Mathematical Sciences And Applications E-Notes
Volume 3 No. 1 pp. 86–93 (2015) ©MSAEN

SPACELIKE CURVES OF CONSTANT BREADTH ACCORDING TO BISHOP FRAME IN MINKOWSKI 3-SPACE

HÜSEYİN KOCAYİĞİT, MUHAMMED ÇETİN

(Communicated by Bayram ŞAHİN)

Abstract. In this paper, the spacelike curves of constant breadth according to Bishop frame in Minkowski 3-space are studied. The differential equations characterizing the spacelike curves of constant breadth in E_1^3 are obtained. Furthermore, It is shown that the spacelike curves of constant breadth are connected with slant helix in Minkowski 3-space E_1^3.

1. Introduction

Firstly, the curves of constant breadth were introduced by Euler [1]. After him, many mathematicians studied about the curves of constant breadth [2-12]. Reuleaux gave the obtaining method some curves of constant breadth and used in kinematics of machinery [13]. The differential equations characterizing space curves of constant breadth were established and a criterion for these curves were given in [14]. Önder and et al gave the diferential equations characterizing the timelike and spacelike curves of constant breadth in Minkowski 3-space in [15]. Furthermore, they gave a criterion for a timelike or spacelike curve to be curve of constant breadth in E_1^3. Also, Kocayiğit and Önder showed that in E_1^3 spacelike and timelike curves of constant breadth were normal curves, helices and spherical curves in some special cases [16].

In this paper, we study the spacelike curves of constant breadth according to Bishop frame in E_1^3. We obtain the differential equations characterizing the spacelike curves of constant breadth. In addition, we demostrate that spacelike curves of constant breadth are connected with slant helix.

2. Preliminaries

The Minkowski 3-space E_1^3 is the real vector space $\mathbb{R}^3$ provided with the standart flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

Date: Received: June 12, 2014, Accepted: February 20, 2015.
2010 *Mathematics Subject Classification.* 53A04, 14H50.
Key words and phrases. Bishop frame, Curve of constant breadth, Differential characterizations of curve, Minkowski space, Spacelike Curve .

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $\overrightarrow{a} = (a_1, a_2, a_3)$ in E_1^3 can have one of three Lorentzian characters; it can be spacelike if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle > 0$ or $\overrightarrow{a} = 0$, timelike if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle < 0$ and null (lightlike) if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle = 0$ and $\overrightarrow{a} \neq 0$. Similarly, an arbitrary curve $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$ can be spacelike, timelike or null (lightlike), if all of its velocity vectors $\overrightarrow{\alpha}'$ are spacelike, timelike or null (lightlike), respectively. In addition, a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively.

For any vectors $\overrightarrow{x} = (x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_1, y_2, y_3)$ in E_1^3, the vector product of $\overrightarrow{x}$ and $\overrightarrow{y}$ is defined by

$$\overrightarrow{x} \wedge \overrightarrow{y} = \begin{vmatrix} \overrightarrow{e_1} & -\overrightarrow{e_2} & -\overrightarrow{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2)$$

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, and $\overrightarrow{e_i} = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $\overrightarrow{e_1} \wedge \overrightarrow{e_2} = -\overrightarrow{e_3}$, $\overrightarrow{e_2} \wedge \overrightarrow{e_3} = \overrightarrow{e_1}$, $\overrightarrow{e_3} \wedge \overrightarrow{e_1} = -\overrightarrow{e_2}$ (See for details [17]).

The parallel transport frame is an allternative approach to defining a moving frame that is well-defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by paralel transporting each component of the frame [18].

Its mathematical properties derive from the observation that, while $\overrightarrow{T}(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\overrightarrow{T}(s)$ at each point. If the derivatives of $(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s))$ depend only on $\overrightarrow{T}(s)$ and not each other, we can make $\overrightarrow{N_1}(s)$ and $\overrightarrow{N_2}(s)$ vary smoothly throughout the path regardless of the curvature. We may therefore choose the alternative frame equations

$$\begin{bmatrix} \dot{\overrightarrow{T}} \\ \dot{\overrightarrow{N_1}} \\ \dot{\overrightarrow{N_2}} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N_1} \\ \overrightarrow{N_2} \end{bmatrix}$$

where $\left\langle \overrightarrow{T}, \overrightarrow{T} \right\rangle = \left\langle \overrightarrow{N_1}, \overrightarrow{N_1} \right\rangle = \left\langle \overrightarrow{N_2}, \overrightarrow{N_2} \right\rangle = 1$ and $\left\langle \overrightarrow{T}, \overrightarrow{N_1} \right\rangle = \left\langle \overrightarrow{N_1}, \overrightarrow{N_2} \right\rangle = \left\langle \overrightarrow{T}, \overrightarrow{N_2} \right\rangle = 0$ [19,20].

One can show that [19]

$$\kappa(s) = \sqrt{k_1^2 + k_2^2}, \qquad \theta(s) = \arctan\left(\frac{k_2}{k_1}\right), \qquad \tau(s) = \frac{d\theta(s)}{ds}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \tau(s)ds$. A fundamental ambiguity in the parallel transport frame compared to the Frenet frame thus aries from the arbitrary choice of an integration constant for θ_0, which disappears from τ due to the differentation [20].

Denote by $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, k_1 and k_2 the moving Bishop frame, the first curvature and the second curvature along the curve $\overrightarrow{\alpha}(s)$, respectively in Minkowski 3-space

E_1^3. If $\overrightarrow{\alpha}(s)$ is a spacelike curve in E_1^3, then Bishop frame is given by

$$\left[\begin{array}{c}\dot{\overrightarrow{T}}\\ \dot{\overrightarrow{N_1}}\\ \dot{\overrightarrow{N_2}}\end{array}\right]=\left[\begin{array}{ccc}0 & k_1 & -k_2\\ \epsilon k_1 & 0 & 0\\ \epsilon k_2 & 0 & 0\end{array}\right]\left[\begin{array}{c}\overrightarrow{T}\\ \overrightarrow{N_1}\\ \overrightarrow{N_2}\end{array}\right] \tag{2.1}$$

where $\left\langle \overrightarrow{T},\overrightarrow{T}\right\rangle = 1$, $\left\langle \overrightarrow{N_1},\overrightarrow{N_1}\right\rangle = -\epsilon$ and $\left\langle \overrightarrow{N_2},\overrightarrow{N_2}\right\rangle = \epsilon$, and [21,22]. Here, ϵ determines the kind of spacelike curve $\overrightarrow{\alpha}(s)$. If $\epsilon = 1$, then $\overrightarrow{\alpha}(s)$ is a spacelike curve with timelike principal normal and spacelike binormal. If $\epsilon = -1$, then $\overrightarrow{\alpha}(s)$ is a spacelike curve with spacelike principal normal and timelike binormal. The relations between κ, τ, θ and k_1, k_2 are given by as follows.

$$\kappa(s)=\sqrt{|k_1^2-k_2^2|},\qquad \theta(s)=\operatorname{arctan h}\left(\frac{k_2}{k_1}\right),\qquad \tau(s)=\frac{d\theta(s)}{ds}.$$

Also, Bishop curvatures are defined by

$$k_1=\kappa\cosh(\theta),\qquad k_2=\sinh(\theta)$$

and

$$\overrightarrow{T}=\overrightarrow{T},\qquad \overrightarrow{N_1}=\overrightarrow{N}\cosh(\theta)-\overrightarrow{B}\sinh(\theta),\qquad \overrightarrow{N_2}=\overrightarrow{N}\sinh(\theta)-\overrightarrow{B}\cosh(\theta)$$

[21,22].

Theorem 2.1. *Let $\alpha : I \longrightarrow E_1^3$ be a unit speed spacelike curve with non-zero natural curvatures. Then $\overrightarrow{\alpha}(s)$ is a spacelike slant helix if and only if $\frac{k_1}{k_2}$ is constant [23].*

3. The Spacelike Curves of Constant Breadth

In this section, we give differential equations characterizing the spacelike curves of constant breadth according to Bishop frame in Minkowski 3-space. In addition, we show that the spacelike curves of constant breadth are related to slant helix in Minkowski 3-space E_1^3.

Definition 3.1. Let (C) be a spacelike curve. If (C) has parallel tangents in opposite directions at the opposite points $\alpha(s)$ and $\alpha^*(s)$, and if the distance between these points is always constant, then (C) is called a spacelike curve of constant breadth. Moreover, a pair of spacelike curves (C) and (C^*) for which the tangents at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ are parallel and in opposite directions, and the distance between these points is always constant are called a spacelike curve pair of constant breadth [16].

Let (C) and (C^*) be a pair of unit-speed spacelike curves with non-zero Bishop curvatures in E_1^3 and let those curves have parallel tangents in opposite directions at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$, respectively. The position vector of the curve (C^*) at the point $\alpha^*(s^*)$ can be expressed as

$$\overrightarrow{\alpha^*}(s^*)=\overrightarrow{\alpha}(s)+\lambda_1(s)\overrightarrow{T}(s)+\lambda_2(s)\overrightarrow{N_1}(s)+\lambda_3(s)\overrightarrow{N_2}(s) \tag{3.1}$$

where $\lambda_i(s)$ $(i = 1, 2, 3)$ are differentiable functions of s which is arc lenght of (C). Denote by $\left\{\overrightarrow{T},\overrightarrow{N_1},\overrightarrow{N_2}\right\}$, k_1 and k_2 the moving Bishop frame, Bishop curvatures along the curve (C) respectively. And denote by $\left\{\overrightarrow{T^*}, \overrightarrow{N_1^*}, \overrightarrow{N_2^*}\right\}$, k_1^* and k_2^* the moving Bishop frame, Bishop curvatures along the curve (C^*), respectively.

Differentiating Eq.(3.1) with respect to s and using the Bishop formulae given by (2.1), we obtain

(3.2)
$$\frac{d\overrightarrow{\alpha^*}}{ds} = \overrightarrow{T^*}\frac{ds^*}{ds} = \left(1 + \frac{d\lambda_1}{ds} + \epsilon k_1\lambda_2 + \epsilon k_2\lambda_3\right)\overrightarrow{T} + \left(k_1\lambda_1 + \frac{d\lambda_2}{ds}\right)\overrightarrow{N_1} + \left(-k_2\lambda_1 + \frac{d\lambda_3}{ds}\right)\overrightarrow{N_2}$$

Since $\overrightarrow{T} = -\overrightarrow{T^*}$ at the corresponding points of (C) and (C^*), we gain the following differential equations system

(3.3)
$$\begin{cases} \frac{d\lambda_1}{ds} = -\frac{ds^*}{ds} - 1 - \epsilon k_1\lambda_2 - \epsilon k_2\lambda_3 \\ \frac{d\lambda_2}{ds} = -k_1\lambda_1 \\ \frac{d\lambda_3}{ds} = k_2\lambda_1 \end{cases}$$

It is well known that the curvature $\kappa(s)$ of the curve (C) is

$$\lim_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s} = \frac{d\varphi}{ds} = \kappa$$

where φ is the angle between the tangent of the spacelike curve (C) and a given fixed direction at the point $\alpha(s)$.

Hence, we can rewrite the system (3.3) as follow.

(3.4)
$$\begin{cases} \frac{d\lambda_1}{ds} = -\epsilon\mu_1\lambda_2 - \epsilon\mu_2\lambda_3 - f \\ \frac{d\lambda_2}{ds} = -\mu_1\lambda_1 \\ \frac{d\lambda_3}{ds} = \mu_2\lambda_1 \end{cases}$$

where

$$\mu_1 = \rho k_1 = \frac{k_1}{\kappa} = \cosh(\theta), \qquad \mu_2 = \rho k_2 = \frac{k_2}{\kappa} = \sinh(\theta), \qquad (\theta = \tau ds)$$

and

$$f(\varphi) = \rho + \rho^*, \qquad \rho = \frac{1}{\kappa}, \qquad \rho^* = \frac{1}{\kappa^*}.$$

Here, ρ and ρ^* indicates the radius of curvatures at the points $\alpha(s)$ and $\alpha^*(s^*)$, respectively.

Eliminating λ_2, λ_3 and their derivatives from the system (3.4), we obtain the following differential equation of third order with respect to λ_1.

(3.5)
$$a_1\lambda_1''' + b_1\lambda_1'' + c_1\lambda_1' + d_1\lambda_1 = e_1$$

where

$$
\begin{aligned}
a_1 &= -\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^3 - \mu_1''\mu_2^3\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\\
b_1 &= -\mu_2\left[(\mu_1'')^2\mu_2^3 + \mu_1''\left(-\mu_2^2\mu_1\mu_2'' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\right) + \mu_2''\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\right]\\
c_1 &= \mu_1''\left[-\mu_2^3\mu_2''\mu_1' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(\epsilon\mu_2^3\mu_1^2 + \mu_1(\mu_2')^2 - \mu_2\mu_1'\mu_2' - \epsilon\mu_2^5\right)\right]\\
&\quad + (\mu_1'')^2\mu_2^3\mu_2' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\left(\mu_2''\mu_2' + \epsilon(\mu_1^2 - \mu_2^2)\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right)\\
d_1 &= \epsilon\mu_2\mu_1''\left[\left(\mu_2^4\mu_1 - \mu_2^2\mu_1^3\right)\mu_2'' + \left(\left(\mu_1^3 + \mu_1^2\mu_2 - 3\mu_2^3\right)\mu_2' - \mu_1\mu_2\mu_1'\left(-2\mu_2 + \mu_1\right)\right)\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right]\\
&\quad + \epsilon\left(\mu_2^4\mu_1^2 - \mu_2^6\right)(\mu_1'')^2 + \epsilon\left[\left(\mu_1^2\mu_2 - \mu_2^3\right)\mu_2'' + 3\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(\mu_1\mu_1' - \mu_2\mu_2'\right)\right]\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\\
e_1 &= -\mu_1''\left[-\mu_2^3\left(f\mu_1' - f'\mu_1\right)\mu_2'' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(-\mu_2^3 f'' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(-\mu_2 f' + \mu_2' f\right)\right)\right]\\
&\quad - (\mu_1'')^2\left(\mu_2^3\mu_2' f - \mu_2^4 f'\right) - \left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\left[\left(-\mu_2 f' + \mu_2' f\right)\mu_2'' - f''\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right].
\end{aligned}
$$

Here and later $(')$ denotes the differentiation with respect to "φ". Similarly, eliminating λ_1, λ_3 and their derivatives from the system (3.4) we obtain the following differential equation of third order with respect to λ_2.

$$a_2\lambda_2''' + b_2\lambda_2'' + c_2\lambda_2' + d_2\lambda_2 = e_2 \tag{3.6}$$

where

$$
\begin{aligned}
a_2 &= -\mu_1^2\mu_2\\
b_2 &= 2\mu_1'\mu_1\mu_2 + \mu_2'\mu_1^2\\
c_2 &= \mu_1''\mu_1\mu_2 - 2(\mu_1')^2\mu_2 + \epsilon\mu_1^4\mu_2 - \mu_2'\mu_1'\mu_1 - \epsilon\mu_2^3\mu_1^2\\
d_2 &= \epsilon\mu_1^3\left(\mu_1'\mu_2 - \mu_2'\mu_1\right)\\
e_2 &= \mu_1^3\left(\mu_2' f - f'\mu_2\right).
\end{aligned}
$$

Furthermore, eliminating λ_1, λ_2 and their derivatives from the system (3.4) we gain the following differential equation of third order with respect to λ_3.

$$a_3\lambda_3''' + b_3\lambda_3'' + c_3\lambda_3' + d_3\lambda_3 = e_3 \tag{3.7}$$

where

$$
\begin{aligned}
a_3 &= -\mu_2^2\mu_1\\
b_3 &= 2\mu_2'\mu_2\mu_1 + \mu_1'\mu_2^2\\
c_3 &= \mu_2''\mu_2\mu_1 - 2(\mu_2')^2\mu_1 - \mu_1'\mu_2'\mu_2 + \epsilon\mu_1^3\mu_2^2 - \epsilon\mu_2^4\mu_1\\
d_3 &= \epsilon\mu_2^3\left(-\mu_2'\mu_1 + \mu_2\mu_1'\right)\\
e_3 &= -\mu_2^3\left(\mu_1' f - f'\mu_1\right).
\end{aligned}
$$

When the curve (C) and the function $f(\varphi)$ are given, from the solving the systems (3.3), (3.4) or the Eqs.(3.5), (3.6), (3.7), we can find the values of $\lambda_i \quad (i = 1, 2, 3)$. Eqs.(3.5), (3.6) and (3.7) express the differential characterizations for the spacelike curves (C) and (C^*) according to coeffcients λ_i.

When the curves (C) and (C^*) are a spacelike curve pair of constant breadth, then the distance d between the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ is constant. Hence,

$$d^2 = \left\|\overrightarrow{d}\right\|^2 = \left\|\overrightarrow{\alpha^*}(s^*) - \overrightarrow{\alpha}(s)\right\|^2 = \left|\lambda_1^2 - \epsilon\lambda_2^2 + \epsilon\lambda_3^2\right| = k^2, \qquad k \in \mathbb{R} \tag{3.8}$$

Differentiating Eq.(3.8) with respect to φ, we gain

$$\frac{1}{2}\frac{d}{d\varphi}\left\|\overrightarrow{d}\right\|^2 = \lambda_1\lambda_1' - \epsilon\lambda_2\lambda_2' + \epsilon\lambda_3\lambda_3' = 0 \tag{3.9}$$

Substituting the equalities given by the system (3.4) into the Eq.(3.9) we obtain the following equality.

$$\lambda_1 f = 0. \tag{3.10}$$

This relation express to be spacelike curve pair of constant breadth of the spacelike curves (C) and (C^*) in E_1^3. Here, there are two main cases.

Case 1. *Let* $f(\varphi) = 0 \quad \left(\frac{ds^*}{ds} + 1 = 0\right)$. *This means that the spacelike curve* (C^*) *is a translation of the spacelike curve* (C) *by the constnat vector*

$$\overrightarrow{d} = \lambda_1\overrightarrow{T} + \lambda_2\overrightarrow{N_1} + \lambda_3\overrightarrow{N_3}. \tag{3.11}$$

In fact, if $f(\varphi) = 0$ then the vector $\overrightarrow{d}$ is constant. To verify this fact, differentiate Eq.(3.11) with respect to φ and use the equalities given by (3.4) for $f = 0$ and Bishop formulae given by (2.1). Hence, we obtain $\frac{d\overrightarrow{d}}{d\varphi} = 0$. Consequently, if $\frac{d\overrightarrow{d}}{d\varphi} = 0$ then the vector $\overrightarrow{d}$ is constant. In this case we can rewrite the systems (3.3), (3.4) and the Eqs.(3.5), (3.6) and (3.7) as follows:

$$\begin{cases} \frac{d\lambda_1}{ds} = -\epsilon k_1\lambda_2 - \epsilon k_2\lambda_3 \\ \frac{d\lambda_2}{ds} = -k_1\lambda_1 \\ \frac{d\lambda_3}{ds} = k_2\lambda_1 \end{cases} \tag{3.12}$$

$$\begin{cases} \frac{d\lambda_1}{ds} = -\epsilon\mu_1\lambda_2 - \epsilon\mu k_2\lambda_3 \\ \frac{d\lambda_2}{ds} = -\mu_1\lambda_1 \\ \frac{d\lambda_3}{ds} = \mu_2\lambda_1 \end{cases} \tag{3.13}$$

$$a_1\lambda_1''' + b_1\lambda_1'' + c_1\lambda_1' + d_1\lambda_1 = 0 \tag{3.14}$$

where

$$\begin{aligned}
a_1 &= -\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^3 - \mu_1''\mu_2^3\left(\mu_1\mu_2' - \mu_2\mu_1'\right) \\
b_1 &= -\mu_2\left[(\mu_1'')^2\mu_2^3 + \mu_1''\left(-\mu_2^2\mu_1\mu_2'' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\right) + \mu_2''\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\right] \\
c_1 &= \mu_1''\left[-\mu_2^3\mu_2''\mu_1' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(\epsilon\mu_2^3\mu_1^2 + \mu_1(\mu_2')^2 - \mu_2\mu_1'\mu_2' - \epsilon\mu_2^5\right)\right] \\
&\quad +(\mu_1'')^2\mu_2^3\mu_2' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\left(\mu_2''\mu_2' + \epsilon(\mu_1^2 - \mu_2^2)\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right) \\
d_1 &= \epsilon\mu_2\mu_1''\left[\left(\mu_2^4\mu_1 - \mu_2^2\mu_1^3\right)\mu_2'' + \left(\left(\mu_1^3 + \mu_1^2\mu_2 - 3\mu_2^3\right)\mu_2' - \mu_1\mu_2\mu_1'\left(-2\mu_2 + \mu_1\right)\right)\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right] \\
&\quad +\epsilon\left(\mu_2^4\mu_1^2 - \mu_2^6\right)(\mu_1'')^2 + \epsilon\left[\left(\mu_1^2\mu_2 - \mu_2^3\right)\mu_2'' + 3\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(\mu_1\mu_1' - \mu_2\mu_2'\right)\right]\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2.
\end{aligned}$$

$$a_2\lambda_2''' + b_2\lambda_2'' + c_2\lambda_2' + d_2\lambda_2 = 0 \tag{3.15}$$

where

$$\begin{aligned} a_2 &= -\mu_1^2\mu_2 \\ b_2 &= 2\mu_1'\mu_1\mu_2 + \mu_2'\mu_1^2 \\ c_2 &= \mu_1''\mu_1\mu_2 - 2(\mu_1')^2\mu_2 + \epsilon\mu_1^4\mu_2 - \mu_2'\mu_1'\mu_1 - \epsilon\mu_2^3\mu_1^2 \\ d_2 &= \epsilon\mu_1^3\left(\mu_1'\mu_2 - \mu_2'\mu_1\right). \end{aligned}$$

$$a_3\lambda_3''' + b_3\lambda_3'' + c_3\lambda_3' + d_3\lambda_3 = 0 \tag{3.16}$$

where

$$\begin{aligned} a_3 &= -\mu_2^2\mu_1 \\ b_3 &= 2\mu_2'\mu_2\mu_1 + \mu_1'\mu_2^2 \\ c_3 &= \mu_2''\mu_2\mu_1 - 2(\mu_2')^2\mu_1 - \mu_1'\mu_2'\mu_2 + \epsilon\mu_1^3\mu_2^2 - \epsilon\mu_2^4\mu_1 \\ d_3 &= \epsilon\mu_2^3\left(-\mu_2'\mu_1 + \mu_2\mu_1'\right). \end{aligned}$$

Theorem 3.1. *The general differential equations and systems characterizing spacelike curve pair of constant breadth according to Bishop frame in E_1^3 are given by (3.12), (3.13) and (3.14), (3.15), (3.16).*

Case 2. *Let $\lambda_1 = 0$. Then, there are three cases here.*

i) We can take $\lambda_2 = const.$ and $\lambda_3 = 0$ (from (3.4)). Then, $f(\varphi) = -\epsilon\mu_1\lambda_2$.

ii) We can take λ_2 =0 and $\lambda_3 = const.$ (from (3.4)). Then, $f(\varphi) = -\epsilon\mu_2\lambda_3$.

iii) Now, we consider the third and important case $\lambda_2 = const.$ and $\lambda_3 = const.$ If $\lambda_2 = const.$, $\lambda_3 = const.$ and $f(\varphi) = 0$ (from (3.4)), then we obtain $\frac{k_1}{k_2} = -\frac{\lambda_3}{\lambda_2} = const.$ This means that the curve (C) is a spacelike slant helix according to Bishop frame. Thus we can give following theorem.

Theorem 3.2. *Let consider the spacelike curve pair of constant breadth which has the sum of curvature radius at corresponding points is zero according to Bishop frame in E_1^3. If the first normal component $\lambda_2 = const.$ and the second normal component $\lambda_3 = const.$ given by Eq.(3.1), then the spacelike curve (C) is a spacelike slant helix in E_1^3.*

References

[1] Euler, L., De Curvis Triangularibus, *Acta Acad. Petropol.*, 3-30, 1778 (1780).

[2] Barbier, E., Note Sur le Probléme de I'aiguille et le jeu du Joint Couvert, *Journal de Mathématiques Pures et Appliquées*, 2 (1860), no. 5, 273-286.

[3] Fujiwara, M., On space Curves of Constant Breadth, *Tohoku Mathematical Journal*, 5 (1914), 180-184.

[4] Blaschke, W., Leibziger Berichte, 67 (1917), 290.

[5] Ball, N.H., On Ovals, *American Mathematical Monthly*, 37 (1930), no. 7, 348-353.

[6] Mellish, A.P., Notes on Differential Geometry, *Annals of Mathematics*, 32 (1931), no. 1, 181-190.

[7] Hammer, P.C., Constant Breadth Curves in the Plane, *Procedings of the American Mathematical Society*, 6 (1955), no. 2, 333-334.

[8] Smakal, S., Curves of Constant Breadth, *Czechoslovak Mathematical Journal*, 23 (1973), no. 1, 86-94.

[9] Köse, Ö., Düzlemde Ovaller ve Sabit Genişlikli Eğrilerin Bazı Özellikleri, *Doğa Bilim Dergisi*, Seri B, 8 (1984), no. 2, 119-126.

[10] Köse, Ö., On Space Curves of Constant Breadth, *Doğa Tr. J. Math*, 10 (1986), no. 1, 11-14.

[11] Mağden, A., and Köse, Ö., On the Curves of Constant Breadth in E^4 Space, *Tr. J. of Mathematics*, 21 (1997), 277-284.

[12] Akdoğan, Z., and Mağden, A., Some Characterization of Curves of Constant Breadth in E^n Space, *Turk J Math*, 25 (2001), 433-444.

[13] Reuleaux, F., The Kinematics of Machinery, Translated by A. B. W. Kennedy, Dover Pub. New York, 1963.

[14] Sezer, M., Differential Equations Characterizing Space Curves of Constant Breadth and a Criterion for These Curves, *Turkish J. of Math*, 13 (1989), no. 2, 70-78.

[15] Önder, M., Kocayiğit, H. and Candan, E., Differential Equations Characterizing Timelike and Spacelike Curves of Constant Breadth in Minkowski 3-Space E_1^3, *J. Korean Math. Soc.* 48 (2011), no. 4, 849-866.

[16] Kocayiğit, H. and Önder, M., Space Curves of Constant Breadth in Minkowski 3-Space, *Annali di Matematica*, 192 (2013), no. 5, 805-814.

[17] O'Neill, B., Semi Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[18] Hanson, A.J. and Ma, H., Parallel Transport Approach to Curve Framing, Indiana University, *Technical Report TR425*, January 11, 1995.

[19] Bishop, R.L., There is More Than One Way to Frame a Curve, *American Mathematical Monthly*, 82 (1975), no. 3, 246-251.

[20] Hanson, A.J., and Ma, H., Quaternion Frame Approach to Streamline Visualization, *IEEE Transactions on Visulation and Computer Graphics*, 1 (1995), no.2, 164-174.

[21] B. Bükçü and M.K. Karacan, Bishop frame of the spacelike curve with a spacelike principal normal in Minkowski 3-space, *Commun. Fac. Sci. Univ. Ank. Series A1*, 57 (2008), no. 1, 13-22.

[22] B. Bükçü and M.K. Karacan, Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3-space, *Selçuk J. Appl. Math*, 11 (2010), no. 1, 15-25.

[23] Bükcü, B. and Karacan, M.K., The Slant Helices according to Bishop Frame of the Spacelike Curve in Lorentzian Space, *Journal of Interdisciplinary Mathematics*, 12 (2009), no. 5, 691-700.

Celal Bayar University, Faculty of Sciences, Department of Mathematics, Manisa, TURKEY

E-mail address: huseyin.kocayigit@cbu.edu.tr

E-mail address: mat.mcetin@hotmail.com