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WELL POSEDNESS OF FIXED POINT PROBLEM FOR MAPPINGS SATISFYING AN IMPLICIT RELATION IN G_p - METRIC SPACES

VALERIU POPA AND ALINA-MIHAELA PATRICIU

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Abstract. The purpose of this paper is to prove a general fixed point theorem in G_p - metric space for mappings satisfying an implicit relation. If G_p - metric is symmetric, we prove that the fixed point problem is well posed.

1. Introduction

In [13], [14], Dhage introduced a new class of generalized metric spaces, named D - metric spaces. Mustafa and Sims [25], [26] proved that most of the claims concerning the fundamental structures on D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric spaces. In fact, Mustafa, Sims and other authors [1], [17], [20], [21], [22], [23], [24], [25], [26], [27], [39], [40], [41], [42], [43], [44] studied many fixed point results for self mappings in G - metric spaces under certain conditions.

In 1994, Mathews [20] introduced the concept of partial metric space as a part of study of denotional semantics of data flows and proved the Banach contraction principle in such spaces. Recently, in [2], [5], [9], [16], [17] and in other papers, some fixed point theorems under various contractive conditions in partial metric spaces are proved.

Quite recently, Zand and Nezhad [46] introduced a generalization and unification of G - metric space and partial metric space, named G_p - metric space. In [8], some fixed point theorems in G_p - metric spaces are proved. Other results are obtained in [9] and [10].

The notion of well posedness of fixed point problem have generate more interest to several mathematicians, for example in [12], [18], [38].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [29], [30] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric

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spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings. Recently, the method is used in the study of fixed points for mappings satisfying contractive/extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces and intuitionistic metric spaces. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for mappings in G - metric spaces is initiated in [33], [34], [35], [36], [37] and in other papers. The study of fixed point for mappings satisfying an implicit relation in partial metric spaces is initiated in [45].

In [3], [4], [5], [6], [31], [32] the authors studied well posedness of fixed point problem for mappings satisfying implicit relations.

The purpose of this paper is to prove a general fixed point theorem for mappings satisfying an implicit relation in G_p - metric space. We prove that for these mappings, if G_p - metric is symmetric, the fixed point problem is well posed.

2. Preliminaries

Definition 2.1 ([28], [46])**.** Let X be a nonempty set. A function $G : X^3 \to \mathbb{R}_+$ is called a G_p - metric on X if the following conditions are satisfied:

$(GP_1) : x = y = z$ if $G_p(x, y, z) = G_p(x, x, x) = G_p(y, y, y) = G_p(z, z, z)$,

$(GP_2) : 0 \le G_p(x, x, x) \le G_p(x, x, y) \le G_p(x, y, z)$ for all $x, y, z \in X$ with $y \ne z$,

$(GP_3) : G_p(x, y, z) = G_p(y, z, x) = ...$ (symmetry in all three variables),

$(GP_4) : G_p(x, y, z) \le G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for all $x, y, z, a \in X$.

The pair (X, G_p) is called a G_p - metric space.

Definition 2.2 ([46])**.** Let (X, G_p) be a G_p - metric space and $\{x_n\}$ a sequence in X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ (x_n is G_p - convergent to x) if $\lim_{m,n\to\infty} G_p(x, x_n, x_m) = G_p(x, x, x)$.

Theorem 2.1 ([8])**.** *Let (X, G_p) be a G_p - partial metric space. Then, for any $\{x_n\} \in X$ and $x \in X$, the following conditions are equivalent:*

a) $\{x_n\}$ is G_p - convergent to x,

b) $G_p(x_n, x_n, x) \to G_p(x, x, x)$ as $n \to \infty$,

c) $G_p(x_n, x, x) \to G_p(x, x, x)$ as $n \to \infty$.

Definition 2.3 ([46])**.** Let (X, G_p) be a G_p - partial metric space.

1) A sequence $\{x_n\}$ of X is called a G_p - Cauchy sequence if $\lim_{m,n\to\infty} G_p(x_n, x_m, x_m)$ exists and is finite,

2) A G_p - metric space is said to be G_p - complete if and only if every G_p - Cauchy sequence in X converges to $x \in X$ such that $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x, x, x)$.

Lemma 2.1 ([8])**.** *Let (X, G_p) be a G_p - metric space. Then:*

1) If $G_p(x, y, z) = 0$ then $x = y = z$,

2) If $x \ne y$ then $G_p(y, x, x) > 0$.

Definition 2.4 ([46])**.** A G_p - metric on X is said to be symmetric if $G_p(x, y, y) = G_p(y, x, x)$. In this case (X, G_p) is said to be symmetric.

Lemma 2.2 ([8])**.** *Let (X, G_p) be a G_p - metric space and $\{x_n\}$ a sequence in X. Assume that $\{x_n\}$ is G_p - convergent to a point $x \in X$ with $G_p(x, x, x) = 0$. Then $\lim_{n\to\infty} G_p(x_n, y, y) = G_p(x, y, y)$ for all $y \in X$.*

Moreover, $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = 0$.

3. Implicit relations

Definition 3.1. Let $\mathfrak{F}_W$ be the set of all continuous functions $F(t_1, ..., t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ satisfying

(F_1) : F is non - increasing in variable t_3, t_4, t_5,

(F_2) : There exists $h_1 \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, v, u, u + v) \leq 0$ implies $u \leq h_1 v$,

(F_3) : There exists $h_2 \in [0, 1)$ such that for all $t, t' > 0$, $F(t, t, t', t, t + t') \leq 0$ implies $t \leq h_2 t'$.

In the following examples, property (F_1) is obviously.

Example 3.1. $F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, where $a, b, c, d \geq 0$ and $a + b + c + 2d < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u + v) = u - av - bv - cu - d(u + v) \leq 0$ which implies $u \leq h_1 v$, where $0 \leq h_1 = \frac{a+b+d}{1-(c+d)} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t, t, t', t, t + t') = t - at - bt' - ct - d(t + t') \leq 0$ which implies $t \leq h_2 t'$, where $0 < h_2 = \frac{b+d}{1-(a+c+d)} < 1$.

Example 3.2. $F(t_1, ..., t_5) = t_1 - k \max\{t_2, t_3, t_4, \frac{t_5}{2}\}$, where $k \in (0, 1)$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u + v) = u - k \max\{u, v, \frac{u+v}{2}\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t, t, t', t, t + t') = t - k \max\{t, t', \frac{t+t'}{2}\} \leq 0$. As in (F_2) it follows that $t \leq h_2 t'$, where $0 < h_2 = k < 1$.

Example 3.3. $F(t_1, ..., t_5) = t_1 - k \max\{t_2, t_3, t_4, t_5\}$, where $k \in \left(0, \frac{1}{2}\right)$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u + v) = u - k(u + v) \leq 0$ which implies $u \leq h_1 v$, where $0 \leq h_1 = \frac{k}{1-k} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t, t, t', t, t + t') = t - k(t + t') \leq 0$ which implies $t \leq h_2 t'$, where $0 < h_2 = \frac{k}{1-k} < 1$.

Example 3.4. $F(t_1, ..., t_5) = t_1 - at_2 - b \max\{t_3, t_4\} - ct_5$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u+v) = u - av - b \max\{u, v\} - c(u+v) \leq 0$. If $u > v$, then $u[1 - (a + b + 2c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \frac{a+b+c}{1-c} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t, t, t', t, t + t') = t - at - b \max\{t, t'\} - c(t + t') \leq 0$. Similar as in (F_2), we obtain $0 < h_2 = \frac{b+c}{1-(a+c)} < 1$.

Example 3.5. $F(t_1, ..., t_5) = t_1^2 - at_2t_3 - bt_4^2 - ct_5^2$, where $a, b, c \geq 0$ and $a+b+4c < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u + v) = u^2 - av^2 - bv^2 - c(u + v)^2 \leq 0$. If $u > v$, then $u^2[1 - (a + b + 4c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \sqrt{a + b + 4c} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t, t, t', t, t + t') = t^2 - att' - bt^2 - c(t + t')^2 \leq 0$. If $t > t'$ then $t^2[1 - (a + b + 4c)] \leq 0$, a contradiction. Hence $t \leq t'$, which implies $t \leq h_2 t'$, where $0 < h_2 = \sqrt{a + b + 4c} < 1$.

Example 3.6. $F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - c \max\{2t_4, t_5\}$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v) = u - av - bv - c\max\{2u, u+v\} \leq 0$. If $u > v$, then $u[1-(a+b+2c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = a+b+2c < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,t',t,t+t') = t - at - bt' - c\max\{2t, t+t'\} \leq 0$. If $t > t'$ then $t[1-(a+b+2c)] \leq 0$, a contradiction. Hence $t \leq t'$, which implies $t \leq h_2 t'$, where $0 < h_2 = a+b+2c < 1$.

Example 3.7. $F(t_1,...,t_5) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5^2$, where $a > 0, b, c, d \geq 0$ and $a+b+c+4d < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v) = u^2 - u(av+bv+cu) - d(u+v)^2 \leq 0$. If $u > v$, then $u^2[1-(a+b+c+4d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \sqrt{a+b+c+4d} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,t',t,t+t') = t^2 - t(at+bt'+ct) - d(t+t')^2 \leq 0$. As in (F_2) we obtain $t \leq h_2 t'$, where $0 < h_2 = \sqrt{a+b+c+4d} < 1$.

Example 3.8. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - c\max\{2t_4 + t_5, t_1 + t_4 + t_5\}$, where $a \geq 0, b > 0, c \geq 0$ and $a+b+4c < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v) = u - av - bv - c(3u+v) \leq 0$ which implies $u \leq h_1 v$, where $0 \leq h_1 = \frac{a+b+c}{1-3c} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,t',t,t+t') = t - at - bt' - c(3t+t') \leq 0$ which implies $t \leq h_2 t'$, where $0 < h_2 = \frac{b+c}{1-(a+3c)} < 1$.

4. Fixed point theorems

Theorem 4.1. *Let (X, G_p) be a G_p - metric space and let $T : X \to X$ be a mapping such that:*

$$\begin{aligned} F(&G_p(Tx,Ty,Ty), G_p(x,y,y), G_p(x,Tx,Tx), \\ &G_p(y,Ty,Ty), G_p(x,Ty,Ty) + G_p(y,Tx,Tx)) \leq 0 \end{aligned} \tag{4.1}$$

for all $x, y \in X$, where F satisfy property (F_3). Then, T has at most a fixed point.

Proof. Suppose that T has two distinct fixed points u and v. Then, by (4.1) we have successively

$$\begin{aligned} F(&G_p(Tu,Tv,Tv), G_p(u,v,v), G_p(u,Tu,Tu), \\ &G_p(v,Tv,Tv), G_p(u,Tv,Tv) + G_p(v,Tu,Tu)) \leq 0, \end{aligned}$$

$$\begin{aligned} F(&G_p(u,v,v), G_p(u,v,v), G_p(u,u,u), \\ &G_p(v,v,v), G_p(u,v,v) + G_p(v,u,u)) \leq 0. \end{aligned}$$

By (GP_2),

$$G_p(u,u,u) \leq G_p(v,u,u)$$

and

$$G_p(v,v,v) \leq G_p(u,v,v)$$

By (F_1) we have

$$\begin{aligned} F(&G_p(u,v,v), G_p(u,v,v), G_p(v,u,u), \\ &G_p(u,v,v), G_p(u,v,v) + G_p(v,u,u)) \leq 0. \end{aligned}$$

By (F_3) we obtain

$$G_p(u,v,v) \leq h_2 G_p(v,u,u).$$

Similarly, we obtain

$$G_p(v,u,u) \leq h_2 G_p(u,v,v).$$

Hence

$$G_p(u,v,v)(1-h_2^2) \leq 0,$$

a contradiction.

Hence, we get $u=v$ by using Lemma 2.1. □

Theorem 4.2. *Let (X,G_p) be a G_p - complete metric space and let $T:X\to X$ be a mapping satisfying inequality (4.1), for all $x,y\in X$, where $F\in\mathfrak{F}_W$. Then, T has a unique fixed point.*

Proof. Let $x_0\in X$ be an arbitrary point of X. We define $x_n=Tx_{n-1}, n=1,2,...$. Then by (4.1) we have successively

$$\begin{gathered}F(G_p(Tx_{n-1},Tx_n,Tx_n),G_p(x_{n-1},x_n,x_n),G_p(x_{n-1},Tx_{n-1},Tx_{n-1}),\\ G_p(x_n,Tx_n,Tx_n),G_p(x_{n-1},Tx_n,Tx_n)+G_p(x_n,Tx_{n-1},Tx_{n-1}))\leq 0,\end{gathered}$$

$$\begin{gathered}F(G_p(x_n,x_{n+1},x_{n+1}),G_p(x_{n-1},x_n,x_n),G_p(x_{n-1},x_n,x_n),\\ G_p(x_n,x_{n+1},x_{n+1}),G_p(x_{n-1},x_{n+1},x_{n+1})+G_p(x_n,x_n,x_n))\leq 0.\end{gathered}$$

By (GP_4) we have

$$G_p(x_{n-1},x_{n+1},x_{n+1})\leq G_p(x_{n-1},x_n,x_n)+G_p(x_n,x_{n+1},x_{n+1})-G_p(x_n,x_n,x_n).$$

By (F_1) we obtain

$$\begin{gathered}F(G_p(x_n,x_{n+1},x_{n+1}),G_p(x_{n-1},x_n,x_n),G_p(x_{n-1},x_n,x_n),\\ G_p(x_n,x_{n+1},x_{n+1}),G_p(x_{n-1},x_n,x_n)+G_p(x_n,x_{n+1},x_{n+1}))\leq 0.\end{gathered}$$

By (F_2) we have

$$G_p(x_n,x_{n+1},x_{n+1})\leq h_1G_p(x_{n-1},x_n,x_n).$$

Therefore,

$$G_p(x_n,x_{n+1},x_{n+1})\leq h_1G_p(x_{n-1},x_n,x_n)\leq ...\leq h_1^nG_p(x_0,x_1,x_1). \tag{4.2}$$

By (4.2) and (GP_4) we obtain for $m>n$ that

$$\begin{aligned}G_p(x_n,x_m,x_m) &\leq G_p(x_n,x_{n+1},x_{n+1})+G_p(x_{n+1},x_{n+2},x_{n+2})+\\ &\quad +...+G_p(x_{m-1},x_m,x_m)\\ &\leq h_1^n(1+h_1+...+h_1^{m-1})G_p(x_0,x_1,x_1)\\ &\leq \frac{h_1^n}{1-h_1}G_p(x_0,x_1,x_1).\end{aligned}$$

Consequently,

$$\lim_{n,m\to\infty}G(x_n,x_m,x_m)=0$$

and thus $\{x_n\}$ is a G_p - Cauchy sequence. Since (X,G_p) is G_p - complete metric space, there exists $z\in X$ such that

$$\lim_{n,m\to\infty}G(x_n,x_m,x_m)=\lim_{n\to\infty}G(z,x_n,x_n)=G_p(z,z,z)=0. \tag{4.3}$$

We prove that z is a fixed point of T.

By (4.1) we have successively

$$\begin{gathered}F(G_p(Tx_n,Tz,Tz),G_p(x_n,z,z),G_p(x_n,Tx_n,Tx_n),\\ G_p(z,Tz,Tz),G_p(x_n,Tz,Tz)+G_p(z,Tx_n,Tx_n))\leq 0,\end{gathered}$$

$$\begin{gathered}F(G_p(Tx_n,Tz,Tz),G_p(x_n,z,z),G_p(x_n,x_{n+1},x_{n+1}),\\ G_p(z,Tz,Tz),G_p(x_n,Tz,Tz)+G_p(z,x_{n+1},x_{n+1}))\leq 0.\end{gathered}$$

By Lemma 2.2, (4.2) and (4.3), letting n tend to infinity we obtain

$$F(G_p(z,Tz,Tz),0,0,G_p(z,Tz,Tz),G_p(z,Tz,Tz)) \leq 0.$$

By (F_2) we obtain $G(z,Tz,Tz) = 0$. By Lemma 2.1 (a), we obtain $z = Tz$. Hence z is a fixed point of T. By Theorem 4.1, z is the unique fixed point of T. □

Corollary 4.1. *Let (X, G_p) be a complete G_p - metric space such that*

$$\begin{aligned} G_p(Tx,Ty,Ty) \leq k\max\{&G_p(x,y,y),G_p(x,Tx,Tx),\\ &G_p(y,Ty,Ty), \tfrac{G_p(x,Ty,Ty)+G_p(y,Tx,Tx)}{2}\}, \end{aligned} \tag{4.4}$$

where $k \in (0,1)$. Then T has an unique fixed point.

Proof. The proof it follows by Theorem 4.1 and Example 3.2. □

Example 4.1. Let $X = [0,\infty)$, $G_p : X^3 \to \mathbb{R}$ defined by $G_p(x,y,z) = \max\{x,y,z\}$. Then (X,G_p) is G_p - complete metric space. Let $T : X \to X$ be defined by $Tx = \frac{x}{x+2}$.

Without loss of generality, we assume that $x \geq y$. Then

$$G_p(Tx,Ty,Ty) = \frac{x}{x+2} \leq \frac{x}{2} = \frac{1}{2}G_p(x,y,y) \leq kG_p(x,y,y),$$

where $k \in \left[\frac{1}{2},1\right)$ which implies

$$\begin{aligned} G_p(Tx,Ty,Ty) \leq k\max\{&G_p(x,y,y),G_p(x,Tx,Tx),\\ &G_p(y,Ty,Ty), \tfrac{G_p(x,Ty,Ty)+G_p(y,Tx,Tx)}{2}\}. \end{aligned}$$

By Corollary 4.1, T has an unique fixed point $x = 0$. Moreover, $G_p(0,0,0) = 0$.

5. Well posedness problem of fixed point in G_p - metric spaces

Definition 5.1 ([38])**.** Let (X,d) be a metric space (X,d) and let $f : (X,d) \to (X,d)$ be a mapping. The fixed point problem of f is said to be well posed if:

1) f has an unique fixed point x_0,

2) for any sequence $\{x_n\} \in X$ with $\lim_{n\to\infty} d(fx_n,x_n) = 0$ we have $\lim_{n\to\infty} d(x_n,x_0) = 0$.

Definition 5.2. A function $F : \mathbb{R}_+^5 \to \mathbb{R}$ has property (F_p) if for all $u,v,w \geq 0$ and $F(u,v,0,w,u+v) \leq 0$, there exists $p \in (0,1)$ such that $u \leq p\max\{v,w\}$.

Example 5.1. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, where $a,b,c,d \geq 0$, $a+c+2d > 0$ and $a+b+c+4d < 1$.

(F_p) : Let $u,v,w \geq 0$ be such that $F(u,v,0,w,u+v) = u - av - cw - d(u+v) \leq 0$. If $u > \max\{v,w\}$, then $u[1-(a+c+2d)] \leq 0$, a contradiction. Hence $u \leq \max\{v,w\}$, which implies $u \leq p\max\{v,w\}$, where $0 < p = a+c+2d < 1$.

Example 5.2. $F(t_1,...,t_5) = t_1 - k\max\{t_2,t_3,t_4,\frac{t_5}{2}\}$, where $k \in (0,1)$.

(F_p) : Let $u,v,w \geq 0$ be such that $F(u,v,0,w,u+v) = u - k\max\{v,w,\frac{u+v}{2}\} \leq 0$. If $u > \max\{v,w\}$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq \max\{v,w\}$, which implies $u \leq p\max\{v,w\}$, where $0 < p = k < 1$.

Example 5.3. $F(t_1,...,t_5) = t_1 - k\max\{t_2,t_3,t_4,t_5\}$, where $k \in \left(0,\frac{1}{2}\right)$.

(F_p) : Let $u,v,w \geq 0$ be such that $F(u,v,0,w,u+v) = u - k\max\{v,w,u+v\} \leq 0$. If $u > \max\{v,w\}$, then $u(1-2k) \leq 0$, a contradiction. Hence $u \leq \max\{v,w\}$, which implies $u \leq p\max\{v,w\}$, where $0 < p = 2k < 1$.

Example 5.4. $F(t_1, ..., t_5) = t_1 - at_2 - b\max\{t_3, t_4\} - ct_5$, where $a, b, c \geq 0$ and $0 < a + b + 2c < 1$.

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u + v) = u - av - bw - c(u + v) \leq 0$. If $u > \max\{v, w\}$, then $u[1 - (a + b + 2c)] \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p\max\{v, w\}$, where $0 < p = a + b + 2c < 1$.

Example 5.5. $F(t_1, ..., t_5) = t_1^2 - at_2t_3 - bt_4^2 - ct_5^2$, where $a, b, c \geq 0$, $a + 2c > 0$ and $a + b + 2c < 1$.

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u+v) = u^2 - bw^2 - c(u+v)^2 \leq 0$. If $u > \max\{v, w\}$, then $u[1-(b+4c)] \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p\max\{v, w\}$, where $0 < p = b + 4c < 1$.

Example 5.6. $F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5\}$, where $a, b, c \geq 0$, $a + 2c > 0$ and $a + b + 2c < 1$.

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u+v) = u - av - c\max\{2w, u + v\} \leq 0$. If $u > \max\{v, w\}$, then $u[1 - (a + 2c)] \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p\max\{v, w\}$, where $0 < p = a + 2c < 1$.

Example 5.7. $F(t_1, ..., t_5) = t_1^2 - t_1[at_2 + bt_3 + ct_4] - dt_5^2$, where $a, b, c, d \geq 0$ and $0 < a + c + 4d < 1$.

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u+v) = u^2 - u[av+cw] - d(u + v)^2 \leq 0$. If $u > \max\{v, w\}$, then $u^2[1 - (a + c + 4d)] \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p\max\{v, w\}$, where $0 < p = a + c + 4d < 1$.

Example 5.8. $F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - c\max\{2t_4 + t_5, t_1 + t_4 + t_5\}$, where $a, b, c \geq 0$, $a + 4c > 0$ and $a + b + 4c < 1$.

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u + v) = u - av - c\max\{2w + u + v, 2u + w + v\} \leq 0$. If $u > \max\{v, w\}$, then $u[1 - (a + 4c)] \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p\max\{v, w\}$, where $0 < p = a + 4c < 1$.

Definition 5.3. Let (X, G_p) be a G_p - metric space and let $T : X \to X$ be a function. The fixed point of T is well posed if

1) T has an unique fixed point x_0,

2) for any sequence $\{x_n\} \in X$ with $\lim_{n\to\infty} G_p(x_n, Tx_n, Tx_n) = 0$ we have $\lim_{n\to\infty} G_p(x_0, x_n, x_n) = 0$.

Theorem 5.1. *Let (X, G_p) be a G_p - symmetric space and $T : X \to X$ a function satisfying the conditions of Theorem 4.2, where F satisfy property (F_p). Then the fixed point problem of T is well posed.*

Proof. By Theorem 4.2, T has an unique fixed point x_0. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} G_p(x_n, Tx_n, Tx_n) = 0$. By (4.1) we have successively

$$F(G_p(Tx_0, Tx_n, Tx_n), G_p(x_0, x_n, x_n), G_p(x_0, Tx_0, Tx_0),$$
$$G_p(x_n, Tx_n, Tx_n), G_p(x_0, Tx_n, Tx_n) + G_p(x_n, Tx_0, Tx_0)) \leq 0,$$

$$F(G_p(x_0, Tx_n, Tx_n), G_p(x_0, x_n, x_n), 0,$$
$$G_p(x_n, Tx_n, Tx_n), G_p(x_0, Tx_n, Tx_n) + G_p(x_n, x_0, x_0)) \leq 0.$$

Since (X, G_p) is symmetric, then $G(x_n, x_0, x_0) = G(x_0, x_n, x_n)$. Hence,

$$F(G_p(x_0, Tx_n, Tx_n), G_p(x_0, x_n, x_n), 0,$$
$$G_p(x_n, Tx_n, Tx_n), G_p(x_0, Tx_n, Tx_n) + G_p(x_0, x_n, x_n)) \leq 0.$$

Since F satisfy property (F_p) then

$$\begin{aligned} G_p(x_0, Tx_n, Tx_n) &\leq p \max\{G_p(x_0, x_n, x_n), G_p(x_n, Tx_n, Tx_n)\} \\ &\leq p \max\{G_p(x_0, x_n, x_n) + G_p(x_n, Tx_n, Tx_n)\}. \end{aligned}$$

By (GP_4):

$$\begin{aligned} G_p(x_0, x_n, x_n) &\leq G_p(x_0, Tx_n, Tx_n) + G_p(Tx_n, x_n, x_n) \\ &\leq p[G_p(x_0, x_n, x_n) + G_p(x_n, Tx_n, Tx_n)] + G_p(x_n, Tx_n, Tx_n), \end{aligned}$$

which implies

$$G_p(x_0, x_n, x_n) \leq \frac{1+p}{1-p} G_p(x_n, Tx_n, Tx_n).$$

Hence,

$$\lim_{n\to\infty} G_p(x_0, x_n, x_n) = 0$$

and the fixed point problem of T is well posed. □

Corollary 5.1. *Let (X, G_p) be a G_p - symmetric space and $T : X \to X$ be a function satisfying the conditions of Corollary 4.1. Then, the fixed point problem of T is well posed.*

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"Vasile Alecsandri" University of Bacău, ROMANIA
E-mail address: vpopa@ub.ro

"Dunărea de Jos" University of Galaţi, Faculty of Sciences and Environment, Department of Mathematics and Computer Sciences, ROMANIA
E-mail address: Alina.Patriciu@ugal.ro