Mathematical Sciences And Applications E-Notes
Volume 3 No. 2 pp. 1–19 (2015) ©MSAEN

SOME HERMITE-HADAMARD AND SIMPSON LIKE INEQUALITIES FOR s-GEOMETRICALLY CONVEX FUNCTION

MEVLÜT TUNÇ*, EBRU YÜKSEL**

(Communicated by Nihal YILMAZ ÖZGÜR)

Abstract. In the paper, the authors establish and generalize some new integral inequalities of Hermite-Hadamard and Simpson type for functions the power of the absolute of whose first derivative is s-geometrically convex.

1. Introduction

In this section, we will present definitions and some known results used in this paper.

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(x\right)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2} \tag{1.2}$$

holds. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions, see the papers and books [2]-[4], [6]-[10], [13]-[16].

The Simpson inequality proposes that the mapping $f : [a, b] \to \mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a, b), that is,

$$\left|\frac{1}{6}\left[f\left(a\right)+f\left(b\right)+4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq \frac{1}{1280}\left\|f^{(4)}\right\|_\infty (b-a)^4, \tag{1.3}$$

where $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(4)}\left(t\right)\right| < \infty$. For some results which generalize, improve and extend the inequality (1.3), see the papers [1], [5], [11], [12].

Date: Received: January 6, 2015; Accepted: September 29, 2015.

2010 *Mathematics Subject Classification.* 26A15, 26A16, 26A33, 26A51, 26D10.

Key words and phrases. Convex function; Hermite-Hadamard integral inequality; Simpson integral inequality; s-geometrically convex function.

*Corresponding author.

Definition 1.1. [8] *Let $s \in (0,1]$. A function $f : I \subset \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex in the second sense if*

$$f\left(tx + (1-t)y\right) \le t^s f(x) + (1-t)^s f(y) \tag{1.4}$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily checked for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Recently, in [15], the concept of geometrically and s-geometrically convex functions was introduced as follows.

Definition 1.2. [15] A function $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^t y^{1-t}\right) \le [f(x)]^t [f(y)]^{1-t} \tag{1.5}$$

for *all $x, y \in I$ and $t \in [0,1]$.*

Definition 1.3. [15] A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a s-geometrically convex function if

$$f\left(x^t y^{1-t}\right) \le [f(x)]^{t^s} [f(y)]^{(1-t)^s} \tag{1.6}$$

for some $s \in (0,1]$, where $x, y \in I$ *and* $t \in [0,1]$.

If $s = 1$ in (1.6), the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

Example 1.1. [15] Let $f(x) = x^s/s$, $x \in (0,1]$, $0 < s < 1$, $q \ge 1$, and then the function

$$|f'(x)|^q = x^{(s-1)q} \tag{1.7}$$

is monotonically decreasing on $(0,1]$. For $t \in [0,1]$, we have

$$(s-1)\,q\,(t^s - t) \le 0, \quad (s-1)\,q\left((1-t)^s - (1-t)\right) \le 0. \tag{1.8}$$

Hence, $|f'(x)|^q$ is s-geometrically convex on $(0,1]$ for $0 < s < 1$.

In recently [16], Zhang *et al.* proved some Hermite-Hadamard type inequalities for s-geometrically convex functions as followings:

Theorem 1.1. [16] *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable on I° such that $f' \in L([a,b])$ for $0 < a < b < \infty$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $q \ge 1$ and $s \in (0,1]$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q} G_1\left(s,q;g_1(\alpha),g_2(\alpha)\right) \tag{1.9}$$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q} G_1\left(s,q;g_2(\alpha),g_1(\alpha)\right) \tag{1.10}$$

where

$$g_1(\alpha) = \begin{cases} \frac{1}{2}, & \alpha = 1, \\ \frac{\alpha\ln\alpha - \alpha + 1}{[\ln\alpha]^2}, & \alpha \ne 1, \end{cases} \qquad g_2(\alpha) = \begin{cases} \frac{1}{2}, & \alpha = 1, \\ \frac{\alpha - \ln\alpha - 1}{[\ln\alpha]^2}, & \alpha \ne 1, \end{cases} \tag{1.11}$$

$$\alpha = \left|\frac{f'(b)}{f'(a)}\right|^{sq/2}, \tag{1.12}$$

and

$$\begin{aligned} &G_1(s,q;g_1(\alpha),g_2(\alpha)) \\ = &\left\{ \begin{array}{cc} |f'(a)|^s [g_1(\alpha)]^{1/q} + |f'(a) f'(b)|^{s/2} [g_2(\alpha)]^{1/q}, & |f'(a)| \le 1, \\ |f'(a)| [g_1(\alpha)]^{1/q} + |f'(a)|^{1-s/2} |f'(b)|^{s/2} [g_2(\alpha)]^{1/q}, & |f'(b)| \le 1 \le |f'(a)|, \\ |f'(a)| |f'(b)|^{1-s} [g_1(\alpha)]^{1/q} + |f'(a) f'(b)|^{1-s/2} [g_2(\alpha)]^{1/q}, & 1 \le |f'(b)|. \end{array} \right. \end{aligned}$$

Theorem 1.2. [16] *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable on I° such that $f' \in L([a,b])$ for $0 < a < b < \infty$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $q > 1$ and $s \in (0,1]$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(s,q;g_3(\alpha)) \tag{1.13}$$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(s,q;g_3(\alpha)) \tag{1.14}$$

where α is the same as in (1.12),

$$g_3(\alpha) = \left\{ \begin{array}{cc} 1, & \alpha = 1, \\ \frac{\alpha-1}{\ln\alpha}, & \alpha \ne 1, \end{array} \right.$$

and

$$G_2(s,q;g_3(\alpha)) = \left\{ \begin{array}{cc} \left[|f'(a)|^s + |f'(a) f'(b)|^{s/2}\right] [g_3(\alpha)]^{1/q}, & |f'(a)| \le 1, \\ \left[|f'(a)| + |f'(a)|^{1-s/2} |f'(b)|^{s/2}\right] [g_3(\alpha)]^{1/q}, & |f'(b)| \le 1 \le |f'(a)|, \\ \left(|f'(a)| |f'(b)|^{1-s} + |f'(a) f'(b)|^{1-s/2}\right) [g_3(\alpha)]^{1/q}, & 1 \le |f'(b)|. \end{array} \right.$$

2. Lemmas

In order to prove our main theorems, we need the following lemmas.

Lemma 2.1. [14] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I°, $a, b \in I$, with $a < b$. If $f' \in L[a,b]$ and $\lambda, \mu \in \mathbb{R}$ then*

$$\begin{aligned} &\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \\ = &\frac{b-a}{4}\int_0^1 \left[(1-\lambda-t) f'\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu - t) f'\left(t\frac{a+b}{2} + (1-t) b\right)\right] dt \end{aligned} \tag{2.1}$$

Lemma 2.2. [14] *For $x > 0$ and $0 \le y \le 1$, one has*

$$\begin{aligned} \int_0^1 |y-t|^x\,dt &= \frac{y^{x+1} + (1-y)^{x+1}}{x+1}, \\ \int_0^1 t\,|y-t|^x\,dt &= \frac{y^{x+2} + (x+1+y)(1-y)^{x+1}}{(x+1)(x+2)}. \end{aligned} \tag{2.2}$$

Lemma 2.3. [3] *If* $0 < \varphi \leq 1 \leq \mu$, $0 < \alpha, \beta \leq 1$, *then*

$$\varphi^{\alpha^{\beta}} \leq \varphi^{\alpha\beta} \text{ and } \mu^{\alpha^{\beta}} \leq \mu^{\beta\alpha+1-\beta}. \tag{2.3}$$

In this study, we will generalize the following theorems for s-geometrically convex functions by using the above three lemma and elementary analytical rules.

3. General inequalities for s-geometrically convex functions

Theorem 3.1. *Let* $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ *be differentiable on* I°, $a, b \in I$, *with* $a < b$, *and* $f' \in L([a,b])$. *If* $|f'(x)|$ *is* s*-geometrically convex and monotonically decreasing on* $[a,b]$, *and* $0 < s \leq 1$, $0 \leq \lambda, \mu \leq 1$, *then*

$$\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4} \times \tag{3.1}$$
$$\times \begin{cases} |f'(a) f'(b)|^{\frac{s}{2}} M_\lambda(s;Z) + |f'(b)|^s E_\mu(s;Z), & |f'(a)| \leq 1; \\ |f'(a)|^{1-\frac{s}{2}} |f'(b)|^{\frac{s}{2}} M_\lambda(s;Z) + |f'(a)|^{1-s} |f'(b)|^s E_\mu(s;Z), & |f'(b)| \leq 1 \leq |f'(a)|; \\ |f'(a) f'(b)|^{1-\frac{s}{2}} M_\lambda(s;Z) + |f'(a) f'(b)|^{1-\frac{s}{2}} E_\mu(s;Z), & 1 \leq |f'(b)|; \end{cases}$$

where

$$\begin{aligned} M_\lambda(s;Z) &= \frac{\left(-1+\lambda+\lambda Z\left(\frac{s}{2},\frac{s}{2}\right)\right)\ln Z\left(\frac{s}{2},\frac{s}{2}\right) - 1 + 2\left[Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^{1-\lambda} - Z\left(\frac{s}{2},\frac{s}{2}\right)}{\left[\ln Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^2} \\ E_\mu(s;Z) &= \frac{\left(-\mu-\mu Z\left(\frac{s}{2},\frac{s}{2}\right) + Z\left(\frac{s}{2},\frac{s}{2}\right)\right)\ln Z\left(\frac{s}{2},\frac{s}{2}\right) - 1 + 2\left[Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^{\mu} - Z\left(\frac{s}{2},\frac{s}{2}\right)}{\left[\ln Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^2} \end{aligned}$$

and $Z(u,v) = |f'(a)|^u |f'(b)|^{-v}$, $u, v > 0$.

Proof. From Lemma 2.1 and since $|f'(x)|$ is s-geometrically convex and monotonically decreasing on $[a,b]$, we get

$$\begin{aligned} &\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t| \left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right| dt + \int_0^1 |\mu - t| \left|f'\left(t\frac{a+b}{2} + (1-t)b\right)\right| dt\right] \\ =\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t| \left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt + \int_0^1 |\mu - t| \left|f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right| dt\right] \\ \leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t| \left|f'\left(a^{\frac{1+t}{2}} b^{\frac{1-t}{2}}\right)\right| dt + \int_0^1 |\mu - t| \left|f'\left(a^{\frac{t}{2}} b^{\frac{2-t}{2}}\right)\right| dt\right] \\ \leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t| |f'(a)|^{\left(\frac{1+t}{2}\right)^s} |f'(b)|^{\left(\frac{1-t}{2}\right)^s} dt + \int_0^1 |\mu - t| |f'(a)|^{\left(\frac{t}{2}\right)^s} |f'(b)|^{\left(\frac{2-t}{2}\right)^s} dt\right]. \end{aligned}$$

When $|f'(a)| \leq 1$, by using (2.3), we get

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\,|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}dt+\int_0^1|\mu-t|\,|f'(a)|^{\left(\frac{t}{2}\right)^s}|f'(b)|^{\left(\frac{2-t}{2}\right)^s}dt\right] \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\,|f'(a)|^{\frac{s+st}{2}}|f'(b)|^{\frac{s-st}{2}}dt+\int_0^1|\mu-t|\,|f'(a)|^{\frac{st}{2}}|f'(b)|^{\frac{2s-st}{2}}dt\right] \\
=\ & \frac{b-a}{4}\left[|f'(a)f'(b)|^{\frac{s}{2}}\int_0^1 |1-\lambda-t|\,|f'(a)/f'(b)|^{\frac{st}{2}}dt+|f'(b)|^s\int_0^1|\mu-t|\,|f'(a)/f'(b)|^{\frac{st}{2}}dt\right]
\end{aligned} \tag{3.2}$$

In a direct calculation yields,

$$\begin{aligned}
&\int_0^1 |1-\lambda-t|\,|f'(a)/f'(b)|^{\frac{st}{2}}dt \\
=\ & \frac{\left(\left(-1+\lambda+\lambda Z\left(\frac{s}{2},\frac{s}{2}\right)\right)\ln Z\left(\frac{s}{2},\frac{s}{2}\right)-1+2\left[Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^{1-\lambda}-Z\left(\frac{s}{2},\frac{s}{2}\right)\right)}{\left[\ln Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^2} \\
=\ & M_\lambda(s;Z)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |\mu-t|\,|f'(a)/f'(b)|^{\frac{st}{2}}dt \\
=\ & \frac{\left(\left(-\mu-\mu Z\left(\frac{s}{2},\frac{s}{2}\right)+Z\left(\frac{s}{2},\frac{s}{2}\right)\right)\ln Z\left(\frac{s}{2},\frac{s}{2}\right)-1+2\left[Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^{\mu}-Z\left(\frac{s}{2},\frac{s}{2}\right)\right)}{\left[\ln Z\left(\frac{s}{2},\frac{s}{2}\right)\right]^2} \\
=\ & E_\mu(s;Z)
\end{aligned}$$

If (3.2) is rewritten, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[|f'(a)f'(b)|^{\frac{s}{2}}M_\lambda(s;Z)+|f'(b)|^s E_\mu(s;Z)\right]
\end{aligned}$$

When $|f'(b)| \leq 1 \leq |f'(a)|$, from (2.3), we get

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\,|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}dt+\int_0^1|\mu-t|\,|f'(a)|^{\left(\frac{t}{2}\right)^s}|f'(b)|^{\left(\frac{2-t}{2}\right)^s}dt\right] \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\,|f'(a)|^{\frac{s+st}{2}+1-s}|f'(b)|^{\frac{s-st}{2}}dt+\int_0^1|\mu-t|\,|f'(a)|^{\frac{st}{2}+1-s}|f'(b)|^{\frac{2s-st}{2}}dt\right] \\
=\ & \frac{b-a}{4}\left[|f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}M_\lambda(s;Z)+|f'(a)|^{1-s}|f'(b)|^s E_\mu(s;Z)\right]
\end{aligned}$$

When $1 \leq |f'(b)|$, by virtue of (2.3), we get

$$\begin{aligned}
&\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\,|f'(a)|^{\left(\frac{1+t}{2}\right)^s} |f'(b)|^{\left(\frac{1-t}{2}\right)^s} dt \right. \\
&\left. + \int_0^1 |\mu-t|\,|f'(a)|^{\left(\frac{t}{2}\right)^s} |f'(b)|^{\left(\frac{2-t}{2}\right)^s} dt\right] \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |1-\lambda-t|\,|f'(a)|^{\frac{s+st}{2}+1-s} |f'(b)|^{\frac{s-st}{2}+1-s} dt \right. \\
&\left. + \int_0^1 |\mu-t|\,|f'(a)|^{\frac{st}{2}+1-s} |f'(b)|^{\frac{2s-st}{2}+1-s} dt\right] \\
=\ & \frac{b-a}{4}\left[|f'(a)f'(b)|^{1-\frac{s}{2}} M_\lambda(s;Z) + |f'(a)f'(b)|^{1-\frac{s}{2}} E_\mu(s;Z)\right].
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

If taking $\lambda = \mu$ in Theorem 3.1, we derive the following corollary.

Corollary 3.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|$ is s-geometrically convex and monotonically decreasing on $[a,b]$, and $0 < s \leq 1$, $0 \leq \lambda \leq 1$, then*

(3.3)

$$\begin{aligned}
&\left| \frac{\lambda[f(a)+f(b)]}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{b-a}{4} \\
&\times \begin{cases} |f'(a)f'(b)|^{\frac{s}{2}} M(\lambda,s;Z) + |f'(b)|^s E(\lambda,s;Z), & |f'(a)| \leq 1; \\ |f'(a)|^{1-\frac{s}{2}} |f'(b)|^{\frac{s}{2}} M(\lambda,s;Z) + |f'(a)|^{1-s} |f'(b)|^s E(\lambda,s;Z), & |f'(b)| \leq 1 \leq |f'(a)|; \\ |f'(a)f'(b)|^{1-\frac{s}{2}} M(\lambda,s;Z) + |f'(a)f'(b)|^{1-\frac{s}{2}} E(\lambda,s;Z), & 1 \leq |f'(b)|; \end{cases}
\end{aligned}$$

where $Z(u,v)$, $M_\lambda(s;Z)$, $E_\lambda(s;Z)$ are defined in Theorem 3.1.

If letting $\lambda = \mu = 1/2$, $2/3$, $1/3$, respectively, in Theorem 3.1, we can deduce the inequalities below.

Corollary 3.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|$ is s-geometrically convex and monotonically decreasing on $[a,b]$, and $0 < s \leq 1$, then*

(3.4)

$$\begin{aligned}
&\left| \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{b-a}{4} \\
&\times \begin{cases} |f'(a)f'(b)|^{\frac{s}{2}} M_{1/2}(s;Z) + |f'(b)|^s E_{1/2}(s;Z), & |f'(a)| \leq 1; \\ |f'(a)|^{1-\frac{s}{2}} |f'(b)|^{\frac{s}{2}} M_{1/2}(s;Z) + |f'(a)|^{1-s} |f'(b)|^s E_{1/2}(s;Z), & |f'(b)| \leq 1 \leq |f'(a)|; \\ |f'(a)f'(b)|^{1-\frac{s}{2}} M_{1/2}(s;Z) + |f'(a)f'(b)|^{1-\frac{s}{2}} E_{1/2}(s;Z), & 1 \leq |f'(b)|; \end{cases}
\end{aligned}$$

(3.5)
$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{4}$$
$$\times\begin{cases}|f'(a)f'(b)|^{\frac{s}{2}}M_{2/3}(s;Z)+|f'(b)|^{s}E_{2/3}(s;Z), & |f'(a)|\le 1;\\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}M_{2/3}(s;Z)+|f'(a)|^{1-s}|f'(b)|^{s}E_{2/3}(s;Z), & |f'(b)|\le 1\le|f'(a)|;\\ |f'(a)f'(b)|^{1-\frac{s}{2}}M_{2/3}(s;Z)+|f'(a)f'(b)|^{1-\frac{s}{2}}E_{2/3}(s;Z), & 1\le|f'(b)|;\end{cases}$$

(3.6)
$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{4}$$
$$\times\begin{cases}|f'(a)f'(b)|^{\frac{s}{2}}M_{1/3}(s;Z)+|f'(b)|^{s}E_{1/3}(s;Z), & |f'(a)|\le 1;\\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}M_{1/3}(s;Z)+|f'(a)|^{1-s}|f'(b)|^{s}E_{1/3}(s;Z), & |f'(b)|\le 1\le|f'(a)|;\\ |f'(a)f'(b)|^{1-\frac{s}{2}}M_{1/3}(s;Z)+|f'(a)f'(b)|^{1-\frac{s}{2}}E_{1/3}(s;Z), & 1\le|f'(b)|;\end{cases}$$

where $Z(u,v)$, $M_\lambda(s;Z)$, $E_\mu(s;Z)$ *are defined in Theorem 3.1.*

Theorem 3.2. *Let* $f: I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ *be differentiable on* I°, $a,b\in I$ *with* $a<b$, $0\le\lambda,\mu\le 1$, *and* f' *is integrable on* $[a,b]$. *If* $|f'(x)|^q$ *is s-geometrically convex and monotonically decreasing on* $[a,b]$, *for* $q\ge 1$ *and* $s\in(0,1]$, *then*

(3.7)
$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{4}\times$$
$$\begin{cases}\begin{array}{c}|f'(a)f'(b)|^{\frac{s}{2}}\left(\frac{(1-\lambda)^2+\lambda^2}{2}\right)^{1-\frac{1}{q}}M_\lambda^{1/q}(s,q;Z)\\ +|f'(a)f'(b)|^{s}\left(\frac{\mu^2+(1-\mu)^2}{2}\right)^{1-\frac{1}{q}}E_\mu^{1/q}(s,q;Z)\end{array}, & |f'(a)|\le 1;\\ \begin{array}{c}|f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}\left(\frac{(1-\lambda)^2+\lambda^2}{2}\right)^{1-\frac{1}{q}}M_\lambda^{1/q}(s,q;Z)\\ +|f'(a)|^{1-s}|f'(b)|^{s}\left(\frac{\mu^2+(1-\mu)^2}{2}\right)^{1-\frac{1}{q}}E_\mu^{1/q}(s,q;Z)\end{array}, & |f'(b)|\le 1\le|f'(a)|;\\ \begin{array}{c}|f'(a)f'(b)|^{1-\frac{s}{2}}\left(\frac{(1-\lambda)^2+\lambda^2}{2}\right)^{1-\frac{1}{q}}M_\lambda^{1/q}(s,q;Z)\\ +|f'(a)|^{1-s}|f'(b)|\left(\frac{\mu^2+(1-\mu)^2}{2}\right)^{1-\frac{1}{q}}E_\mu^{1/q}(s,q;Z)\end{array}, & 1\le|f'(b)|;\end{cases}$$

where

$$\begin{aligned}&M_\lambda(s,q;Z)\\ =\ &\frac{\left(-1+\lambda+\lambda Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1+2\left[Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^{1-\lambda}-Z\left(\frac{sq}{2},\frac{sq}{2}\right)}{\left[\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^2}\\ &E_\mu(s,q;Z)\\ =\ &\frac{\left(-\mu-\mu Z\left(\frac{sq}{2},\frac{sq}{2}\right)+Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1+2\left[Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^{\mu}-Z\left(\frac{sq}{2},\frac{sq}{2}\right)}{\left[\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^2}\end{aligned}$$

and $Z(u,v)=|f'(a)|^u|f'(b)|^{-v}$, $u,v>0$.

Proof. From Lemma 2.1 and since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, by using well known power mean inequality, we get

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{4}\left[\left(\int_0^1 |1-\lambda-t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-\lambda-t|\left(\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|\right)^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 |\mu-t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu-t|\left(\left|f'\left(a^{\frac{t}{2}}b^{\frac{2-t}{2}}\right)\right|\right)^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{b-a}{4}\left[\left(\frac{(1-\lambda)^2+\lambda^2}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-\lambda-t|\,|f'(a)|^{q\left(\frac{1+t}{2}\right)^s}|f'(b)|^{q\left(\frac{1-t}{2}\right)^s}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{\mu^2+(1-\mu)^2}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu-t|\,|f'(a)|^{q\left(\frac{t}{2}\right)^s}|f'(b)|^{q\left(\frac{2-t}{2}\right)^s}dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

When $|f'(a)|\leq 1$, by (2.3), we get

$$\begin{aligned}
&\int_0^1 |1-\lambda-t|\,|f'(a)|^{q\left(\frac{1+t}{2}\right)^s}|f'(b)|^{q\left(\frac{1-t}{2}\right)^s}dt \\
\leq\ & \int_0^1 |1-\lambda-t|\,|f'(a)|^{sq\left(\frac{1+t}{2}\right)}|f'(b)|^{sq\left(\frac{1-t}{2}\right)}dt \\
=\ & |f'(a)f'(b)|^{\frac{sq}{2}}\int_0^1 |1-\lambda-t|\,|f'(a)/f'(b)|^{\frac{sqt}{2}}dt \\
=\ & |f'(a)f'(b)|^{\frac{sq}{2}} \\
&\times\frac{\left(-1+\lambda+\lambda Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1+2\left[Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^{1-\lambda}-Z\left(\frac{sq}{2},\frac{sq}{2}\right)}{\left[\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^2} \\
=\ & |f'(a)f'(b)|^{\frac{sq}{2}}M_\lambda(s,q;Z)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |\mu-t|\,|f'(a)|^{q\left(\frac{t}{2}\right)^s}|f'(b)|^{q\left(\frac{2-t}{2}\right)^s}dt \\
\leq\ & \int_0^1 |\mu-t|\,|f'(a)|^{sq\left(\frac{t}{2}\right)}|f'(b)|^{sq\left(\frac{2-t}{2}\right)}dt \\
=\ & |f'(a)f'(b)|^{sq}\int_0^1 |\mu-t|\,|f'(a)/f'(b)|^{\frac{sqt}{2}}dt \\
=\ & |f'(a)f'(b)|^{sq} \\
&\times\frac{\left(-\mu-\mu Z\left(\frac{sq}{2},\frac{sq}{2}\right)+Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1+2\left[Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^{\mu}-Z\left(\frac{sq}{2},\frac{sq}{2}\right)}{\left[\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)\right]^2} \\
=\ & |f'(a)f'(b)|^{sq}E_\mu(s,q;Z).
\end{aligned}$$

When $|f'(b)| \leq 1 \leq |f'(a)|$, by (2.3), we get

$$\begin{aligned}
& \int_0^1 |1-\lambda-t| \, |f'(a)|^{q\left(\frac{1+t}{2}\right)^s} |f'(b)|^{q\left(\frac{1-t}{2}\right)^s} dt \\
\leq & \int_0^1 |1-\lambda-t| \, |f'(a)|^{q\left[s\left(\frac{1+t}{2}\right)+1-s\right]} |f'(b)|^{sq\left(\frac{1-t}{2}\right)} dt \\
= & |f'(a)|^{q\left(1-\frac{s}{2}\right)} |f'(b)|^{\frac{sq}{2}} \int_0^1 |1-\lambda-t| \, |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\
= & |f'(a)|^{q\left(1-\frac{s}{2}\right)} |f'(b)|^{\frac{sq}{2}} M_\lambda(s,q;Z)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 |\mu-t| \, |f'(a)|^{q\left(\frac{t}{2}\right)^s} |f'(b)|^{q\left(\frac{2-t}{2}\right)^s} dt \\
\leq & \int_0^1 |\mu-t| \, |f'(a)|^{q\left[s\left(\frac{t}{2}\right)+1-s\right]} |f'(b)|^{sq\left(\frac{2-t}{2}\right)} dt \\
= & |f'(a)|^{q(1-s)} |f'(b)|^{sq} \int_0^1 |\mu-t| \, |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\
= & |f'(a)|^{q(1-s)} |f'(b)|^{sq} E_\mu(s,q;Z).
\end{aligned}$$

When $1 \leq |f'(b)|$, by (2.3), we get

$$\begin{aligned}
& \int_0^1 |1-\lambda-t| \, |f'(a)|^{q\left(\frac{1+t}{2}\right)^s} |f'(b)|^{q\left(\frac{1-t}{2}\right)^s} dt \\
\leq & \int_0^1 |1-\lambda-t| \, |f'(a)|^{q\left[s\left(\frac{1+t}{2}\right)+1-s\right]} |f'(b)|^{q\left[s\left(\frac{1-t}{2}\right)+1-s\right]} dt \\
= & |f'(a) f'(b)|^{q\left(1-\frac{s}{2}\right)} \int_0^1 |1-\lambda-t| \, |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\
= & |f'(a) f'(b)|^{q\left(1-\frac{s}{2}\right)} M_\lambda(s,q;Z)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 |\mu-t| \, |f'(a)|^{q\left(\frac{t}{2}\right)^s} |f'(b)|^{q\left(\frac{2-t}{2}\right)^s} dt \\
\leq & \int_0^1 |\mu-t| \, |f'(a)|^{q\left[\frac{st}{2}+1-s\right]} |f'(b)|^{q\left[s\left(\frac{2-t}{2}\right)+1-s\right]} dt \\
= & |f'(a)|^{q(1-s)} |f'(b)|^{q} \int_0^1 |\mu-t| \, |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\
= & |f'(a)|^{q(1-s)} |f'(b)|^{q} E_\mu(s,q;Z).
\end{aligned}$$

As a result, the proof of Theorem 3.2 is complete. □

If taking $\lambda = \mu$ in Theorem 3.2, we derive the following corollary.

Corollary 3.3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically* *decreasing on* $[a,b]$, *for* $q\geq 1$ *and* $0<s\leq 1$, $0\leq\lambda\leq 1$, *then*

(3.8)

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left(\frac{(1-\lambda)^2+\lambda^2}{2}\right)^{1-\frac{1}{q}}\times$$
$$\begin{cases} \begin{array}{c}|f'(a)f'(b)|^{\frac{s}{2}}M_{\lambda}^{1/q}(s,q;Z)\\ +|f'(a)f'(b)|^{s}E_{\lambda}^{1/q}(s,q;Z)\end{array}, & |f'(a)|\leq 1;\\ \begin{array}{c}|f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}M_{\lambda}^{1/q}(s,q;Z)\\ +|f'(a)|^{1-s}|f'(b)|^{s}E_{\lambda}^{1/q}(s,q;Z)\end{array}, & |f'(b)|\leq 1\leq|f'(a)|;\\ \begin{array}{c}|f'(a)f'(b)|^{1-\frac{s}{2}}M_{\lambda}^{1/q}(s,q;Z)\\ +|f'(a)|^{1-s}|f'(b)|E_{\lambda}^{1/q}(s,q;Z)\end{array}, & 1\leq|f'(b)|,\end{cases}$$

where $Z(u,v)$, $M_{\lambda}(s,q;Z)$, $E_{\mu}(s,q;Z)$ *are defined in Theorem 3.2.*

If letting $\lambda=\mu=1/2$, $2/3$, $1/3$, respectively, in Theorem 3.2, we can deduce the inequalities below.

Corollary 3.4. *Let* $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ *be differentiable on* I°, $a,b\in I$, *with* $a<b$, *and* $f'\in L([a,b])$. *If* $|f'(x)|^q$ *is s-geometrically convex and monotonically decreasing on* $[a,b]$, *for* $q\geq 1$ *and* $0<s\leq 1$, *then*

(3.9)

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{4^{1/q}(b-a)}{16}\times$$
$$\begin{cases} \begin{array}{c}|f'(a)f'(b)|^{\frac{s}{2}}M_{1/2}^{1/q}(s,q;Z)\\ +|f'(a)f'(b)|^{s}E_{1/2}^{1/q}(s,q;Z),\end{array} & , \quad |f'(a)|\leq 1;\\ \begin{array}{c}|f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}M_{1/2}^{1/q}(s,q;Z)\\ +|f'(a)|^{1-s}|f'(b)|^{s}E_{1/2}^{1/q}(s,q;Z),\end{array} & , \quad |f'(b)|\leq 1\leq|f'(a)|;\\ \begin{array}{c}|f'(a)f'(b)|^{1-\frac{s}{2}}M_{1/2}^{1/q}(s,q;Z)\\ +|f'(a)|^{1-s}|f'(b)|E_{1/2}^{1/q}(s,q;Z),\end{array} & , \quad 1\leq|f'(b)|;\end{cases}$$

(3.10)

$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{5(b-a)}{72}\left(\frac{18}{5}\right)^{1/q}\times$$
$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}}M_{2/3}^{1/q}(s,q;Z)+|f'(a)f'(b)|^{s}E_{2/3}^{1/q}(s,q;Z), & |f'(a)|\leq 1;\\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}M_{2/3}^{1/q}(s,q;Z)+|f'(a)|^{1-s}|f'(b)|^{s}E_{2/3}^{1/q}(s,q;Z), & |f'(b)|\leq 1\leq|f'(a)|;\\ |f'(a)f'(b)|^{1-\frac{s}{2}}M_{2/3}^{1/q}(s,q;Z)+|f'(a)|^{1-s}|f'(b)|E_{2/3}^{1/q}(s,q;Z), & 1\leq|f'(b)|;\end{cases}$$

(3.11)

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{5(b-a)}{72}\left(\frac{18}{5}\right)^{1/q}\times$$
$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}} M_{1/3}^{1/q}(s,q;Z)+|f'(a)f'(b)|^{s} E_{1/3}^{1/q}(s,q;Z), & |f'(a)|\leq 1;\\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}} M_{1/3}^{1/q}(s,q;Z)+|f'(a)|^{1-s}|f'(b)|^{s} E_{1/3}^{1/q}(s,q;Z), & |f'(b)|\leq 1\leq |f'(a)|;\\ |f'(a)f'(b)|^{1-\frac{s}{2}} M_{1/3}^{1/q}(s,q;Z)+|f'(a)|^{1-s}|f'(b)| E_{1/3}^{1/q}(s,q;Z), & 1\leq |f'(b)|, \end{cases}$$

where $Z(u,v)$, $M_\lambda(s,q;Z)$, $E_\mu(s,q;Z)$ *are defined in Theorem 3.2.*

Theorem 3.3. *Let* $f: I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ *be differentiable on* I°, $a,b\in I$ *with* $a<b$, $0\leq\lambda,\mu\leq 1$, *and* f' *is integrable on* $[a,b]$. *If* $|f'(x)|^q$ *is* s*-geometrically convex and monotonically decreasing on* $[a,b]$, *for* $p,q\geq 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $s\in(0,1]$, *then*

(3.12)

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4}T^{1/q}(u)$$
$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}} F_\lambda^{1/p}+|f'(b)|^{s} F_\mu^{1/p}, & |f'(a)|\leq 1;\\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}} F_\lambda^{1/p}+|f'(a)|^{1-s}|f'(b)|^{s} F_\mu^{1/p}, & |f'(b)|\leq 1\leq|f'(a)|;\\ |f'(a)f'(b)|^{1-s/2} F_\lambda^{1/p}+|f'(a)|^{1-s}|f'(b)| F_\mu^{1/p}, & 1\leq|f'(b)|, \end{cases}$$

where

$$T(u)=\begin{cases}\frac{u-1}{\ln u}, & u\neq 1\\ 1, & u=1\end{cases},\quad u=|f'(a)/f'(b)|^{\frac{sq}{2}}$$

and

$$F_\lambda=\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1},\ F_\mu=\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}.$$

Proof. From Lemma 2.1 and since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, by using Hölder's inequality, we get

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left[\left(\int_0^1|1-\lambda-t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1|\mu-t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a^{\frac{t}{2}}b^{\frac{2-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4}\left[\left(\int_0^1|1-\lambda-t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}\right)^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1|\mu-t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left(|f'(a)|^{\left(\frac{t}{2}\right)^s}|f'(b)|^{\left(\frac{2-t}{2}\right)^s}\right)^q dt\right)^{\frac{1}{q}}\right].$$

It is know that for $0 \leq \lambda, \mu \leq 1$, by using Lemma 2.2, we have

$$\int_0^1 |(1-\lambda-t)|^p \, dt = \frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1} = F_\lambda$$

and

$$\int_0^1 |(\mu-t)|^p \, dt = \frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1} = F_\mu.$$

Therefore, we have

(3.13)

$$\begin{aligned}
&\left| \frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
\leq\;& \frac{b-a}{4}\left[\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 |f'(a)|^{q\left(\frac{1+t}{2}\right)^s} |f'(b)|^{q\left(\frac{1-t}{2}\right)^s} dt\right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 |f'(a)|^{q\left(\frac{t}{2}\right)^s} |f'(b)|^{q\left(\frac{2-t}{2}\right)^s} dt\right)^{\frac{1}{q}} \right].
\end{aligned}$$

When $|f'(a)| \leq 1$, from (2.3), we get

$$\begin{aligned}
\int_0^1 |f'(a)|^{q\left(\frac{1+t}{2}\right)^s} |f'(b)|^{q\left(\frac{1-t}{2}\right)^s} dt &\leq \int_0^1 |f'(a)|^{sq\left(\frac{1+t}{2}\right)} |f'(b)|^{sq\left(\frac{1-t}{2}\right)} dt \\
&= |f'(a) f'(b)|^{\frac{sq}{2}} \int_0^1 |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\
&= |f'(a) f'(b)|^{\frac{sq}{2}} \frac{Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1}{\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 |f'(a)|^{q\left(\frac{t}{2}\right)^s} |f'(b)|^{q\left(\frac{2-t}{2}\right)^s} dt &\leq \int_0^1 |f'(a)|^{\frac{sqt}{2}} |f'(b)|^{sq-\frac{sqt}{2}} dt \\
&= |f'(b)|^{sq} \int_0^1 |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\
&= |f'(b)|^{sq} \frac{Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1}{\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)}.
\end{aligned}$$

When $|f'(b)| \leq 1 \leq |f'(a)|$, by virtue of (2.3), we get

$$\begin{aligned}
\int_0^1 |f'(a)|^{q\left(\frac{1+t}{2}\right)^s} |f'(b)|^{q\left(\frac{1-t}{2}\right)^s} dt &\leq \int_0^1 |f'(a)|^{q[s((1+t)/2)+1-s]} |f'(b)|^{sq(1-t)/2} dt \\
&= |f'(a)|^{(1-s/2)q} |f'(b)|^{sq/2} \int_0^1 |f'(a)/f'(b)|^{sqt/2} dt \\
&= |f'(a)|^{(1-s/2)q} |f'(b)|^{sq/2} \frac{Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1}{\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)},
\end{aligned}$$

and

$$\begin{aligned}\int_0^1 |f'(a)|^{q\left(\frac{t}{2}\right)^s} |f'(b)|^{q\left(\frac{2-t}{2}\right)^s} dt &\leq \int_0^1 |f'(a)|^{q[st/2+1-s]} |f'(b)|^{sq-sqt/2} dt \\ &= |f'(a)|^{q(1-s)} |f'(b)|^{sq} \int_0^1 |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\ &= |f'(a)|^{q(1-s)} |f'(b)|^{sq} \frac{Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1}{\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)}.\end{aligned}$$

When $1 \leq |f'(b)|$, by virtue of (2.3), we get

$$\begin{aligned}\int_0^1 |f'(a)|^{q\left(\frac{1+t}{2}\right)^s} |f'(b)|^{q\left(\frac{1-t}{2}\right)^s} dt &\leq \int_0^1 |f'(a)|^{q[s((1+t)/2)+1-s]} |f'(b)|^{q[s(1-t)/2+1-s]} dt \\ &= |f'(a) f'(b)|^{(1-s/2)q} \int_0^1 |f'(a)/f'(b)|^{sqt/2} dt \\ &= |f'(a) f'(b)|^{(1-s/2)q} \frac{Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1}{\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)},\end{aligned}$$

and

$$\begin{aligned}\int_0^1 |f'(a)|^{q\left(\frac{t}{2}\right)^s} |f'(b)|^{q\left(\frac{2-t}{2}\right)^s} dt &\leq \int_0^1 |f'(a)|^{q[st/2+1-s]} |f'(b)|^{q[1-st/2]} dt \\ &= |f'(a)|^{q(1-s)} |f'(b)|^{q} \int_0^1 |f'(a)/f'(b)|^{\frac{sqt}{2}} dt \\ &= |f'(a)|^{q(1-s)} |f'(b)|^{q} \frac{Z\left(\frac{sq}{2},\frac{sq}{2}\right)-1}{\ln Z\left(\frac{sq}{2},\frac{sq}{2}\right)}.\end{aligned}$$

As a result, the proof of Theorem 3.3 is complete. □

If taking $\lambda = \mu$ in Theorem 3.3, we derive the following corollary.

Corollary 3.5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|$ is s-geometrically convex and monotonically decreasing on $[a,b]$, for $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < s \leq 1$, $0 \leq \lambda \leq 1$, then*

$$(3.14)\qquad \left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4} F_\lambda^{1/p} T^{1/q}(u) \times \begin{cases} |f'(a) f'(b)|^{\frac{s}{2}} + |f'(b)|, & |f'(a)| \leq 1; \\ |f'(a)|^{1-\frac{s}{2}} |f'(b)|^{\frac{s}{2}} + |f'(a)|^{1-s} |f'(b)|^{s}, & |f'(b)| \leq 1 \leq |f'(a)|; \\ |f'(a) f'(b)|^{1-s/2} + |f'(a)|^{1-s} |f'(b)|, & 1 \leq |f'(b)|, \end{cases}$$

where u, $T(u)$, F_λ are defined in Theorem 3.3.

If letting $\lambda = \mu = 1/2,\ 2/3,\ 1/3$, respectively, in Theorem 3.3, we can deduce the inequalities below.

Corollary 3.6. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|$ is s-geometrically convex and monotonically* *decreasing on* $[a,b]$, *for* $p,q\geq 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $0<s\leq 1$, $0\leq\lambda\leq 1$, *then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.15}$$

$$\leq \frac{(b-a)}{8(p+1)^{\frac{1}{p}}}T^{1/q}(u)\times$$

$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}}+|f'(b)|, & |f'(a)|\leq 1; \\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}+|f'(a)|^{1-s}|f'(b)|^{s}, & |f'(b)|\leq 1\leq |f'(a)|; \\ |f'(a)f'(b)|^{1-s/2}+|f'(a)|^{1-s}|f'(b)|, & 1\leq |f'(b)|; \end{cases}$$

$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.16}$$

$$\leq \frac{(b-a)}{4}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}T^{1/q}(u)\times$$

$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}}+|f'(b)|, & |f'(a)|\leq 1; \\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}+|f'(a)|^{1-s}|f'(b)|^{s}, & |f'(b)|\leq 1\leq |f'(a)|; \\ |f'(a)f'(b)|^{1-s/2}+|f'(a)|^{1-s}|f'(b)|, & 1\leq |f'(b)|; \end{cases}$$

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.17}$$

$$\leq \frac{(b-a)}{4}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}T^{1/q}(u)\times$$

$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}}+|f'(b)|, & |f'(a)|\leq 1; \\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}+|f'(a)|^{1-s}|f'(b)|^{s}, & |f'(b)|\leq 1\leq |f'(a)|; \\ |f'(a)f'(b)|^{1-s/2}+|f'(a)|^{1-s}|f'(b)|, & 1\leq |f'(b)|, \end{cases}$$

where u, $T(u)$ *are defined in Theorem 3.3.*

Theorem 3.4. *Let* $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ *be differentiable on* I°, $a,b\in I$ *with* $a<b$, $0\leq\lambda,\mu\leq 1$, *and* f' *is integrable on* $[a,b]$. *If* $|f'(x)|$ *is s-geometrically convex and monotonically decreasing on* $[a,b]$, *for* $s\in(0,1]$ *and* $\omega,\beta>0$ *with* $\omega+\beta=1$, *then*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4}\times \tag{3.18}$$

$$\begin{cases} \begin{array}{c} F(\omega,\lambda)+F(\omega,\mu) \\ +\beta T(v;s,\beta)\left[|f'(a)f'(b)|^{\frac{s}{2\beta}}+|f'(b)|^{\frac{s}{\beta}}\right] \end{array}, & |f'(a)|\leq 1; \\ \begin{array}{c} F(\omega,\lambda)+F(\omega,\mu) \\ +\beta T(v;s,\beta)\left[|f'(a)|^{\frac{2-s}{2\beta}}|f'(b)|^{\frac{s}{2\beta}}+|f'(a)|^{\frac{1-s}{\beta}}|f'(b)|^{\frac{s}{\beta}}\right] \end{array}, & |f'(b)|\leq 1\leq|f'(a)|; \\ \begin{array}{c} F(\omega,\lambda)+F(\omega,\mu) \\ +\beta T(v;s,\beta)\left[|f'(a)|^{\frac{2-s}{2\beta}}|f'(b)|^{\frac{2-s}{2\beta}}+|f'(a)|^{\frac{1-s}{\beta}}|f'(b)|^{\frac{1}{\beta}}\right] \end{array}, & 1\leq|f'(b)|; \end{cases}$$

where

$$T(v;s,\beta)=\begin{cases}\frac{v-1}{\ln v}, & v\neq 1\\ 1, & v=1\end{cases}\ ,\ v=\left|f'(a)/f'(b)\right|^{\frac{s}{2\beta}},$$

and

$$F(\omega,\lambda)=\frac{\omega^2}{\omega+1}\left(\lambda^{\frac{1}{\omega}+1}+(1-\lambda)^{\frac{1}{\omega}+1}\right),\ F(\omega,\mu)=\frac{\omega^2}{\omega+1}\left(\mu^{\frac{1}{\omega}+1}+(1-\mu)^{\frac{1}{\omega}+1}\right).$$

Proof. From Lemma 2.1, and since $|f'(x)|$ is s-geometrically convex and monotonically decreasing on $[a,b]$, and by using Cauchy's inequality, we get

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\ &\frac{b-a}{4}\left[\omega\int_0^1|1-\lambda-t|^{\frac{1}{\omega}}\,dt+\beta\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^{\frac{1}{\beta}}dt\right.\\
&\left.\omega\int_0^1|\mu-t|^{\frac{1}{\omega}}\,dt+\beta\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\,b\right)\right|^{\frac{1}{\beta}}dt\right]\\
\leq\ &\frac{b-a}{4}\left[\omega\int_0^1|1-\lambda-t|^{\frac{1}{\omega}}\,dt+\beta\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^{\frac{1}{\beta}}dt\right.\\
&\left.\omega\int_0^1|\mu-t|^{\frac{1}{\omega}}\,dt+\beta\int_0^1\left|f'\left(a^{\frac{t}{2}}b^{\frac{2-t}{2}}\right)\right|^{\frac{1}{\beta}}dt\right]\\
\leq\ &\frac{b-a}{4}\left[\omega\int_0^1|1-\lambda-t|^{\frac{1}{\omega}}\,dt+\beta\int_0^1\left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}\right)^{\frac{1}{\beta}}dt\right.\\
&\left.+\,\omega\int_0^1|\mu-t|^{\frac{1}{\omega}}\,dt+\beta\int_0^1\left(|f'(a)|^{\left(\frac{t}{2}\right)^s}|f'(b)|^{\left(\frac{2-t}{2}\right)^s}\right)^{\frac{1}{\beta}}dt\right].
\end{aligned}$$

On the other hand, we have

$$\omega\int_0^1|1-\lambda-t|^{\frac{1}{\omega}}\,dt=\frac{\omega^2}{\omega+1}\left(\lambda^{\frac{1}{\omega}+1}+(1-\lambda)^{\frac{1}{\omega}+1}\right)=F(\omega,\lambda)$$

and

$$\omega\int_0^1|\mu-t|^{\frac{1}{\omega}}\,dt=\frac{\omega^2}{\omega+1}\left(\mu^{\frac{1}{\omega}+1}+(1-\mu)^{\frac{1}{\omega}+1}\right)=F(\omega,\mu)$$

Then, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\ &\frac{b-a}{4}\left[F(\omega,\lambda)+\beta\int_0^1\left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}\right)^{\frac{1}{\beta}}dt\right.\\
&\left.+\,F(\omega,\mu)+\beta\int_0^1\left(|f'(a)|^{\left(\frac{t}{2}\right)^s}|f'(b)|^{\left(\frac{2-t}{2}\right)^s}\right)^{\frac{1}{\beta}}dt\right].
\end{aligned}$$

When $|f'(a)| \leq 1$, from (2.3), we get

$$\begin{aligned} \beta \int_0^1 \left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s} |f'(b)|^{\left(\frac{1-t}{2}\right)^s} \right)^{\frac{1}{\beta}} dt &\leq \beta \int_0^1 |f'(a)|^{\frac{s+st}{2\beta}} |f'(b)|^{\frac{s-st}{2\beta}} dt \\ &= \beta |f'(a) f'(b)|^{\frac{s}{2\beta}} \int_0^1 |f'(a)/f'(b)|^{\frac{st}{2\beta}} dt \\ &= \beta |f'(a) f'(b)|^{\frac{s}{2\beta}} T(v; s, \beta), \end{aligned}$$

and

$$\begin{aligned} \beta \int_0^1 \left(|f'(a)|^{\left(\frac{t}{2}\right)^s} |f'(b)|^{\left(\frac{2-t}{2}\right)^s} \right)^{\frac{1}{\beta}} dt &\leq \beta \int_0^1 |f'(a)|^{\frac{st}{2\beta}} |f'(b)|^{\frac{2s-st}{2\beta}} dt \\ &= \beta |f'(b)|^{\frac{s}{\beta}} \int_0^1 |f'(a)/f'(b)|^{\frac{st}{2\beta}} dt \\ &= \beta |f'(b)|^{\frac{s}{\beta}} T(v; s, \beta). \end{aligned}$$

When $|f'(b)| \leq 1 \leq |f'(a)|$, by virtue of (2.3), we get

$$\begin{aligned} \beta \int_0^1 \left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s} |f'(b)|^{\left(\frac{1-t}{2}\right)^s} \right)^{\frac{1}{\beta}} dt &\leq \beta \int_0^1 |f'(a)|^{\frac{st+2-s}{2\beta}} |f'(b)|^{\frac{s-st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{s}{2\beta}} \int_0^1 |f'(a)/f'(b)|^{\frac{st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{s}{2\beta}} T(v; s, \beta), \end{aligned}$$

and

$$\begin{aligned} \beta \int_0^1 \left(|f'(a)|^{\left(\frac{t}{2}\right)^s} |f'(b)|^{\left(\frac{2-t}{2}\right)^s} \right)^{\frac{1}{\beta}} dt &\leq \beta \int_0^1 |f'(a)|^{\frac{st+2-2s}{2\beta}} |f'(b)|^{\frac{2s-st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{s}{\beta}} \int_0^1 |f'(a)/f'(b)|^{\frac{st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{s}{\beta}} T(v; s, \beta). \end{aligned}$$

When $1 \leq |f'(b)|$, by using (2.3), we get

$$\begin{aligned} \beta \int_0^1 \left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s} |f'(b)|^{\left(\frac{1-t}{2}\right)^s} \right)^{\frac{1}{\beta}} dt &\leq \beta \int_0^1 |f'(a)|^{\frac{st+2-s}{2\beta}} |f'(b)|^{\frac{2-s-st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{2-s}{2\beta}} \int_0^1 |f'(a)/f'(b)|^{\frac{st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{2-s}{2\beta}} T(v; s, \beta), \end{aligned}$$

and

$$\begin{aligned} \beta \int_0^1 \left(|f'(a)|^{\left(\frac{t}{2}\right)^s} |f'(b)|^{\left(\frac{2-t}{2}\right)^s} \right)^{\frac{1}{\beta}} dt &\leq \beta \int_0^1 |f'(a)|^{\frac{st+2-2s}{2\beta}} |f'(b)|^{\frac{2-st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{1}{\beta}} \int_0^1 |f'(a)/f'(b)|^{\frac{st}{2\beta}} dt \\ &= \beta |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{1}{\beta}} T(v; s, \beta). \end{aligned}$$

The proof of Theorem 3.4 is complete. □

If taking $\lambda = \mu$ in Theorem 3.4, we derive the following corollary.

Corollary 3.7. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda \leq 1$, and f' is integrable on $[a, b]$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a, b]$, for $s \in (0, 1]$ and $\omega, \beta > 0$ with $\omega + \beta = 1$, then*

(3.19)

$$\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x)\,dx\right| \leq \frac{b - a}{4}$$

$$\times \begin{cases} 2F(\omega, \lambda) + \beta T(v; s, \beta)\left[|f'(a) f'(b)|^{\frac{s}{2\beta}} + |f'(b)|^{\frac{s}{\beta}}\right], & |f'(a)| \leq 1; \\ 2F(\omega, \lambda) + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{s}{\beta}}\right], & |f'(b)| \leq 1 \leq |f'(a)|; \\ 2F(\omega, \lambda) + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{2-s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{1}{\beta}}\right], & 1 \leq |f'(b)|, \end{cases}$$

where $T(v; s, \beta)$, $F(\omega, \lambda)$ are defined in Theorem 3.4.

If letting $\lambda = \mu = 1/2,\ 2/3,\ 1/3$, respectively, in Theorem 3.4, we can deduce the inequalities below.

Corollary 3.8. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and f' is integrable on $[a, b]$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a, b]$, for $s \in (0, 1]$ and $\omega, \beta > 0$ with $\omega + \beta = 1$, then*

(3.20)

$$\left|\frac{1}{2}\left[\frac{f(a) + f(b)}{2} + f\left(\frac{a + b}{2}\right)\right] - \frac{1}{b - a}\int_a^b f(x)\,dx\right| \leq \frac{b - a}{4}$$

$$\times \begin{cases} \frac{\omega^2}{2^{1/\omega - 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a) f'(b)|^{\frac{s}{2\beta}} + |f'(b)|^{\frac{s}{\beta}}\right], & |f'(a)| \leq 1; \\ \frac{\omega^2}{2^{1/\omega - 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{s}{\beta}}\right], & |f'(b)| \leq 1 \leq |f'(a)|; \\ \frac{\omega^2}{2^{1/\omega - 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{2-s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{1}{\beta}}\right], & 1 \leq |f'(b)|; \end{cases}$$

(3.21)

$$\left|\frac{1}{3}\left[f(a) + f(b) + f\left(\frac{a + b}{2}\right)\right] - \frac{1}{b - a}\int_a^b f(x)\,dx\right| \leq \frac{b - a}{4}$$

$$\times \begin{cases} \frac{2\omega^2\left(2^{\frac{1}{\omega}+1} + 1\right)}{3^{1/\omega + 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a) f'(b)|^{\frac{s}{2\beta}} + |f'(b)|^{\frac{s}{\beta}}\right], & |f'(a)| \leq 1; \\ \frac{2\omega^2\left(2^{\frac{1}{\omega}+1} + 1\right)}{3^{1/\omega + 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{s}{\beta}}\right], & |f'(b)| \leq 1 \leq |f'(a)|; \\ \frac{2\omega^2\left(2^{\frac{1}{\omega}+1} + 1\right)}{3^{1/\omega + 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{2-s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{1}{\beta}}\right], & 1 \leq |f'(b)|; \end{cases}$$

(3.22)

$$\left|\frac{1}{6}\left[f(a) + f(b) + 4f\left(\frac{a + b}{2}\right)\right] - \frac{1}{b - a}\int_a^b f(x)\,dx\right| \leq \frac{b - a}{4}$$

$$\times \begin{cases} \frac{2\omega^2\left(2^{\frac{1}{\omega}+1} + 1\right)}{3^{1/\omega + 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a) f'(b)|^{\frac{s}{2\beta}} + |f'(b)|^{\frac{s}{\beta}}\right], & |f'(a)| \leq 1; \\ \frac{2\omega^2\left(2^{\frac{1}{\omega}+1} + 1\right)}{3^{1/\omega + 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{s}{\beta}}\right], & |f'(b)| \leq 1 \leq |f'(a)|; \\ \frac{2\omega^2\left(2^{\frac{1}{\omega}+1} + 1\right)}{3^{1/\omega + 1}(\omega + 1)} + \beta T(v; s, \beta)\left[|f'(a)|^{\frac{2-s}{2\beta}} |f'(b)|^{\frac{2-s}{2\beta}} + |f'(a)|^{\frac{1-s}{\beta}} |f'(b)|^{\frac{1}{\beta}}\right], & 1 \leq |f'(b)|; \end{cases}$$

where $T(v;s,\beta)$ *as in the Theorem 3.4.*

If letting $\lambda=\mu=1/2,\ 2/3,\ 1/3$, and $\omega,\beta=1/2$ respectively, in Theorem 3.4, we can deduce the inequalities below.

Corollary 3.9. *Let* $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ *be differentiable on* I°, $a,b\in I$ *with* $a<b$, *and* f' *is integrable on* $[a,b]$. *If* $|f'(x)|^q$ *is* s*-geometrically convex and monotonically decreasing on* $[a,b]$, *for* $s\in(0,1]$, *then*

(3.23)

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{8}$$
$$\times\begin{cases}\frac{1}{6}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)f'(b)|^s+|f'(b)|^{2s}\right], & |f'(a)|\le 1;\\ \frac{1}{6}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)|^{2-s}|f'(b)|^s+|f'(a)|^{2-2s}|f'(b)|^{2s}\right], & |f'(b)|\le 1\le|f'(a)|;\\ \frac{1}{6}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)f'(b)|^{2-s}+|f'(a)|^{2-2s}|f'(b)|^{2}\right], & 1\le|f'(b)|;\end{cases}$$

(3.24)

$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{8}$$
$$\times\begin{cases}\frac{2}{9}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)f'(b)|^s+|f'(b)|^{2s}\right], & |f'(a)|\le 1;\\ \frac{2}{9}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)|^{2-s}|f'(b)|^s+|f'(a)|^{2-2s}|f'(b)|^{2s}\right], & |f'(b)|\le 1\le|f'(a)|;\\ \frac{2}{9}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)f'(b)|^{2-s}+|f'(a)|^{2-2s}|f'(b)|^{2}\right], & 1\le|f'(b)|;\end{cases}$$

(3.25)

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{8}$$
$$\times\begin{cases}\frac{2}{9}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)f'(b)|^s+|f'(b)|^{2s}\right], & |f'(a)|\le 1;\\ \frac{2}{9}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)|^{2-s}|f'(b)|^s+|f'(a)|^{2-2s}|f'(b)|^{2s}\right], & |f'(b)|\le 1\le|f'(a)|;\\ \frac{2}{9}+T\left(v;s,\frac{1}{2}\right)\left[|f'(a)f'(b)|^{2-s}+|f'(a)|^{2-2s}|f'(b)|^{2}\right], & 1\le|f'(b)|;\end{cases}$$

where $T(v;s,\beta)$ *as in the Theorem 3.4.*

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*Department of Mathematics, Faculty of Science and Arts, Mustafa Kemal University, 31000, Hatay, Turkey

E-mail address: mevluttttunc@gmail.com

**Department of Mathematics, Faculty of Science and Arts, Ağri İbrahim Çeçen University, 04000, Ağri, Turkey

E-mail address: yuksel.ebru90@hotmail.com