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RADICAL SCREEN TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. We study screen transversal lightlike submanifolds of semi-Riemannian product manifolds. We give examples, investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. We also obtain characterization of screen transversal anti-invariant lightlike submanifolds.

1. Introduction

A submanifold M of a semi-Riemannian manifold $\bar{M}$ is called lightlike (degenerate) submanifold if the induced metric on M is degenerate. Lightlike submanifolds of a semi-Riemannian manifold have been studied by Duggal-Bejancu and Kupeli in [5] and [12], respectively. Kupeli's approach is intrinsic while Duggal-Bejancu's approach is extrinsic. Lightlike submanifolds have their applications in mathematical physics. Indeed, lightlike submanifolds appear in general relativity as some smooth parts of event horizons of the Kruskal and Keer black holes[10] . In [5], Duggal and Bejancu introduced CR- lightlike submanifolds of indefinite Kaehler manifold and discussed lightlike version of non-degenerate CR- submanifolds. They show that such lightlike submanifolds do not contain invariant and anti-invariant submanifolds contrary to the non-degenerate CR-submanifolds.Therefore, in [8] (see also[9]), Duggal and Sahin introduced screen CR-lightlike submanifolds, and showed that such lightlike submanifolds include invariant lightlike submanifolds as well as anti-invariant(screen real) submanifolds of indefinite Kaehler manifolds.

They also show that there is no inclusion relation between CR-lightlike submanifolds and SCR-lightlike submanifolds. On the other hand, lightlike submanifolds of almost para Hermitian manifolds were investigated by Bejan in [2]. She mainly studied invariant lightlike submanifolds of para-Hermitian manifolds in that paper.

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As an analogue of CR-lightlike submanifolds, semi-invariant lightlike submanifolds of semi-Riemannian product manifolds were introduced by Atceken and Kilic in [1]. Kilic and Sahin[11] introduced radical anti invariant lightlike submanifolds of semi-Riemannian product manifolds and studied their geometry. On the other hand, screen transversal lightlike submanifolds of indefinite Kaehler manifolds were introduced in [15] and such submanifolds were also studied in indefinite contact geometry. They show that such submanifolds include real curves of indefinite contact manifolds. They investigated the geometry of distributions and obtained necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. They also check the existence of screen transversal lightlike submanifolds in indefinite Sasakian manifolds.

Considering above information on radical anti-invariant lightlike submanifolds of semi-Riemannian product manifolds and screen transversal lightlike submanifolds of indefinite Sasakian manifolds, similar research is needed for the screen transversal lightlike submanifolds of semi-Riemannian product manifolds. Therefore, as a first step, in this paper, we introduce screen transversal lightlike submanifolds of semi-Riemannian product manifolds and study their geometry.

The paper is organized as follows: In section 2, we give basic information needed for this paper. In section 3 and section 4, we introduce semi-Riemannian product manifold along with its subclasses (radical screen transversal, screen transversal anti-invariant lightlike submanifolds) and obtain a characterization of screen transversal anti-invariant lightlike submanifolds. We investigate the geometry of distributions and obtain necessary and sufficient condition for induced connection to be a metric connection. In section 5, we study radical screen transversal lightlike submanifolds and find the integrability of distributions. We give two examples.

2. Preliminaries

A submanifold M^m immersed in a semi-Riemannian manifold $\left(\bar{M}^{m+k}, \bar{g}\right)$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution which is a semi-Riemannian complementary distribution of $RadTM$ is of rank r , where$1 \leq r \leq m.RadTM = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \cup_{x\in M} \left\{ u \in T_x\bar{M} \mid \bar{g}\left(u, v\right) = 0, \forall v \in T_xM \right\}. \tag{2.1}$$

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM . i.e., $TM = RadTM \perp S(TM)$.

We consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. It is known that, for any local basis $\{\xi_i\}$of $RadTM$, there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [5]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}^{\perp} \mid_M$. Then

$$\begin{aligned} tr(TM) &= ltrTM \perp S(TM^{\perp}), \\ T\bar{M}\mid_M &= S(TM) \perp [RadTM \oplus ltrTM] \perp S(TM^{\perp}). \end{aligned}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/RadTM$ [12]. The following result is important to this paper.

Proposition 2.1. [5]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$.*

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is
Case1: r-lightlike if $r < \min\{m, k\}$;
Case2: Co-isotropic if $r = k < m; S(TM^{\perp}) = \{0\}$;
Case3: Isotropic if $r = m = k$; $S(TM) = \{0\}$;
Case4: Totally lightlike if $r = k = m$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{2.2}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{2.3}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y) + h^s(X, Y), \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\ell}_X N + D^s(X, N), N \in \Gamma(ltr(TM)), \tag{2.5}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^{\ell}(X, W), W \in \Gamma\left(S(TM^{\perp})\right). \tag{2.6}$$

Denote the projection of TM on $S(TM)$ by $\bar{P}$.Then by using (2.2), (2.4)-(2.6) and a metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}\left(Y, D^{\ell}(X, W)\right) = g(A_W X, Y), \tag{2.7}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X). \tag{2.8}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*\left(X, \bar{P}Y\right), \tag{2.9}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.10}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. By using above equations, we obtain

$$\begin{aligned}
\bar{g}\left(h^{\ell}\left(X, \bar{P}Y\right), \xi\right) &= g\left(A^*_{\xi} X, \bar{P}Y\right),\\
\bar{g}\left(h^s\left(X, \bar{P}Y\right), N\right) &= g\left(A_N X, \bar{P}Y\right),\\
\bar{g}\left(h^{\ell}(X, \xi), \xi\right) &= 0, A^*_{\xi}\xi = 0.
\end{aligned}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.4) we get

$$(\nabla_X g)(Y, Z) = \bar{g}\left(h^{\ell}(X, Y), Z\right) + \bar{g}\left(h^{\ell}(X, Z), Y\right). \tag{2.11}$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$. We recall that the Gauss equation of lightlike submanifolds is given by

$$\begin{aligned}
\bar{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^{\ell}(X,Z)}Y - A_{h^{\ell}(Y,Z)}X + A_{h^s(X,Z)}Y\\
& - A_{h^s(Y,Z)}X + \left(\nabla_X h^{\ell}\right)(Y, Z) - \left(\nabla_Y h^{\ell}\right)(X, Z)\\
& + D^{\ell}(X, h^s(Y, Z)) - D^{\ell}(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z)\\
& - (\nabla_Y h^s)(X, Z) + D^s(X, h^{\ell}(Y, Z)) - D^s(Y, h^{\ell}(X, Z))
\end{aligned}$$

for $\forall X, Y, Z \in \Gamma(TM)$.

3. Semi-Riemannian Product Manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2 -dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0, q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$, respectively.

We denote the product manifold by $\bar{M} = (M_1 \times M_2, \bar{g})$, where

$$\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $\forall X, Y \in \Gamma(T\bar{M})$. Then we have

$$\pi_*^2 = \pi_*, \pi_*\sigma_* = \sigma_*\pi_* = 0,$$

$$\sigma_*^2 = \sigma_*, \pi_* + \sigma_* = I,$$

where I is identity transformation. Thus $(\bar{M}, \bar{g})$ is an $(m_1 + m_2)$- dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\bar{M} = M_1 \times M_2$ is characterized by M_1 and M_2 are totally geodesic submanifolds of $\bar{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that

$$F^2 = \pi_*^2 - \pi_*\sigma_* - \sigma_*\pi_* + \sigma_*^2 = I$$

$$\bar{g}(FX, Y) = \bar{g}(X, FY) \tag{3.1}$$

for any $X, Y \in \Gamma(T\bar{M})$. If we denote the Levi-civita connection on $\bar{M}$ by $\bar{\nabla}$, then it can be seen that

$$\left(\bar{\nabla}_X F\right) Y = 0, \tag{3.2}$$

for any $X, Y \in \Gamma(T\bar{M})$, that is, F is parallel with respect to $\bar{\nabla}$ [16].

Let M be a submanifold of a Riemannian (or semi-Riemannian) product manifold $\bar{M} = M_1 \times M_2$. If $F(TM) = TM$, then M is called an invariant submanifold[16].

4. Screen Transversal Lightlike Submanifolds of semi-Riemannian Product Manifolds

In this section, we introduce screen transversal lightlike submanifolds of semi-Riemannian product manifolds.We investigate the integrability of distributions, give a necessary and sufficient condition for the induced connection to be a metric connection.

Lemma 4.1. *Let M be a lightlike submanifold of semi-Riemannian product manifolds $(\bar{M}, \bar{g})$ and let $FRadTM \subset S(TM^{\perp})$. Then, $FltrTM$ is also subvector bundle of screen transversal bundle.*

Proof. Let us assume that $ltrTM$ is invariant with respect to F , i.e., $F(ltrTM) = ltrTM$. By the definition of a lightlike submanifold, there exist vector fields $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltrTM)$ such that $\bar{g}(\xi, N) = 1$. Also from (3.1) we get

$$\bar{g}(\xi, N) = \bar{g}(F\xi, FN) = 1.$$

However, if $FN \in \Gamma(ltrTM)$ then by hypothesis, we get $\bar{g}(F\xi, FN) = 0$.Hence, we obtain a contradiction which implies that FN does not belong to $ltrTM$. Now, suppose that $FN \in \Gamma(S(TM))$. Then, in a similar way, from (3.1) we have

$$1 = \bar{g}(\xi, N) = \bar{g}(F\xi, FN) = 0$$

since $F\xi \in \Gamma\left(S(TM^{\perp})\right)$ and $FN \in \Gamma\left(S(TM)\right)$. Thus, FN does not belong to $S(TM)$.We can also obtain that FN does not belong to $RadTM$. Then, from the decomposition of a lightlike submanifold, we conclude that $FN \in \Gamma(S(TM^{\perp}))$. $\square$

Definition 4.1. Let M be a lightlike submanifold of semi-Riemannian product manifolds $\left(\bar{M}, \bar{g}\right)$ .Then

$$F(RadTM) \subset S(TM^{\perp})$$

we say that M is a screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$.

Definition 4.2. Let M be a screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then

(i) We say that M is a radical screen transversal lightlike submanifold if $F(S(TM)) = S(TM)$.

(ii) We say that M is a screen transversal anti-invariant lightlike submanifold of $\bar{M}$ if $F(S(TM)) \subset S(TM^{\perp})$.

If M is screen transversal anti-invariant submanifold, we have

$$S(TM^{\perp}) = FRadTM \oplus FltrTM \perp FS(TM) \perp D_0.$$

where D_0 is a non degenerate orthogonal complementary distribution to

$$FRadTM \oplus FltrTM \perp FS(TM)$$

in $S(TM^{\perp})$. For the distribution D_0, we have the following.

Proposition 4.1. *Let M be screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then , the distribution D_0 is invariant with respect to F.*

Proof. Using (3.1), we have

$$g(FX, \xi) = g(\xi, FN) = 0,$$

which show that FX does not belong to $ltrTM$

$$\begin{aligned} g(FX, N) &= g(X, FN) = 0, \\ g(FX, F\xi) &= g(X, F^2\xi) = g(X, \xi) = 0, \\ g(FX, FN) &= g(X, F^2N) = g(X, N) = 0, \\ g(FX, Z) &= g(X, FZ) = 0, \\ g(FX, FZ) &= g(X, F^2Z) = g(X, Z) = 0 \end{aligned}$$

for $X \in \Gamma\left(D_0\right)$, $\xi \in \Gamma(RadTM)$, $N \in \Gamma(ltrTM)$, $Z \in \Gamma(S(TM))$. Hence, the distribution D_0 is invariant with respect to F. $\square$

Let M be screen transversal anti-invariant lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Let S and R be projection transformations of $S(TM)$ and $RadTM$, respectively. Then, for $X \in \Gamma(TM)$ we have

$$X = SX + RX \tag{4.1}$$

on the other hand, if we apply F to (4.1), we find

$$\begin{aligned} FX &= FSX + FRX, \\ FX &= S_1X + S_2X \end{aligned}$$

for $S_1X \in \Gamma\left(S(TM)\right)$, $S_2X \in \Gamma\left(S(TM^{\perp})\right)$.

Also, for any $v \in \Gamma(tr(TM))$ we can find

$$FV = BV + CV$$

where $BV \in \Gamma(TM)$ and $CV \in \Gamma(tr(TM))$.

Let F_1, F_2, F_3, F_4 be the projection morphisms on $FRadTM$, $FS(TM)$, $F(ltr(TM))$ and D_0, respectively. Then, for $V \in \Gamma(S(TM^{\perp}))$ we have

$$V = F_1V + F_2V + F_3V + F_4V. \tag{4.2}$$

Applying F to (4.2), we get

$$FV = FF_1V + FF_2V + FF_3V + FF_4V. \tag{4.3}$$

If we put $FF_1 = B_1$, $FF_2 = B_2$, $FF_3 = C_1$, $FF_4 = C_2$, we can rewrite (4.3) as

$$FV = B_1V + B_2V + C_1V + C_2V, \tag{4.4}$$

where $B_1V \in \Gamma(RadTM)$, $B_2V \in \Gamma(S(TM))$, $C_1V \in \Gamma(ltr(TM))$, $C_2V \in \Gamma(D_0)$.

It is known that the induced connection on a screen transversal anti-invariant lightlike submanifold immersed in semi-Riemannian product manifolds is not a metric connection. The condition under which the induced connection on the submanifold is a metric connection is given by the following theorem.

Theorem 4.1. *Let M be a screen transversal anti-invariant lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then , the induced connection ∇ on M is a metric connection if and only if*

$$B_2\nabla^s_X F\xi = 0$$

for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.

Proof. From (3.2), we have

$$\bar{\nabla}_X FY = F\bar{\nabla}_X Y.$$

Taking $Y = E$ in the above equation, we find

$$-A_{F\xi}X + \nabla^s_X F\xi + D^{\ell}(X, F\xi) = F(\nabla_X \xi + h^{\ell}(X,\xi) + h^s(X,\xi)).$$

If we apply F to this equation and from (4.4), we have

$$-FA_{F\xi}X + F\nabla^s_X F\xi + FD^{\ell}(X, F\xi) = \nabla_X \xi + h^{\ell}(X,\xi) + h^s(X,\xi)$$

$$\begin{pmatrix} -FA_{F\xi}X + B_1\nabla^s_X F\xi + B_2\nabla^s_X F\xi \\ +C_1\nabla^s_X F\xi + C_2\nabla^s_X F\xi + FD^{\ell}(X,F\xi) \end{pmatrix} = \nabla_X \xi + h^{\ell}(X,\xi) + h^s(X,\xi)$$

Then, taking the tangential parts of the above equation, we obtain

$$\nabla_X \xi = B_1\nabla^s_X F\xi + B_2\nabla^s_X F\xi.$$

Thus, proof is completed. □

Theorem 4.2. *Let M be a screen transversal anti-invariant lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then the radical distribution is integrable if and only if*

$$\nabla^s_X FY = \nabla^s_Y FX$$

$\forall X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Proof. By the definition of a screen transversal anti-invariant lightlike submanifold, $\Gamma(RadTM)$ is integrable if and only if $g([X,Y],Z)=0$ for $X,Y\in\Gamma(RadTM)$ and $Z\in\Gamma(S(TM))$. Since $\bar{M}$ is a semi-Riemannian product manifold, using (3.1) and (3.2) we find

$$\begin{aligned} g([X,Y],Z) &= g(\bar{\nabla}_X Y,Z)-g(\bar{\nabla}_Y X,Z)\\ &= g(F\bar{\nabla}_X Y,FZ)-g(F\bar{\nabla}_Y X,FZ),\\ &= g(\bar{\nabla}_X FY,FZ)-g(\bar{\nabla}_Y FX,FZ). \end{aligned}$$

Using (2.6), we have

$$\begin{aligned} g([X,Y],Z) &= g(-A_{FY}X+\nabla^s_X FY+D^\ell(X,FY),FZ)\\ &\quad -g(-A_{FX}Y+\nabla^s_Y FX+D^\ell(Y,FX),FZ)\\ &= g(\nabla^s_X FY,FZ)-g(\nabla^s_Y FX,FZ)\\ &= g(\nabla^s_X FY-\nabla^s_Y FX,FZ). \end{aligned}$$

Thus, the proof is completed. □

In the similar way, we have the following.

Theorem 4.3. *Let M be a screen transversal anti-invariant lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then screen distribution is integrable if and only if*

$$\nabla^s_X FY=\nabla^s_Y FX, g(\nabla^s_X FY-\nabla^s_Y FX,FN)=0$$

$\forall X,Y\in\Gamma(S(TM))$ *and* $N\in\Gamma(ltr(TM))$.

5. Radical Screen Transversal Lightlike Submanifolds of semi Riemannian Product Manifolds

In this section, we study radical screen transversal lightlike submanifolds of a semi-Riemannian product manifold. We first investigate the integrability of distributions, give a necessary and sufficient condition for the induced connection to be a metric connection. We also study the geometry of totally radical screen transversal lightlike submanifolds.

Theorem 5.1. *Let M be a radical screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then the screen distribution is integrable if and only if*

$$h^s(X,FY)=h^s(Y,FX)$$

$\forall X,Y\in\Gamma(S(TM))$.

Proof. Screen distribution is integrable if and only if $g([X,Y],N)=0$ and $\forall X,Y\in\Gamma(S(TM))$,we find

$$\begin{aligned} g([X,Y],N) &= g(\bar{\nabla}_X Y,N)-g(\bar{\nabla}_Y X,N)\\ &= g(F\bar{\nabla}_X Y,FN)-g(F\bar{\nabla}_Y X,FN)\\ &= g(\bar{\nabla}_X FY,FN)-g(\bar{\nabla}_Y FX,FN) \end{aligned}$$

Using (2.4), we have

$$\begin{aligned} g([X,Y],N) &= g(h^s(X,FY),FN)-g(h^s(Y,FX),FN)\\ &= g(h^s(X,FY)-h^s(Y,FX),FN). \end{aligned}$$

Thus, the proof is completed. □

Theorem 5.2. *Let M be a radical screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then the radical distribution is integrable if and only if*

$$A_{FX}Y = A_{FY}X$$

$\forall X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Proof. By the definition of radical screen transversal lightlike submanifold, $\Gamma(RadTM)$ is integrable if and only if $g([X,Y], Z) = 0$ for $Z \in \Gamma(S(TM))$ and $\forall X, Y \in \Gamma(RadTM)$. Then, from (3.1) and (2.6) we have

$$\begin{aligned} g([X,Y], Z) &= g(F[X,Y], FZ) \\ &= g(F\bar{\nabla}_X Y, FZ) - g(F\bar{\nabla}_Y X, FZ), \\ &= g(\bar{\nabla}_X FY, FZ) - g(\bar{\nabla}_Y FX, FZ), \end{aligned}$$

using (2.6) we have

$$\begin{aligned} g([X,Y], Z) &= g(-A_{FY}X + \nabla^s_X FY + D^\ell(X, FY), FZ) \\ &\quad -g(-A_{FX}Y + \nabla^s_Y FX + D^\ell(Y, FX), FZ) \end{aligned}$$

thus we find

$$g([X,Y], Z) = g(A_{FX}Y - A_{FY}X, FZ) = 0.$$

Thus the proof is completed. □

Proposition 5.1. *Let M be a radical screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then the distribution D_0 is invariant with respect to F.*

Proof is similar to those given in section4.

Theorem 5.3. *Let M be a radical screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then, the screen distribution defines a totally geodesic foliation if and only if $h^s(X, FY)$ has no components in $FRadTM$ for $X, Y \in \Gamma(S(TM))$.*

Proof. By the definition of radical screen transversal lightlike submanifold $S(TM)$)defines totally geodesic foliation if and only if $g(\nabla_X Y, N) = 0$ for $\forall X, Y \in \Gamma(S(TM))$, $N \in \Gamma(ltr(TM))$. Using (3.1) , we find

$$\begin{aligned} g(\nabla_X Y, N) &= g\left(\bar{\nabla}_X Y, N\right) \\ &= g\left(F\bar{\nabla}_X Y, FN\right) \end{aligned}$$

using (2.4), we have

$$\begin{aligned} g(\nabla_X Y, N) &= g\Big(\nabla_X FY + h^\ell(X, FY) + h^s(X, FY), FN\Big) \\ &= g(h^s(X, FY), FN) \end{aligned}$$

Thus, proof is completed. □

Theorem 5.4. *Let M be a radical screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then, the radical distribution defines a totally geodesic foliation on M if and only if $h^s(X, FY)$ has no components in $FltrTM$, $\forall X, Y \in \Gamma(S(TM))$.*

Proof. $g\left(\nabla_X Y, Z\right) = 0$ for $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma\left(S(TM)\right)$.From (3.1) and (2.6), we find

$$\begin{aligned} g\left(\nabla_X Y, Z\right) &= g\left(\bar{\nabla}_X Y, Z\right) \\ &= g\left(F\bar{\nabla}_X Y, FZ\right) \\ &= g\left(\bar{\nabla}_X FY, FZ\right) \\ &= g\left(-A_{FY}X + \nabla^s_X FY + D^\ell\left(X, FY\right), FZ\right) \\ &= -g(A_{FY}X, FZ). \end{aligned}$$

From (2.7), we have

$$\begin{aligned} g\left(\nabla_X Y, Z\right) &= -g\left(A_{FY}X, FZ\right) \\ &= -g\left(h^s\left(X, FZ\right), FY\right) - g\left(FZ, D^\ell\left(X, FY\right)\right) \\ &= -g\left(h^s\left(X, FZ\right), FY\right). \end{aligned}$$

□

The condition under which the induced connection on the submanifold is a metric connection is given by the following theorem.

Theorem 5.5. *Let M be a radical screen transversal lightlike submanifold of semi-Riemannian product manifolds $\bar{M}$. Then the induced connection ∇ on M is a metric connection if and only if $h^s\left(Y, Z\right)$ has no components in $FltrTM$, $\forall Y, Z \in \Gamma\left(S(TM)\right)$.*

Proof. Since $\bar{M}$ is a semi-Riemannian product manifold, we have $\left(\bar{\nabla}_Z F\right)X = 0$ for $Z \in \Gamma\left(S(TM)\right)$ and $X \in \Gamma\left(RadTM\right)$, from which, we have

$$\bar{\nabla}_Z FX = F\bar{\nabla}_Z X. \tag{5.1}$$

Taking inner product of equation (5.1) with $V \in \Gamma S(TM))$, we find

$$g\left(\bar{\nabla}_Z FX, Y\right) = g\left(F\bar{\nabla}_Z X, Y\right) = g\left(\bar{\nabla}_Z X, FY\right).$$

From the above equation, we have

$$\begin{aligned} g\left(-A_{FX}Z + \nabla^s_Z FX + D^\ell(Z, FX), Y\right) &= g(\nabla_Z X, FY) - g\left(A_F Z, Y\right) \\ &= g\left(\nabla_Z X, FY\right) \end{aligned}$$

from which, it follows that

$$g\left(A_W X, Y\right) = g\left(h^s\left(X, Y\right), W\right) + g\left(Y, D^\ell\left(X, W\right)\right) - g\left(h^s(Z, Y), FX\right) = g\left(\nabla_Z X, FY\right).$$

Thus, proof is completed. □

We now give two examples for screen transversal lightlike submanifolds.

Example 5.1. Let $\bar{M} = \mathbb{R}^4_2 \times \mathbb{R}^4_2$ be a semi-Riemannian product manifold with semi-Riemannian product metric tensor $\bar{g} = \pi^* g_1 \otimes \sigma^* g_2$, i=1,2 where g_i denote standard metric tensor of $\mathbb{R}^4_2$. Let we get

$$\begin{aligned} f : M &\to \bar{M}, \\ (x_1, x_2, x_3) &\to \left(x_1, x_2 + x_3, x_1, 0, x_1, 0, x_2 - x_3, x_1\right). \end{aligned}$$

Then, we find

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_3 - \partial x_5 - \partial x_8 \\ Z_2 &= \partial x_2 + \partial x_7 \\ Z_3 &= \partial x_2 - \partial x_7. \end{aligned}$$

The radical bundle $RadTM$ is spanned by Z_1 and $S(TM) = Span\{Z_2, Z_3\}$ for $FZ_2 = Z_3$. If we choose

$$N = -\frac{1}{2}(\partial x_1 - \partial x_5),$$

we find

$$g(Z_1, N) = 1.$$

Also we obtain

$$\begin{aligned} FZ_1 &= \partial x_1 + \partial x_3 + \partial x_5 + \partial x_8 = W_1, \\ FN &= -\frac{1}{2}(\partial x_1 + \partial x_5) = W_2. \end{aligned}$$

If we choose

$$W_3 = \partial x_4 + \partial x_6,$$

we obtain

$$FW_3 = W_4 = \partial x_4 - \partial x_6.$$

Thus we have $FRadTM \subset S(TM^{\perp})$ and $FS(TM) = S(TM)$. Then M is a radical screen transversal lightlike submanifold of $\bar{M}$ semi-Riemannian product manifold.

Example 5.2. Let $\bar{M} = \mathbb{R}^4_2 \times \mathbb{R}^4_2$ be a semi-Riemannian product manifold with semi-Riemannian product metric tensor $\bar{g} = \pi^* g_1 \otimes \sigma^* g_2$, i=1,2 where g_i denote standard metric tensor of $\mathbb{R}^4_2$. Let we get

$$\begin{aligned} f : M &\to \bar{M}, \\ (x_1, x_2, x_3) &\to \left(-x_1, -x_2, x_1, \sqrt{2}x_2, x_1, 0, 0, x_1, -x_2\right). \end{aligned}$$

Then, we find

$$\begin{aligned} Z_1 &= -\partial x_1 + \partial x_3 + \partial x_5 + \partial x_7 \\ Z_2 &= -\partial x_2 - \sqrt{2}\partial x_4 - \partial x_8. \end{aligned}$$

The radical bundle $RadTM$ is spanned by Z_1 and $S(TM) = Span\{Z_2\}$. If we choose

$$N = \frac{1}{2}(\partial x_1 - \partial x_5),$$

we find

$$g(Z_1, N) = 1.$$

Thus, we obtain

$$\begin{aligned} FZ_1 &= -\partial x_1 + \partial x_3 - \partial x_5 - \partial x_7 = W_1, \\ FZ_2 &= -\partial x_2 - \sqrt{2}\partial x_4 + \partial x_8 = W_2, \\ FN &= \frac{1}{2}(\partial x_1 + \partial x_5) = W_3. \end{aligned}$$

If we choose

$$W_4 = -\sqrt{2}\partial x_2 - \sqrt{2}\partial x_4 + \partial x_6,$$

we obtain

$$FW_4 = W_5 = -\sqrt{2}\partial x_2 - \sqrt{2}\partial x_4 - \partial x_6.$$

Thus we have $FRadTM \subset S\left(TM^{\perp}\right)$ and $FS(TM) = S(TM^{\perp})$. Then M is a screen transversal anti-invariant lightlike submanifold of $\bar{M}$ semi-Riemannian product manifold.

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