Mathematical Sciences And Applications E-Notes
Volume 3 No. 1 pp. 103–107 (2015) ©MSAEN

WEAK INSERTION OF A PERFECTLY CONTINUOUS FUNCTION BETWEEN TWO REAL-VALUED FUNCTIONS

MAJID MIRMIRAN

(Communicated by Ishak ALTUN)

Abstract. A sufficient condition in terms of lower cut sets are given for the weak insertion of a perfectly continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets are open.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [11]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i. e., sets that are intersection of open sets are called $\Lambda-$sets [11].

Recall that a real-valued function f defined on a topological space X is called $A-$*continuous* [13] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subset of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [2, 6].

Hence, a real-valued function f defined on a topological space X is called *perfectly continuous*[12] (resp. *contra-continuous* [3]) if the preimage of every open subset of $\mathbb{R}$ is a clopen (i. e., open and closed) (resp. closed) subset of X.

A function is perfectly continuous if and only if it is continuous and contra-continuous.

A real-valued function f defined on a topological space X is called *regular set-connected* [4] if the preimage of every regular open subset of $\mathbb{R}$ is a clopen subset of X.

Recall, a subset A is said to be *regular open* (resp. *regular closed)* if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$).

If a function is perfectly continuous then it is regular set-connected.

Date: Received: June 1, 2014; Accepted: October 1, 2014.

2010 *Mathematics Subject Classification.* Primary 54C08, 54C10, 54C50; Secondary 26A15, 54C30.

Key words and phrases. Weak insertion, Strong binary relation, Perfectly continuous function, Regular set-connected function, Contra-continuous function, Clopen sets, $\Lambda-$sets, Lower cut set.

This research was partially supported by Centre of Excellence for Mathematics (University of Isfahan).

Results of Katětov [7, 8] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give a sufficient condition for the weak insertion of a perfectly continuous function between two comparable real-valued functions on the topological spaces that $\Lambda-$sets are open [11].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [9].

A property P defined relative to a real-valued function on a topological space is a *pc−property* provided that any constant function has property P and provided that the sum of a function with property P and any perfectly continuous function also has property P. If P_1 and P_2 are *pc*−property, the following terminology is used: A space X has the *weak pc−insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a perfectly continuous function h such that $g \leq h \leq f$.

In this paper, a sufficient condition for the weak *pc*−insertion property is given. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a perfectly continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all open, closed and clopen will be denoted by $O(X, \tau)$, $C(X, \tau)$ and $Clo(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in O(X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F \in C(X, \tau)\}$.
In [5, 10], A^Λ is called the *kernel* of A.

Definition 2.2. Let A be a subset of a topological space (X, τ). Respectively, we define the *closure, interior, clo-closure* and *clo-interior* of a set A, denoted by $Cl(A), Int(A), clo(Cl(A))$ and $clo(Int(A))$ as follows:

$Cl(A) = \cap\{F : F \supseteq A, F \in C(X, \tau)\}$,
$Int(A) = \cup\{O : O \subseteq A, O \in O(X, \tau)\}$,
$clo(Cl(A)) = \cap\{F : F \supseteq A, F \in Clo(X, \tau)\}$ and
$clo(Int(A)) = \cup\{O : O \subseteq A, O \in Clo(X, \tau)\}$.

If (X, τ) is a topological space whose $\Lambda-$sets are open, then respectively, we have $A^V, clo(Cl(A))$ are closed, clopen and $A^\Lambda, clo(Int(A))$ are open, clopen.

The following first two definitions are modifications of conditions considered in [7, 8].

Definition 2.3. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition 2.4. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \ \bar{\rho} \ B$.

3) If $A \ \rho \ B$, then $clo(Cl(A)) \subseteq B$ and $A \subseteq clo(Int(B))$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition 2.5. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X, in which $\Lambda-$sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$, then there exists a perfectly continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \ \bar{\rho} \ F(t_2), G(t_1) \ \bar{\rho} \ G(t_2)$, and $F(t_1) \ \rho \ G(t_2)$. By Lemmas 1 and 2 of [8] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \ \rho \ H(t_2), H(t_1) \ \rho \ H(t_2)$ and $H(t_1) \ \rho \ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = clo(Int(H(t_2))) \setminus clo(Cl(H(t_1)))$. Hence $h^{-1}(t_1, t_2)$ is a clopen subset of X, i. e., h is a perfectly continuous function on X. The above proof used the technique of proof of Theorem 1 of [7].

3. Applications

The abbreviations c, pc, rc and cc are used for continuous, perfectly continuous, regular set-connected and contra-continuous, respectively.

Before stating the consequences of Theorem 2.1, we suppose that X is a topological space that $\Lambda-$sets are open.

Corollary 3.1. If for each pair of disjoint closed (resp. open) sets F_1, F_2 of X , there exist clopen sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X has the weak $pc-$insertion property for (c, c) (resp. (cc, cc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are c (resp. cc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $Cl(A) \subseteq Int(B)$ (resp. $A^\Lambda \subseteq B^V$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a closed (resp. open) set and since $\{x \in X : g(x) < t_2\}$ is an open (resp. closed) set, it follows that $Cl(A(f, t_1)) \subseteq Int(A(g, t_2))$ (resp. $A(f, t_1)^\Lambda \subseteq A(g, t_2)^V$). Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1.

Corollary 3.2. If for each pair of disjoint closed (resp. open) sets F_1, F_2, there exist clopen sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every continuous (resp. contra-continuous) function is perfectly continuous.

Proof. Let f be a real-valued continuous (resp. contra-continuous) function defined on the X. By setting $g = f$, then by Corollary 3.1, there exists a perfectly continuous function h such that $g = h = f$.

Corollary 3.3. If for each pair of disjoint subsets F_1, F_2 of X , such that F_1 is closed and F_2 is open, there exist clopen subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X have the weak $pc-$insertion property for (c, cc) and (cc, c).

Proof. Let g and f be real-valued functions defined on the X, such that g is c (resp. cc) and f is cc (resp. c), with $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $A^\Lambda \subseteq Int(B)$ (resp. $Cl(A) \subseteq B^V$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an open (resp. closed) set and since $\{x \in X : g(x) < t_2\}$ is an open (resp. closed) set, it follows that $A(f, t_1)^\Lambda \subseteq Int(A(g, t_2))$ (resp. $Cl(A(f, t_1)) \subseteq A(g, t_2)^V$). Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1.

Corollary 3.4. X has the weak $pc-$insertion property for (rc, rc).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are rc, and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $clo(Cl(A)) \subseteq clo(Int(B))$, then ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ and $\{x \in X : g(x) < t_2\}$ are clopen set, it follows that $clo(Cl(A(f, t_1))) \subseteq clo(Int(A(g, t_2)))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1.

Corollary 3.5. Every regular set-connected function is perfectly continuous.

Proof. Let f be a regular set-connected function defined on the X. By setting $g = f$, then by Corollary 3.4, there exists a perfectly continuous function h such that $g = h = f$.

Corollary 3.6. *X* has the weak $pc-$insertion property for (c, rc), (rc, c), (cc, rc) and (rc, cc).

Proof. The proof follows from Corollary 3.5.

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Department of Mathematics, University of Isfahan, Isfahan 81746-73441, IRAN.
E-mail address: `mirmir@sci.ui.ac.ir`