Research Article GU J Sci, Part A, 12(2): 562-582 (2025) 10.54287/gujsa.1655950

Gazi University
Journal of Science
PART A: ENGINEERING AND INNOVATION
http://dergipark.org.tr/gujsa

Some Applications of Berezin Radius Inequalities

Hamdullah BAŞARAN[1*]

[1] *Antalya, Türkiye*

Keywords

Reproducing Kernel Hilbert Space

Operator Norm

Berezin Symbol

Polar Decomposition

Contraction Operator

Abstract

We show a number of inequalities in the reproducing kernel Hilbert space (RKHS) in this study. Stronger boundaries between the Berezin radius and the numerical radius on the Berezin number of the bounded linear operator described in the RKHS than those found in the literature are obtained by using a few auxiliary theorems while analyzing the Berezin radius.

Cite

Başaran, H. (2025). Some Applications of Berezin Radius Inequalities. *GU J Sci, Part A*, *12*(2), 562-582. doi:10.54287/gujsa.1655950

Author ID (ORCID Number)

0000-0002-9864-9515 Hamdullah BAŞARAN

Article Process

Submission Date 11.03.2025
Revision Date 12.05.2025
Accepted Date 28.05.2025
Published Date 30.06.2025

1. INTRODUCTION

We represent complex-valued functions on an existing set X in a Hilbert space $\mathcal{H} = \mathcal{H}(X)$ which means evaluation at any given point in X. As a reproducing kernel Hilbert space, we establish a Hilbert space $\mathcal{H} = \mathcal{H}(X)$ of complex-valued functions on a given set X such that evaluation at each point of X is a continuous functional on H. The Hilbert function space $\mathcal{H}$ has been shown to have a reproducing kernel according to the Riesz representation theorem. This means that for all $\rho \in X$, is present a function $k_\rho(z) \in \mathcal{H}$ which means $\langle f, k_\rho \rangle = f(\rho)$ for all $f \in \mathcal{H}$ and $\rho \in X$. The reproducing kernel of the space H is the term presented to this function. We demonstrate the normalized reproducing kernel by $\hat{k}_\rho = \frac{k_\rho}{\|k_\rho\|}$. The Dirichlet space $\mathcal{D}^2(\mathbb{D})$, Hardy space $\mathcal{H}^2(\mathbb{D})$, Bergman space $\mathcal{L}_\alpha^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ is the unit disc, and the Fock space $\mathcal{F}(\mathbb{C})$ are the important RKHSs. For instance, Aronzajn (1950) extends a comprehensive account of the theory of RKHSs and reproducing kernels. Reproducing kernels plays an essential role in a number of pure and applied mathematics fields, which includes wavelets, frame theory, fractal theories, and signals (for example, see Jorgensen's book (Jorgensen, 2006) and its references).

The C^*-algebra of any bounded linear operator described on a complex Hilbert space $(\mathcal{H}, \langle .,. \rangle)$ is shown with symbol $\mathcal{L}(\mathcal{H})$. For each of the bounded linear operator N on $\mathcal{H}$ (that is, for $N \in \mathcal{L}(\mathcal{H})$) its Berezin symbol $\widetilde{N}$

*Corresponding Author, e-mail: 07hamdullahbasaran@gmail.com

is denoted by (can observe, Berezin (1972) and Karaev (2006))

$$\widetilde{N} = \langle N\hat{k}_\rho, \hat{k}_\rho \rangle, \rho \in X.$$

Because of the Cauchy-Schwarz inequality $|\widetilde{N}(\rho)| \leq \|N\hat{k}_\rho\| \leq \|N\|$ for all $\rho \in X$, that scalar-valued function is bounded on X. An operator's Berezin symbol constitutes significant details with regard to the operator. As an example, has become commonly accepted that each of the operator within the RKHSs that is analytic functions (like the Dirichlet, Fock, Hardy, and Bergman spaces) is uniquely established by the Berezin symbol, i.e., $N_1 = N_2$ iff $\widetilde{N}_1 = \widetilde{N}_2$ (see, for example, Engliš (1995) and Zhu (1990)).

The Berezin set corresponding to the operator $N \in \mathcal{L}(\mathcal{H})$) is described

$$Ber(N) = Range(\widetilde{N})$$

and Berezin radius of N is the number defined by

$$ber(N) = sup\{|\xi|: \xi \in Ber(N)\}$$

(see Karaev (2006, 2013)). The Berezin norm of operators $N \in \mathcal{L}(\mathcal{H})$ is denoted by

$$\|N\|_{Ber} = sup_{\rho \in X}\|N\hat{k}_\rho\|.$$

It becomes simplicity to demonstrate that actually $\|N\|_{Ber}$ chooses a new operator norm in $\mathcal{L}(\mathcal{H}(X))$ (since the set of reproducing kernels $\{k_\rho: \rho \in X\}$ span the space $\mathcal{H}(X)$). It is straightforward that $ber(N) \leq \|N\|_{Ber} \leq \|N\|$ (refer to Aronzajn (1950) and Berezin (1972) for further details on reproducing kernel Hilbert spaces and the Berezin symbol).

The following are given $W(N)$ and also $w(N)$ which stand for the numerical range and numerical radius of N:

$$W(N) = \{\langle Nf, f \rangle: f \in \mathcal{H} \ and \ \|f\|_{\mathcal{H}} = 1\}$$

and

$$w(N) = sup\{|\langle Nf, f \rangle|: f \in \mathcal{H} \ and \ \|f\|_{\mathcal{H}} = 1\}.$$

It is clear that $Ber(N) \subseteq W(N)$ and $ber(N) \leq w(N)$ (the numerical radius of operator N). There are some intriguing qualities of an operator's numerical range. As an instance, it is widely acknowledged that the closure of an operator's numerical range includes its spectrum. For essential characteristics of the numerical radius, we refer to Abu-Omar and Kittaneh (2015), Bhatia (2007), Bhunia et al. (2022), Bhunia et al. (2023a), Furuta, (2001), Kittaneh (1952), Kittaneh et al. (2015), Stojiljkovic and Gürdal (2025), Sababheh and Moradi (2023), Yamakazi (2007), Yang and Xu (2023).

It is often known that

$$\frac{1}{2}\|N\| \leq w(N) \leq \|N\| \tag{1}$$

and

$$ber(N) \leq w(N) \leq \|N\| \tag{2}$$

for any $N \in \mathcal{L}\big(\mathcal{H}(X)\big)$. Hence, studying of the recently discovered numerical properties $Ber(N)$, $ber(N)$, and $\|N\|_{Ber}$, we will be essential for a detailed examination of both this numerical radius and range of operators based the RKHSs.

For $N \in \mathcal{L}(\mathcal{H})$, the Berezin norm of N is defined as:

$$\|N\|_{ber} = sup\{\big|\langle N\hat{k}_\rho, \hat{k}_\varphi\rangle\big|, \rho, \varphi \in X\},$$

Here, $\hat{k}_\rho$ and $\hat{k}_\varphi$ represent normalized reproducing kernels of the space $\mathcal{H}$. It is essential to remember that the norm $\|.\|_{ber}$ does not always imply the submultiplicative property. See Bhunia et al. (2023b), Karaev et al. (2016), Karaev et al. (2011) and Yamancı et al. (2020) for an examination of the fundamental properties and facts with respect to Berezin number.

The Cauchy-Schwarz inequality, which indicates that

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \qquad x, y \in \mathcal{H} \tag{3}$$

is belongs to the foremost significant and useful inequalities in operator theory. Buzano (1974) established an extension (3) involving it has been shown that

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|) \tag{4}$$

for any $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. It is easy to see that (4) may be expressed for any $e \in \mathcal{H}$ in the form as follows:

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{\|e\|^2}{2}(\|x\|\|y\| + |\langle x, y\rangle|). \tag{5}$$

Using the (3) yields

$$|\langle x, e\rangle\langle e, y\rangle| \leq \|e\|^2\|x\|\|y\|,$$

which is important to remember here. As a consequence, (5) includes a more appealing bound than the one which ensues from using (3) twice. In close connection with Buzano's inequality (4), Dragomir (2016) presented that

$$|\langle Tx, Ty\rangle| \leq \frac{\|T\|^2}{2}(\|x\|\|y\| + |\langle x, y\rangle|).$$

for any $T \in \mathcal{L}(\mathcal{H})$.

In the present article, we prove several inequalities in RKHS. When analyzing the radius of Berezin we find stronger bounds on $ber(N)$ than those obtained in inequality (2). Following that, that is utilized Lemma 2.8 for demonstrating the final inequality.

2. MATERIAL AND METHOD

The following lemmas from the literature will be required in order to reach our conclusions. The first lemma is proven by Sababheh et al. (2024).

Lemma 2.1. LetP, R, S $\in \mathcal{L}(\mathcal{H})$, where P and R are positive. Then the next statements are equivalent to one another:

(i) $\begin{bmatrix} P & S \\ S^* & R \end{bmatrix}$ is a positive operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

(ii) $\begin{bmatrix} P & S^* \\ S & R \end{bmatrix}$ is a positive operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

(iii) $|\langle Sx, y\rangle| \leq \langle Px, x\rangle\langle Ry, y\rangle$ for all , $x, y \in \mathcal{H}$.

(iv) There exists a contraction O (i.e. $\|O\| \leq 1$) such that $S = R^{\frac{1}{2}}OP^{\frac{1}{2}}$.

The following two lemmas are proven by Kittaneh (1997; 1988).

Lemma 2.2. Let $P, R \in \mathcal{L}(\mathcal{H})$ be a positive operators. Then

$$\|P+R\| \leq max\{\|P\|, \|R\|\} + \left\|R^{\frac{1}{2}}P^{\frac{1}{2}}\right\|.$$

Lemma 2.3. Let $P, R, S \in \mathcal{L}(\mathcal{H})$, where P and R are positive and RS = SP. If $\begin{bmatrix} P & S^* \\ S & R \end{bmatrix}$ is positive in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, then $\begin{bmatrix} f^2(P) & S^* \\ S & g^2(R) \end{bmatrix}$ is also positive, where f, g are nonnegative continuous functions on $[0, \infty)$, such that $f(t)g(t) = t$ for every $t \in [0, \infty)$.

Since $P|P| = |P^*|P$ (see, (Pečarić et al., 2005)) Lemma 2.3, implies

$$\begin{bmatrix} f^2(|P|) & P^* \\ P & g^2(|P^*|) \end{bmatrix} \geq 0 \text{ for any } P \in \mathcal{L}(\mathcal{H}). \tag{6}$$

The following lemma is obtained by use to inequality (7) and a more eleborate version inequality (6) which is associated with the product of three operators.

Lemma 2.4. Let $P, R, S \in \mathcal{L}(\mathcal{H})$ and S positive. Then

$$|\langle PSRx, y\rangle| \leq \frac{\|S\|^2}{2}(|\langle PRx, y\rangle| + \|Rx\|\|P^*y\|) \tag{7}$$

for any $x, y \in \mathcal{H}$ (see, (Sababheh et al., 2024)).

We provide a new equivalent assertion about the positivity of a specific operator matrix in the following theorem, which may be added to the claims in Lemma 2.1.

Lemma 2.5. Let $P, R, S \in \mathcal{L}(\mathcal{H})$ and P and R positive and any $x, y \in \mathcal{H}$. Then $\begin{bmatrix} P & S^* \\ S & R \end{bmatrix}$ is a positive operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ iff

$$|\langle Sx, y\rangle| \leq \sqrt{\langle R^{\frac{1}{2}}|O|R^{\frac{1}{2}}x, x\rangle \langle P^{\frac{1}{2}}|O^*|P^{\frac{1}{2}}y, y\rangle} \tag{8}$$

for some contraction O (see, (Sababheh et al., 2024)).
This is the conclusion that follows directly from Lemma 2.5 and inequality (6).

Lemma 2.6. Let $N \in \mathcal{L}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. If f, g are nonnegative continuous on $[0, \infty)$ obtaining

$f(t)g(t) = t \ (t > 0)$, then

$$|\langle Nx, y\rangle| \leq \sqrt{\langle g(|N^*|)|O|^{\frac{1}{2}} g(|N^*|)x, x\rangle \langle f(|N|)|O^*|^{\frac{1}{2}} f(|N|)y, y\rangle} \tag{9}$$

for some contraction O (see, (Sababheh et al., 2024)).

The following two lemmas are proven by using Lemma 2.1.

Lemma 2.7. Let $N \in \mathcal{L}(\mathcal{H})$ with the polar decomposition $N = U|N|$, where U is a partial isometry, with $0 \leq r \leq 1$. Then for any $x, y \in \mathcal{H}$,

$$|\langle Nx, y\rangle| \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(\left|\langle U|N|^{\frac{1}{2}}x, y\rangle\right| + \sqrt{\langle |N|^r x, x\rangle\langle |N^*|^{1-r} y, y\rangle}\right) \tag{10}$$

and

$$|\langle Nx, y\rangle| \leq \frac{\|N\|^{1-r}}{2}\left(|\langle U|N|^r x, y\rangle| + \sqrt{\langle |N|^r x, x\rangle\langle |N^*|^r y, y\rangle}\right) \tag{11}$$

(see, (Sababheh et al., 2024)).

Lemma 2.8. Let $N \in \mathcal{M}_n$. If f, g are nonnegative continuous based $[0, \infty)$ obtaining $f(t)g(t) = t \ (t > 0)$, then for all vectors $x, y \in \mathbb{C}$,

$$\text{(i)} \qquad |\langle \Re N x, y\rangle| \leq \frac{1}{2}\sqrt{\langle (f^2(|N|) + f^2(|N^*|))x, x\rangle\langle (g^2(|N|) + g^2(|N^*|))y, y\rangle} \tag{12}$$

and

$$\text{(ii)} \qquad |\langle \Im N x, y\rangle| \leq \frac{1}{2}\sqrt{\langle (f^2(|N|) + f^2(|N^*|))x, x\rangle\langle (g^2(|N|) + g^2(|N^*|))y, y\rangle} \tag{13}$$

(see, (Sababheh et al., 2024)).

3. MAIN RESULTS

In this part of the paper, we obtain Berezin radius inequality. We offer some bounds for the Berezin radius.

Let's present the first theorem.

Theorem 3.1. Assume $\mathcal{H} = \mathcal{H}(\mathrm{X})$ to be an RKHS. Given the polar decomposition $\mathrm{N} = \mathrm{U}|\,\mathrm{N}|$, let $\mathrm{N} \in \mathcal{H}$. Then for any $0 \leq \mathrm{r} \leq 1$,

$$\mathrm{ber}(\mathrm{N}) \leq \frac{\|\mathrm{N}\|^{\frac{1}{2}}}{2}\left(\mathrm{ber}\left(\mathrm{U}|\,\mathrm{N}|^{\frac{1}{2}}\right) + \frac{1}{2}\||\mathrm{N}|^{\mathrm{r}} + |\mathrm{N}^*|^{1-\mathrm{r}}\|_{\mathrm{ber}}\right) \tag{14}$$

and

$$\mathrm{ber}(\mathrm{N}) \leq \frac{\|\mathrm{N}\|^{1-\mathrm{r}}}{2}\left(\mathrm{ber}(\mathrm{U}|\,\mathrm{N}|^{\mathrm{r}}) + \frac{1}{2}\||\mathrm{N}|^{\mathrm{r}} + |\mathrm{N}^*|^{\mathrm{r}}\|_{\mathrm{ber}}\right), \tag{15}$$

where U is a partial isometry.

Proof. Suppose $\hat{k}_\rho$ be a reproducing kernel that has been normalized. If we write $x = y = \hat{k}_\rho$ in the inequality (10), then we get

$$\begin{aligned}
\left|\langle \mathrm{N}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right| &\leq \frac{\|\mathrm{N}\|^{\frac{1}{2}}}{2}\left(\left|\langle \mathrm{U}|\,\mathrm{N}|^{\frac{1}{2}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right| + \sqrt{\langle |\mathrm{N}|^{\mathrm{r}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\langle |\mathrm{N}^*|^{1-\mathrm{r}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle}\right) \\
&\leq \frac{\|\mathrm{N}\|^{\frac{1}{2}}}{2}\left(\left|\langle \mathrm{U}|\,\mathrm{N}|^{\frac{1}{2}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right| + \frac{\langle |\mathrm{N}|^{\mathrm{r}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle + \langle |\mathrm{N}^*|^{1-\mathrm{r}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle}{2}\right) \\
&\leq \frac{\|\mathrm{N}\|^{\frac{1}{2}}}{2}\left(\left|\langle \mathrm{U}|\,\mathrm{N}|^{\frac{1}{2}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right| + \frac{1}{2}\langle (|\mathrm{N}|^{\mathrm{r}} + |\mathrm{N}^*|^{1-\mathrm{r}})\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right)
\end{aligned}$$

Thus, we obtain

$$\left|\langle \mathrm{N}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right| \leq \frac{\|\mathrm{N}\|^{\frac{1}{2}}}{2}\left(\left|\langle \mathrm{U}|\,\mathrm{N}|^{\frac{1}{2}}\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right| + \frac{1}{2}\langle (|\mathrm{N}|^{\mathrm{r}} + |\mathrm{N}^*|^{1-\mathrm{r}})\hat{\mathrm{k}}_\rho, \hat{\mathrm{k}}_\rho\rangle\right).$$

Using the inequality above, by calculating the supremum over $\rho \in X$, we obtain

$$ber(N) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(ber\left(U|\,N|^{\frac{1}{2}}\right) + \frac{1}{2}\||N|^{r} + |N^*|^{1-r}\|_{ber}\right).$$

If we write $x = y = \hat{k}_\rho$ in the inequality (11), then we get

$$\left|\langle N\hat{k}_\rho,\hat{k}_\rho\rangle\right| \le \frac{\|N\|^{1-r}}{2}\left(\left|\langle U|\,N|^r\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \sqrt{\langle |N|^r\hat{k}_\rho,\hat{k}_\rho\rangle\langle |N^*|^r\hat{k}_\rho,\hat{k}_\rho\rangle}\right) \tag{16}$$

$$\le \frac{\|N\|^{1-r}}{2}\left(\left|\langle U|\,N|^r\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \frac{\langle |N|^r\hat{k}_\rho,\hat{k}_\rho\rangle+\langle |N^*|^r\hat{k}_\rho,\hat{k}_\rho\rangle}{2}\right)$$

(by arithmetic-geometric mean inequality)

$$\le \frac{\|N\|^{1-r}}{2}\left(\left|\langle U|\,N|^r\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \frac{1}{2}\langle(|N|^r+|N^*|^r)\hat{k}_\rho,\hat{k}_\rho\rangle\right).$$

Thus, we have

$$\left|\langle N\hat{k}_\rho,\hat{k}_\rho\rangle\right| \le \frac{\|N\|^{1-r}}{2}\left(\left|\langle U|\,N|^r\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \frac{1}{2}\langle(|N|^r+|N^*|^r)\hat{k}_\rho,\hat{k}_\rho\rangle\right).$$

Taking the supremum over $\rho \in X$ in the above inequality, we get

$$ber(N) \le \frac{\|N\|^{1-r}}{2}\left(ber(U|\,N|^r) + \frac{1}{2}\||N|^r+|N^*|^r\|_{ber}\right).$$

The proof has become completed.

Remark *3.2.* If we put r=1/2 and apply Theoem 3.1 in Huban et al. (2022a), we get the following inequalities.

(i) $\mathrm{ber(N)} \le \frac{\|N\|^{\frac{1}{2}}}{2}\left(\mathrm{ber}\left(\mathrm{U}|\,\mathrm{N}|^{\frac{1}{2}}\right) + \frac{1}{2}\left\||\mathrm{N}|^{\frac{1}{2}}+|\mathrm{N}^*|^{\frac{1}{2}}\right\|_{\mathrm{ber}}\right)$

(ii) $\mathrm{ber(N)} \le \frac{1}{2}\left(\mathrm{ber(U|\,N|)} + \frac{1}{2}\||\mathrm{N}|+|\mathrm{N}^*|\|_{\mathrm{ber}}\right).$

Now, we will introduce theorem related to three operator.

Theorem 3.3. Assume $\mathcal{H} = \mathcal{H}(\mathrm{X})$ to be an RKHS. Let $\mathrm{P, R, S} \in \mathcal{L}(\mathcal{H})$ and S be positive. Then

$$\mathrm{ber(PSR)} \le \frac{\|S\|}{2}\left(\mathrm{ber(PR)} + \frac{1}{2}\||\mathrm{P}^*|^2+|\mathrm{R}|^2\|_{\mathrm{ber}}\right) \tag{17}$$

and

$$\mathrm{ber(PSR)} \le \frac{\|S\|}{2}\left(\mathrm{ber(PR)} + \frac{1}{2}(\|\mathrm{P}\|_{\mathrm{ber}}\|\mathrm{R}\|_{\mathrm{ber}} + \|\mathrm{PR}\|_{\mathrm{ber}})\right). \tag{18}$$

Proof. From inequality (6), we have

$$\begin{aligned}
\left|\langle PSR\hat{k}_\rho,\hat{k}_\rho\rangle\right| &\le \frac{\|S\|}{2}\left(\left|\langle PR\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \left\|R\hat{k}_\rho\right\|\left\|P^*\hat{k}_\rho\right\|\right) \\
&\le \frac{\|S\|}{2}\left(\left|\langle PR\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \sqrt{\langle |P^*|^2\hat{k}_\rho,\hat{k}_\rho\rangle\langle |R|^2\hat{k}_\rho,\hat{k}_\rho\rangle}\right) \\
&\le \frac{\|S\|}{2}\left(\left|\langle PR\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \frac{\langle |P^*|^2\hat{k}_\rho,\hat{k}_\rho\rangle + \langle |R|^2\hat{k}_\rho,\hat{k}_\rho\rangle}{2}\right) \\
&\text{(by arithmetic-geometric mean inequality)} \\
&\le \frac{\|S\|}{2}\left(\left|\langle PR\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \frac{1}{2}\langle (|P^*|^2+|R|^2)\hat{k}_\rho,\hat{k}_\rho\rangle\right)
\end{aligned} \tag{19}$$

Here, we get

$$\left|\langle PSR\hat{k}_\rho,\hat{k}_\rho\rangle\right| \le \frac{\|S\|}{2}\left(\left|\langle PR\hat{k}_\rho,\hat{k}_\rho\rangle\right| + \frac{1}{2}\langle (|P^*|^2+|R|^2)\hat{k}_\rho,\hat{k}_\rho\rangle\right).$$

Using the inequality above, by calculating the supremum over $\rho \in X$, we obtained

$$ber(PSR) \le \frac{\|S\|}{2}\left(ber(PR) + \frac{1}{2}\||P^*|^2 + |R|^2\|_{ber}\right).$$

Now, lets prove the second inequality. If $P = 0$ or $R = 0$, the result follows trivally. Therefore, assume that P and R are nonzero. In inequality (15), if we replace P and R by $\sqrt{\frac{\|R\|_{ber}}{\|P\|_{ber}}}P$ and $\sqrt{\frac{\|P\|_{ber}}{\|R\|_{ber}}}R$, respectively, we reach

$$\begin{aligned}
ber(PSR) &\le \frac{\|S\|}{2}\left(ber(PR) + \frac{1}{2}\left\|\frac{\|R\|_{ber}}{\|P\|_{ber}}|P^*|^2 + \frac{\|P\|_{ber}}{\|R\|_{ber}}|R|^2\right\|_{ber}\right). \\
&\le \frac{\|S\|}{2}\left(ber(PR) + \frac{1}{2}(\|P\|_{ber}\|R\|_{ber} + \||P^*|R\|_{ber})\right) \\
&\text{(by Lemma 2.2)} \\
&\le \frac{\|S\|}{2}\left(ber(PR) + \frac{1}{2}(\|P\|_{ber}\|R\|_{ber} + \|RP\|_{ber})\right),
\end{aligned}$$

in where

$$\||P^*||R|\|_{ber}^2 = \|(|P^*||R|)^*(|P^*||R|)\|_{ber} = \||R||P^*|^2|R|\|_{ber} = \||R|PP^*|R|\|_{ber} = \|P^*|R|\|_{ber}^2$$

and

$$\|P^*|R|\|_{ber}^2 = \|P^*|R||R|P\|_{ber} = \|P^*R^*RP\|_{ber} = \||RP|\|_{ber}^2 = \|RP\|_{ber}^2.$$

This completes the proof.

Corollary ***3.4.*** *Let $P, R, S \in \mathcal{L}(\mathcal{H})$ and S be positive. Then*

(i) $ber(PSR) \leq \frac{\|S\|}{2}\left(ber(PR) + \frac{1}{2}\||P^*|^2 + |R|^2\|_{ber}\right)$

(ii) $ber(PSR) \leq \frac{\|S\|}{2}\left(ber(PR) + \frac{1}{2}(\|P\|_{ber}\|R\|_{ber} + \|PR\|_{ber})\right).$

Corollary ***3.5.*** *Let $P, R, S \in \mathcal{L}(\mathcal{H})$ and S be positive. Then*

$$ber(PSR) \leq \frac{\|S\|}{2}(ber(PR) + \|R\|_{ber}\|P^*\|_{ber}).$$

Proof Taking the supremum over $\rho \in X$ in inequality (19), we have desired inequality.

Corollary ***3.6.*** *Let $P, R \in \mathcal{L}(\mathcal{H})$. Then*

$$ber(PR) \leq \frac{1}{2}\||P^*|^2 + |R|^2\|_{ber}. \tag{20}$$

Proof. When we write $S = 1$ in inequality (17), we have

$$ber(PR) \leq \frac{1}{2}\left(ber(PR) + \frac{1}{2}\||P^*|^2 + |R|^2\|_{ber}\right).$$

Here, we get

$$ber(PR) \leq \frac{1}{2}\||P^*|^2 + |R|^2\|_{ber}.$$

Remark 3.7. *If we replace P by P^* in inequality (20), we have*

$$ber(P^*R) \leq \frac{1}{2}\||P|^2 + |R|^2\|_{ber}.$$

(see, (Huban al et., 2022b)).

Corollary *3.8. Let $N, M \in \mathcal{L}(\mathcal{H})$. Then*

$$ber(NM) \leq \frac{\|N\|^{\frac{1}{2}}}{2} \||N^*| + |M|^2\|_{ber}$$

and

$$ber(NM) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(\frac{1}{2}\||N^*| + |M|^2\|_{ber} + \frac{1}{2}\|M\|_{ber}\|N\|_{ber}^{\frac{1}{2}}\right).$$

Proof. Let $N = U|N|$ be the polar decomposition of N. $P = U|N|^{\frac{1}{2}}$, $S = |N|^{\frac{1}{2}}$ and $R = M$, in Theorem 3.4, we have

$$ber(NM) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(ber\left(U|N|^{\frac{1}{2}}M\right) + \frac{1}{2}\||N^*| + |M|^2\|_{ber}\right) \tag{21}$$

and

$$\begin{aligned} ber(NM) &\leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(ber\left(U|N|^{\frac{1}{2}}M\right) + \frac{1}{2}\left(\|U|N|\|_{ber}\|M\|_{ber} + \left\|MU|N|^{\frac{1}{2}}\right\|_{ber}\right)\right) \\ &\leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(ber\left(U|N|^{\frac{1}{2}}M\right) + \frac{1}{2}\left(\|N\|_{ber}^{\frac{1}{2}} + \|M\|_{ber}\right)\right) \end{aligned} \tag{22}$$

In addition, we get

$$\begin{aligned} \left|\langle U|N|^{\frac{1}{2}}M\hat{k}_\rho, \hat{k}_\rho\rangle\right| &= \left|\langle M\hat{k}_\rho, |N|^{\frac{1}{2}}U\hat{k}_\rho\rangle\right| \\ &\leq \left\|M\hat{k}_\rho\right\| \left\||N|^{\frac{1}{2}}U\hat{k}_\rho\right\| \end{aligned} \tag{23}$$

(by Cauchy-Schwarz inequality)

$$\leq \sqrt{\langle |M|^2\hat{k}_\rho, \hat{k}_\rho\rangle\langle U|N|U^*\hat{k}_\rho, \hat{k}_\rho\rangle}$$

$$\leq \frac{\langle |M|^2\hat{k}_\rho, \hat{k}_\rho\rangle + \langle U|N|U^*\hat{k}_\rho, \hat{k}_\rho\rangle}{2}$$

(by arithmetic-geometric mean inequality)

$$\leq \frac{1}{2}\langle (|M|^2 + |N^*|)\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Here, we reach

$$\left|\langle U|\,N|^{\frac{1}{2}}M\hat{k}_\rho,\hat{k}_\rho\rangle\right| \leq \frac{1}{2}\langle(|M|^2+|N^*|)\hat{k}_\rho,\hat{k}_\rho\rangle.$$

Using the inequality above, by calculating the supremum over $\rho \in X$, we get

$$ber(NM) \leq \frac{1}{2}\||N^*|+|M|^2\|_{ber}.$$

We now come together inequality (21) and inequality (22).to present the desired outcome.

Corollary ***3.9.*** *Let* $N, M \in \mathcal{L}(\mathcal{H})$*. Then*

$$ber\left(U|\,N|^{\frac{1}{2}}M\right) \leq \|M\|_{Ber}\left\||\,N|^{\frac{1}{2}}U\right\|_{Ber} \leq \|M\|_{Ber}\left\||\,N|^{\frac{1}{2}}U\right\|_{Ber} \leq \|M\|_{Ber}\|N\|_{Ber}.$$

Proof. From inequality (23), we have

$$\left|\langle U|\,N|^{\frac{1}{2}}M\hat{k}_\rho,\hat{k}_\rho\rangle\right| \leq \left\|M\hat{k}_\rho\right\|\left\||\,N|^{\frac{1}{2}}U\hat{k}_\rho\right\|.$$

Given the inequality above, by calculating the supremum over $\rho \in X$, we get

$$ber\left(U|\,N|^{\frac{1}{2}}M\right) \leq \|M\|_{Ber}\left\||\,N|^{\frac{1}{2}}U\right\|_{Ber}$$

(by $|U|^2 \leq 1$)

$$\leq \|M\|_{Ber}\left\||\,N|^{\frac{1}{2}}U\right\|_{Ber}$$

$$\leq \|M\|_{Ber}\|N\|_{Ber}.$$

Corollary ***3.10.*** *Let* $N \in \mathcal{L}(\mathcal{H})$ *with the polar decomposition* $N = U|\,N|$*. Then for any* $0 \leq r \leq 1$*,*

$$ber(N) \leq \frac{\|N\|^{1-r}}{2}\left(ber(U|\,N|^r)+\|N\|_{ber}^r\right). \tag{24}$$

Proof. Let $\hat{k}_\rho$ be a normalized reproducing kernel. If we write $x = y = \hat{k}_\rho$ in the inequality (11), then we have

$$|\langle N\hat{k}_\rho,\hat{k}_\rho\rangle \leq| \frac{\|N\|^{1-r}}{2}\left(|\langle U|\,N|^r\hat{k}_\rho,\hat{k}_\rho\rangle| + \sqrt{\langle|N|^r\hat{k}_\rho,\hat{k}_\rho\rangle\langle|N^*|^r\hat{k}_\rho,\hat{k}_\rho\rangle}\right).$$

In the inequality above, by computing the supremum over $\rho \in X$, we have

$$ber(N) \leq \frac{\|N\|^{1-r}}{2}\left(ber(U|\,N|^r) + \sqrt{\||N|^r\|_{ber}\||N^*|^r\|_{ber}}\right)$$
$$\leq \frac{\|N\|^{1-r}}{2}\left(ber(U|\,N|^r) + \sqrt{\|N\|_{ber}^r\|N\|_{ber}^r}\right)$$
$$\text{(by } \||N|\|_{ber} = \||N^*|\|_{ber} = \|N\|_{ber}\text{)}$$
$$\leq \frac{\|N\|^{1-r}}{2}\left(ber(U|\,N|^r) + \|N\|_{ber}^r\right).$$

Thus, we obtain

$$ber(N) \leq \frac{\|N\|^{1-r}}{2}\left(ber(U|\,N|^r) + \|N\|_{ber}^r\right).$$

Remark 3.11. *Let $N \in \mathcal{L}(\mathcal{H})$ with the polar decomposition $N = U|\,N|$. When we write $r = \frac{1}{2}$ in inequality (24), it has been demonstrated by Başaran and Gürdal (2023) with the following inequality*

$$ber(N) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(ber\left(U|\,N|^{\frac{1}{2}}\right) + \|N\|_{ber}^{\frac{1}{2}}\right). \tag{25}$$

Corollary ***3.12.*** *Let $N \in \mathcal{L}(\mathcal{H})$. Then for any $0 \leq r \leq 1$,*

$$ber(N) \leq \frac{\|N\|^{1-r}}{2}\||N|^r + |N^*|^r\|_{ber}. \tag{26}$$

In particular

$$ber(N) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left\||N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right\|_{ber}.$$

Proof. Suppose $\hat{k}_\rho$ be a reproducing kernel that has been normalized. In the inequality (9), if we choose $f(t) = g(t) = t^{\frac{r}{2}}$ and $x = y = \hat{k}_\rho$, then we reach

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \sqrt{\langle |N^*|^{\frac{r}{2}}|O|^{\frac{1}{2}}|N^*|^{\frac{r}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle\langle |N|^{\frac{r}{2}}|O^*|^{\frac{1}{2}}|N|^{\frac{r}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle}$$
$$\leq \sqrt{\langle |N^*|^{\frac{r}{2}}|N^*|^{\frac{r}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle\langle |N|^{\frac{r}{2}}|N|^{\frac{r}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle}$$

$$\leq \sqrt{\langle |N|^r \hat{k}_\rho, \hat{k}_\rho\rangle \langle |N^*|^r \hat{k}_\rho, \hat{k}_\rho\rangle}.$$

Here, we get

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \sqrt{\langle |N|^r \hat{k}_\rho, \hat{k}_\rho\rangle \langle |N^*|^r \hat{k}_\rho, \hat{k}_\rho\rangle}$$

From the above inequality, if $N = U|\,N|$ is the polar decomposition of $N \in \mathcal{L}(\mathcal{H})$, then we get

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| = \left|\langle U|\,N|\hat{k}_\rho, \hat{k}_\rho\rangle\right|$$
$$\leq \sqrt{\langle |N|^r \hat{k}_\rho, \hat{k}_\rho\rangle \langle |N^*|^r \hat{k}_\rho, \hat{k}_\rho\rangle}$$
$$\leq \frac{\langle |N|^r \hat{k}_\rho, \hat{k}_\rho\rangle + \langle |N^*|^r \hat{k}_\rho, \hat{k}_\rho\rangle}{2}$$

(by arithmetic-geometric mean inequality)

$$\leq \frac{1}{2}\langle (|N|^r + |N^*|^r)\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Here, we obtain

$$\left|\langle U|\,N|\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \frac{1}{2}\langle (|N|^r + |N^*|^r)\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Given the inequality shown above, by computing the supremum $\rho \in X$, We obtain

$$ber(N) \leq \frac{1}{2}\| |N|^r + |N^*|^r \|_{ber}.$$

Given that we combine this with inequlity (15), we find the desired result. Since $|N|^{\frac{1}{2}} \leq \|N\|_{ber}^{\frac{1}{2}} I$, we get $|N|^{\frac{1}{2}} \leq \|N\|^{\frac{1}{2}} |N|^{\frac{1}{2}}$. Similarly, $|N^*|^{\frac{1}{2}} \leq \|N\|^{\frac{1}{2}} |N^*|^{\frac{1}{2}}$. Hence,

$$\langle |N|\hat{k}_\rho, \hat{k}_\rho\rangle \leq \|N\|^{\frac{1}{2}} \langle |N|^{\frac{1}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle \text{ and } \langle |N^*|\hat{k}_\rho, \hat{k}_\rho\rangle \leq \|N\|^{\frac{1}{2}} \langle |N^*|^{\frac{1}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Here, we have

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \sqrt{\langle |N|\hat{k}_\rho, \hat{k}_\rho\rangle \langle |N^*|\hat{k}_\rho, \hat{k}_\rho\rangle}$$

(by mixed Cauchy-Schwarz inequality of Kato (1952))

$$\leq \sqrt{\langle |N|^{\frac{1}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle \langle |N^*|^{\frac{1}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle}$$

$$\leq \|N\|^{\frac{1}{2}}\sqrt{\langle |N|^{\frac{1}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle \langle |N^*|^{\frac{1}{2}}\hat{k}_\rho, \hat{k}_\rho\rangle}$$

$$\leq \|N\|^{\frac{1}{2}} \frac{\langle |N|^{\frac{1}{2}}\hat{k}_\rho,\hat{k}_\rho\rangle + \langle |N^*|^{\frac{1}{2}}\hat{k}_\rho,\hat{k}_\rho\rangle}{2}$$

$$\leq \|N\|^{\frac{1}{2}} \frac{1}{2}\langle\left(|N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right)\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Thus, we get

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \|N\|^{\frac{1}{2}} \frac{1}{2}\langle\left(|N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right)\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Using the inequality shown above, by computing the supremum $\rho \in X$, We are arriving at

$$ber(N) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left\||N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right\|_{ber}.$$

Remark 3.13. *Let $N \in \mathcal{L}(\mathcal{H})$ with the polar decomposition $N = U|N|$. When we write $r = \frac{1}{2}$ in inequality (24), it has been demonstrated by Başaran and Gürdal (2023) with the following inequality*

$$ber(N) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(\left\||N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right\|_{ber}\right).$$

The next statement has been given by Theorem 3.1 in the case $T = \frac{1}{2}$.

Corollary ***3.14.*** *Let $N \in \mathcal{L}(\mathcal{H})$ with the polar decomposition $N = U|N|$. Then*

$$ber(N) \leq \frac{\|N\|^{\frac{1}{2}}}{2}\left(ber\left(U|N|^{\frac{1}{2}}\right) + \left\||N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right\|_{ber}\right). \tag{27}$$

The inequality (25) is refined by the inequality (27). Because $\left\||N|^{\frac{1}{2}} + |N^*|^{\frac{1}{2}}\right\|_{ber} \leq 2\|N\|_{ber}^{\frac{1}{2}}$.

Theorem 3.15. Assume $\mathcal{H} = \mathcal{H}(\mathrm{X})$ to be an RKHS. Suppose that $\mathrm{N} \in \mathcal{L}(\mathcal{H})$. If f, g are nonnegative continuous function on $[0, \infty)$ obtaining $\mathrm{f(t)g(t) = t}$ $(\mathrm{t} > 0)$, then

$$ber(N) \leq \frac{1}{2}\left\| g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|) + f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\right\|_{ber}$$

for some contraction O.

Proof From the inequality (9), we get

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \sqrt{\langle g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|)\hat{k}_\rho, \hat{k}_\rho\rangle \langle f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\hat{k}_\rho, \hat{k}_\rho\rangle}$$

$$\leq \frac{\langle g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|)\hat{k}_\rho, \hat{k}_\rho\rangle + \langle f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\hat{k}_\rho, \hat{k}_\rho\rangle}{2}$$

(by arithmetic-geometric mean inequality)

$$\leq \frac{1}{2}\langle\left[g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|) + f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\right]\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Thus, we have

$$\left|\langle N\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \frac{1}{2}\langle\left[g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|) + f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\right]\hat{k}_\rho, \hat{k}_\rho\rangle.$$

Taking the supremum over $\rho \in X$ in the above inequality, we have

$$ber(N) \leq \frac{1}{2}\left\| g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|) + f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\right\|_{ber}.$$

Recall that $|O|^{\frac{1}{2}} \leq \left\| |O|^{\frac{1}{2}}\right\|_{ber} \leq I$. Hence

$$g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|) \leq g^2(|N^*|). \tag{28}$$

Similarly, we get

$$f(|N|)|O^*|^{\frac{1}{2}}f(|N|) \leq f^2(|N|). \tag{29}$$

There, we takes utilization of the fact that O^* becomes a contraction also if $|N|$ becomes contraction operator. The combination of inequality (28) with inequality (29), The next are the results we obtain.

Corollary ***3.16.*** *Let* $N \in \mathcal{L}(\mathcal{H})$. If f, g are nonnegative continuous function on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t > 0)$, then

$$ber(N) \leq \frac{1}{2}\left\| g(|N^*|)|O|^{\frac{1}{2}}g(|N^*|) + f(|N|)|O^*|^{\frac{1}{2}}f(|N|)\right\|_{ber} \tag{30}$$
$$\leq \frac{1}{2}\|g^2(|N^*|) + f^2(|N|)\|_{ber}.$$

for some contraction O.

Remark 3.17. The inequality (30) provides a improvement of (Huban et al., 2022a).

Remark 3.18. *If,* $N, M \in \mathcal{L}(\mathcal{H})$*, then*

$$ber(N) \leq \frac{1}{2}\left\| |N||O|^{\frac{1}{2}}|N| + |M||O^*|^{\frac{1}{2}}|M|\right\|_{ber}$$
$$\leq \frac{1}{2}\||M|^2 + |N|^2\|_{ber}$$

for some contraction O. The above inequality from Lemma 2.7 and positivity $\begin{bmatrix} |N|^2 & N^*M \\ M^*N & |M|^2 \end{bmatrix}$. Recall that the above result refines (Huban et al, 2022b).

Now, we prove the following theorem by using The Lemma 2.8.

Theorem 3.19. Assume $\mathcal{H} = \mathcal{H}(X)$ to be an RKHS. Suppose that $N \in \mathcal{L}(\mathcal{H})$. If f, g are nonnegative continuous function on $[0,\infty)$ obtaining $f(t)g(t) = t$ $(t > 0)$, then

$$ber(N) \leq \frac{1}{2}\sqrt{\|f^2(|N|) + f^2(|N^*|)\|_{ber}\|g^2(|N|) + g^2(|N^*|)\|_{ber}} \tag{31}$$

and

$$ber(N) \leq \frac{1}{4}\|f^2(|N|) + f^2(|N^*|) + g^2(|N|) + g^2(|N^*|)\|_{ber}.$$

Proof. Depending on Lemma 8(i), we have

$$|\langle \Re N x, y\rangle| \leq \frac{1}{2}\sqrt{\langle\big(f^2(|N|) + f^2(|N^*|)\big)x, x\rangle\langle\big(g^2(|N|) + g^2(|N^*|)\big)y, y\rangle}$$

Putting $x = y = \hat{k}_\rho$ and $N = e^{i\theta}N$ in the above inequality, we reach

$$\left|\langle \Re(e^{i\theta}N)\hat{k}_\rho, \hat{k}_\rho\rangle\right| \leq \frac{1}{2}\sqrt{\langle (f^2(|N|) + f^2(|N^*|))\hat{k}_\rho \hat{k}_\rho\rangle \langle (g^2(|N|) + g^2(|N^*|))\hat{k}_\rho, \hat{k}_\rho\rangle}.$$

In the inequality above, by computing the supremum over $\rho \in X$, we have

$$\left\|\Re(e^{i\theta}N)\right\| \leq \frac{1}{2}\sqrt{\|f^2(|N|) + f^2(|N^*|)\|_{ber}\|g^2(|N|) + g^2(|N^*|)\|_{ber}}.$$

If we take supremum over $\theta \in \mathbb{R}$ and use the fact that $ber(N) = sup_{\theta\in\mathbb{R}}\left\|\Re(e^{i\theta}N)\right\|$, we reach

$$ber(N) \leq \frac{1}{2}\sqrt{\|f^2(|N|) + f^2(|N^*|)\|_{ber}\|g^2(|N|) + g^2(|N^*|)\|_{ber}}$$

as desired.

Let's prove the second inequality. If we use utilize the arithmetic-geometric mean inequality for norm inequality (31), we have

$$ber(N) \leq \frac{1}{4}\|f^2(|N|) + f^2(|N^*|) + g^2(|N|) + g^2(|N^*|)\|_{ber}.$$

This completes the proof.

Remark 3.20. Among the celebrated upper bounds for $\mathrm{ber}(\mathrm{N})$, we get Huban et al. (2022a)

$$ber(N) \leq \tfrac{1}{2}\||N| + |N^*|\|_{ber}. \tag{32}$$

In Theorem 3.19, if we $\mathrm{f(t)} = \mathrm{g(t)} = t^{\frac{1}{2}}$, then the inequality (31) follows as a particular case of Theorem 3.19. Thus, this last theorem presents a generalizations of (31).
Başaran (2024), Başaran et al. (2019), Başaran and Gürdal (2023), Başaran et al. (2022), Gürdal et al. (2025), Gürdal and Başaran (2023), Gürdal et al. (2024), Garayev et al. (2021), and Huban et al. (2021) contain the fundamental characteristics and background knowledge on these creative concepts.

CONFLICT OF INTEREST

The author declares no conflict of interest.

ACKNOWLEDGMENT

I would like to thank the referees who contributed to the preparation of this article.

REFERENCES

Abu-Omar, A., & Kittaneh F. (2015). Upper and lower bounds for the numerical radius with an application to involution operators. *The Rocky Mountain Journal of Mathematics*, 45(4), 1055-065. https://doi.org/10.1216/RMJ-2015-45-4-1055

Aronzajn, N. (1950). Theory of reproducing kernels. *Transactions of the American Mathematical Society*, 68, 337-404.

Başaran H. (2024). Berezin radius inequalities related to the contraction operator. *Montes Taurus Journal of Pure and Applied Mathematics,* 6 (1), 102-109. Article ID:MTJPAM-D-23-00044.

Başaran, H., Gürdal, M., & Güncan, A. N. (2019). Some operator inequalities associated with Kantorovich and Hö lder-McCarthy inequalities and their applications. *Turkish Journal of Mathematics*, 43(1) (2019), 523-532. http://doi.org/10.3906/mat-1811-10

Başaran, H., & Gürdal, V. (2023). Berezin radius and Cauchy-Schwarz inequality. *Montes Taurus Journal of Pure and Applied Mathematics*, 5(3), 16-22. Article ID: MTJPAM-D-21-00068

Başaran H., Huban M. B., Gürdal M. (2022). Inequalities related to Berezin norm and Berezin number of operators. *Bulletin of Mathematical Analysis and Applications*, 14(2), 1-11. https://doi.org/10.54671/bmaa-2022-2-1

Berezin, F.A. (1972). Covariant and contravariant symbols for operators. *Mathematics of the USSR-Izvestiya,* 6(5), 1117-1151. http://dx.doi.org/10.1070/IM1972v006n05ABEH001913

Bhunia, P., Dragomir, S. S., Moslehian, M. S., & Paul, K. (2022). *Lectures on Numerical Radius Inequalities*. Springer Cham.

Bhatia, R. (2007). *Positive define matrices*. Princeton, NJ: Princeton University Press.

Bhunia, P., Jana, S., & Paul, K. (2023a). Numerical radius inequalities and estimation of zeros of polynomials. Georgian Mathematical Journal, 30(5), 671-682. https://doi.org/10.1515/gmj-2023-2037

Bhunia, P., Gürdal, M., Paul, K., Sen, A., & Tapdigoglu, R. (2023b). On a new norm on the space of reproducing kernel Hilbert space operators and Berezin radius inequalities. *Numerical Functional Analysis and Optimization*, 44(9):970-986. https://doi.org/10.1080/01630563.2023.2221857

Buzano, M. L. (1974). *Generalizzatione della disuguaglianza di Cauchy-Schwarz*. Rendiconti del Seminario Matematico. Universit_a e Politecnico Torino, 31 (1971/73), 405-409.

Dragomir, S. S. (2016). Buzano's inequality holds for any projection. *Bulletin of the Australian Mathematical* Society, 3, 504-510. https://doi.org/10.1017/S0004972715001525

Engliš, M. (1995). Toeplitz operators and the Berezin transform on H^2. *Linear Algebra and Applications*, 223-224, 171-204. https://doi.org/10.1016/0024-3795(94)00056-J

Furuta, T. (2001). *Invilation to linear operators*, London: Taylor and Francis.

Garayev, M. T., Guedri, H., Gürdal M., & Alsahli, G. M. (2021). On some problems for operators on the reproducing kernel Hilbert space. *Linear and Multilinear Algebra*, 69(11), 2059-2077. https://doi.org/10.1080/03081087.2019.1659220

Gürdal, M., Alomari, M.W., Başaran, H. (2025). Berezin radius inequalities via Orlicz function. *Palestine Journal of Mathematics*, 14(1), 928-945.

Gürdal, M., Erkan, G. G., & Garayev, M. (2024). Berezin norm and Berezin radius inequalities of product and sums with Selberg inequality. *Proceedings of the Institute of Mathematics and Mechanics*, 50(2), 258-273. https://doi.org/10.30546/2409-4994.2024.50.2.258

Gürdal, V., & Başaran H. (2023). On Berezin radius inequalities via Cauchy-Schwarz type inequalities. *Malaya Journal of Matematik*, 11(02) (2023), 127-141. https://doi.org/10.26637/mjm1102/002

Huban, M.B., Başaran, H. & Gürdal, M. (2021). New upper bounds related to the Berezin number inequalities. *Journal of Inequalities and Special Functions*, 12(3), 1-12.

Huban, M.B., Başaran, H. & Gürdal, M. (2022a). Some new inequalities via Berezin numbers. *Turkish Journal of Mathematics and Computer Science*, 14(1), 129-137. https://doi.org/10.47000/tjmcs.1014841

Huban, M.B., Başaran, H. & Gürdal, M. (2022b). Berezin number inequalities via convex functions. *Filomat*, 36(7), 2333-2344. https://doi.org/10.2298/FIL2207333H

Jorgensen, P. (2006). *Analysis and probability: wavelets, signals, fractals*, Spinger.

Karaev, M.T. (2006). Berezin symbol and invertibility of operators on the functional Hilbert spaces. *Journal of Functional Analysis*, 238, 181-192. https://doi.org/10.1016/j.jfa.2006.04.030

Karaev, M.T. (2013). Reproducing kernels and Berezin symbols techniques in various questions of operator theory. *Complex Analysis and Operator Theory*, 7, 983-1018. https://doi.org/10.1007/s11785-012-0232-z

Karaev, M.T., Gürdal, M., & Huban, M.B. (2016). Reproducing kernels, Englis algebras and some applications. *Studia Mathematica*, 232(2), 113-141. https://doi.org/10.4064/sm8385-4-2016

Karaev, M., Gürdal, M., & Saltan, S. (2011). Some applications of Banach algebra techniques. *Mathematische Nachrichten*, 284(13):1678-1689. https://doi.org/10.1002/mana.200910129

Kato, T. (1952). Notes on some inequalities for linear operators. *Mathematische Annalen,* 125, 208-212. https://doi.org/10.1007/BF01343117

Kittaneh, F. (1988). Notes on some inequalities for Hilbert space operators. *Publications of the Research Institute for Mathematical Sciences*, 24, 283-293. https://doi.org/10.2977/PRIMS/1195175202

Kittaneh, F. (1997). Norm inequalities for certain operators sums. *Journal of Functional Analysis,* 143, 337-348. https://doi.org/10.1006/jfan.1996.2957

Kittaneh, F., Moslehian, M. S., & Yamakazi, T. (2015). Caretesian decomposition and numerical radius inequalities. *Linear Algebra and Applications*, 471, 46-53. https://doi.org/10.1016/j.laa.2014.12.016

Pečarić, J., Furuta, T., Hot, M.H., & Seo, Y. (2005). *Mond-Pečarić method in operator inequalities*. Zagreb; Elements.

Sababheh, M., Conde, C., & Moradi, H. R. (2023). On the Cauchy-Schwarz inequality. *Operators and Matrisces*, 17(2) (2023), 525-538. https://doi.org/10.7153/oam-2023-17-34

Sababheh, M., & Moradi, H. R. (2023). Numerical radius of kronecker product of matrices. *Journal of Applied Analysis & Computation,* 16(3), 2943-2954. https://www.jaac-online.com/article/doi/10.11948/20230064

Sababheh, M., Moradi, H. R., & Sahoo, S. (2024). Inner product inequalities with application. *Linear and Multilinear Algebra*, (2024), 1-14. https://doi.org/10.1080/03081087.2024.2312444

Stojiljkovic, V., & Gürdal, M. (2025). Estimates of the numerical radius utilizing various function properties. *Turkish Journal of Mathematics*, 49(1), 49-64. https://doi.org/10.55730/1300-0098.3573

Yamakazi, T. (2007). On upper and lower bounds of the numerical radius and an equality condition. *Studia Mathematica*, 178, 83-89. https://doi.org/10.4064/sm178-1-5

Yamancı, U., Tunç, R., & Gürdal, M. (2020). Berezin number, Grüss-type inequalities and their applications. *Bulletin of the Malaysian Mathematical Sciences Society*, 43(3), 2287-2296. https://doi.org/10.1007/s40840-019-00804-x

Yang, C., & Xu, M. (2023). Some new numerical radius and Hilbert-Schmitd numerical radius inequalities for Hilbert space operators. *Journal of Mathematical Inequalities*, 17(1), 269-282. https://doi.org/10.7153/jmi-2023-17-19

Zhu, K., (1990). *Operator Theory in Function Spaces*, Marcel Dekker Inc., New York, USA Series: Pure and Applied Mathematicsa Series of Monographs and Textbooks, Volume 139.