number λ such that

$$\lambda\,(\kappa\tau' - \kappa'\tau) - \tau' = 0 \tag{4.1}$$

and

ii) α is a Mannheim curve of type 3 or 4 if and only if there is a nonzero real number λ such that

$$\lambda\,(\kappa'\tau - \kappa\tau') - \tau' = 0. \tag{4.2}$$

Theorem 4.2. *Let α be a Mannheim curve of osculating order* 3 *in E_1^3.If α is a $AW(1)-$type Mannheim curve, then α is of type 2,4 or 5.*

Proof. Let α be a $AW(1)-$type Mannheim curve of type 1 in E_1^3. Then, α is a spacelike curve with a timelike principal normal. From Theorem 3.4, we have

$$\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s) = 0 \tag{4.3}$$

and

$$\tau(s) = \frac{c}{\kappa^2(s)}(c = \text{constant}). \tag{4.4}$$

By differentiating the equation (4.4), we get

$$\tau'(s) = -\frac{2\kappa'(s)}{\kappa^3(s)}c. \tag{4.5}$$

If we substitute the equations (4.4) and (4.5) in the equation (4.1), we obtain

$$\kappa(s) = \frac{2}{3\lambda}. \tag{4.6}$$

Moreover; from the equations (4.3), (4.4) and (4.6), we find

$$\lambda^6 = -\frac{64}{(27c)^2}$$

which gives us $\lambda^6 < 0$. Since λ is a real number, contraction is obtained. Thus, α can't be a $AW(1)-$type Mannheim curve of type 1.

Now assume that α is a $AW(1)-$type Mannheim curve of type 2 in E_1^3. Then, α is a timelike curve. From Theorem 3.7, we have the equations

$$\kappa''(s) + \kappa^3(s) - \kappa(s)\tau^2(s) = 0 \tag{4.7}$$

and (4.4). After substituting the equations (4.4) and (4.5) into (4.1), we obtain the equation (4.6). From the equations (4.7), (4.4) and (4.6), we find

$$\lambda = \pm \frac{2}{(27c)^{1/3}}.$$

For the other types, the proof can be given similarly. □

Theorem 4.3. *Let α be a Mannheim curve of osculating order 3 in E_1^3.*

i) Let α be a Mannheim curve of type 1. Then, α is a $AW(2)-$type curve if and only if

$$(\kappa'(s))^2\tau(s)(2-\lambda\kappa(s))+\lambda\kappa^2(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s)+\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

ii) Let α be a Mannheim curve of type 2. Then, α is a $AW(2)-$type curve if and only if

$$(\kappa'(s))^2\tau(s)(-2+\lambda\kappa(s))-\lambda\kappa^2(s)\kappa'(s)\tau'(s) = -\kappa(s)\kappa''(s)\tau(s)-\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

iii) Let α be a Mannheim curve of type 3. Then, α is a $AW(2)-$type curve if and only if

$$(\kappa'(s))^2\tau(s)(2+\lambda\kappa(s))-\lambda\kappa^2(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s)+\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

iv) Let α be a Mannheim curve of type 4. Then, α is a $AW(2)-$type curve if and only if

$$-(\kappa'(s))^2\tau(s)(2+\lambda\kappa(s))+\lambda\kappa^2(s)\kappa'(s)\tau'(s) = -\kappa(s)\kappa''(s)\tau(s)-\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

v) Let α be a Mannheim curve of type 5. Then, α is a $AW(2)-$type curve if and only if

$$(\kappa'(s))^2\tau(s)(2-\lambda\kappa(s))+\lambda\kappa^2(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s)+,\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

Proof. *i)* Let α be a Mannheim curve of type 1. From Theorem 3.5, we have

$$2(\kappa'(s))^2\tau(s) + \kappa(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s) + \kappa^4(s)\tau(s) + \kappa^2(s)\tau^3(s).$$

If we substitute (4.1) into the last equation, it is easily seen that

$$(\kappa'(s))^2\tau(s)(2-\lambda\kappa(s))+\lambda\kappa^2(s)\kappa'(s)\tau'(s) = \kappa(s)\kappa''(s)\tau(s)+\kappa^4(s)\tau(s)+\kappa^2(s)\tau^3(s).$$

The converse assertion is trivial. Thus, the proof is completed.

The proofs of the statements *ii)*, *iii)*, *iv)*, and *v)* in Theorem 3 can be given in a similar way of the proof of statement *i)*. □

Theorem 4.4. *Let α be a Mannheim curve of osculating order* 3 *in E_1^3. The curve α is of type $AW(3)$ if and only if α is a circular helix.*

Proof. Let α be a $AW(3)-$type Mannheim curve of type 1 in E_1^3. Then, α is a spacelike curve with a timelike principal normal. From Theorem 3.6, we have

$$\tau(s) = \frac{c}{\kappa^2(s)}(c = \text{constant}). \tag{4.8}$$

By differentiating the equation (4.8) and using the equation (4.1), we find

$$\kappa(s) = \frac{2}{3\lambda} = \text{constant.}$$

Substituting the last equation in (4.8), the following equation is obtained

$$\tau(s) = \frac{9\lambda^2 c}{4} = \text{constant.}$$

Since $\kappa(s)$ and $\tau(s)$ are nonzero constants, α is a circular helix.

The converse statement is trivial. Hence, theorem is proved.

For the other types, the proof can be given similarly. □

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