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ON DETERMINATION OF k-FIBONACCI AND k-LUCAS NUMBERS

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Abstract. In this study we investigate some properties of the k-Fibonacci and k-Lucas sequences which are generalize the classical Fibonacci and Lucas sequences. Moreover, two efficient tests are introduced as to whether or not a positive integer is k-Fibonacci or k-Lucas.

1. Introduction

Let k and t be nonzero real numbers such that $k^2+4t > 0$. Generalized Fibonacci sequence $\{U_n\}$ is defined by

$$U_0 = 0, U_1 = 1 \text{ and } U_{n+1} = kU_n + tU_{n-1} \text{ for } n \geq 1, \tag{1.1}$$

and generalized Lucas sequence $\{V_n\}$ is defined by

$$V_0 = 2, V_1 = k, \text{ and } V_{n+1} = kV_n + tV_{n-1} \text{ for } n \geq 1, \tag{1.2}$$

(see [3], and [9] for more details about these sequences).

Let $k \geq 1$ be any integer. In (1.1) and (1.2) if we take $t = 1$, we obtain k-Fibonacci and k-Lucas numbers $F_{k,n}$ and $L_{k,n}$, respectively. Basic properties of these sequences were investigated in the several papers (see [1], [2], [12] and [2]). After complex factorizations of these numbers for $k = 1$ were found by the autors in [6], these results were generalized for k-Fibonacci and k-Lucas numbers in [10]. In [7], using the Chebyshev polynomials of the first and second kinds, one-parameter generalizations of the Fibonacci and Lucas numbers were given.

In this paper we introduce two tests as to whether or not a positive integer is k-Fibonacci or k-Lucas. Our results generalize the results for the case $k = 1$ given by I. Gessel, P. James and G. Wulczyn, respectively (see [8] and [4]).

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2. When is a number k-Fibonacci?

Let r_1 and r_2 be the roots of the characteristic equation $r^2 = kr+1$ and $r_1 > r_2$. It is known that

$$r_1 = \frac{k+\sqrt{k^2+4}}{2},\ r_2 = \frac{k-\sqrt{k^2+4}}{2}$$

and

$$r_1.r_2 = -1. \tag{2.1}$$

Before giving our results, we recall the following equations (see [8], [11], [12])

$$L_{k,n} = F_{k,n-1} + F_{k,n+1} \text{ for } n \geq 1, \tag{2.2}$$

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \text{ (Binet's Formula)}, \tag{2.3}$$

$$L_{k,n} = r_1^n + r_2^n, \tag{2.4}$$

and

$$F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r}F_{k,r}^2 \text{ (Catalan's Identity)}. \tag{2.5}$$

We begin the following lemmas.

Lemma 2.1. *The equation*

$$y^2 - kxy - x^2 = \mp 1 \tag{2.6}$$

is satisfied by $(x, y) = (F_{k,n}, F_{k,n+1})$ for all $n \geq 0$, $x, y \in \mathbb{N}$.

Proof. For $n = 0$, using $(F_{k,0}, F_{k,1}) = (0, 1)$ we have

$$1^2 - k.0.1 - 0^2 = 1,$$

that is, the equation (2.6) is true.

Now, suppose that the equation (2.6) is true for n. Then by definition we obtain

$$\begin{aligned} & F_{k,n+2}^2 - kF_{k,n+1}F_{k,n+2} - F_{k,n+1}^2 \\ = {} & (kF_{k,n+1} + F_{k,n})^2 - kF_{k,n+1}(kF_{k,n+1} + F_{k,n}) - F_{k,n+1}^2 \\ = {} & -(F_{k,n+1}^2 - kF_{k,n+1}F_{k,n} - F_{k,n}^2) \\ = {} & \mp 1. \end{aligned}$$

□

Lemma 2.2. *If (x, y) is a pair of positive integers satisfying the equation (2.6) then $(x, y) = (F_{k,n}, F_{k,n+1})$ or $(x, y) = (F_{k,n}, F_{k,n-1})$ for some $n \geq 0$.*

Proof. If we take $kx = y$, then substituting $\frac{y}{k}$ for x we get

$$y^2 - kxy - x^2 = y^2 - y^2 - \left(\frac{y}{k}\right)^2 = -\frac{y^2}{k^2} = \mp 1,$$

hence setting $(x, y) = (F_{k,1}, F_{k,2})$, satisfies this equation. Now assume that $kx < y$ and $y^2 - kxy - x^2 = \mp 1$. We see that the pair of positive integers $(a, b) = (y - kx, x)$ satisfies the equation (2.6) since we have

$$\begin{aligned} b^2 - kab - a^2 &= x^2 - k(y-kx)x - (y-kx)^2 \\ &= -(y^2 - kxy - x^2) = \mp 1. \end{aligned}$$

By induction, $(a,b)=(F_{k,n},F_{k,n+1})$ by some n. Thus we get

$$x=b=F_{k,n+1}$$

and

$$y=kx+a=kF_{k,n+1}+F_{k,n}.$$

This means that $(x,y)=(F_{k,n+1},F_{k,n+2})$. Similarly if $kx>y$ then it can be seen that the pair of positive integers $(a,b)=(x,kx-y)$ satisfies the equation (2.6) since we have

$$\begin{aligned} b^2-kab-a^2 &= (kx-y)^2-k(kx-y)x-x^2 \\ &= y^2-kyx-x^2=\mp 1. \end{aligned}$$

Hence we get

$$x=a=F_{k,n}$$

and

$$y=kx-b=kF_{k,n}-F_{k,n+1}.$$

This means that $(x,y)=(F_{k,n},F_{k,n-1})$. □

From [2], we have the following theorem.

Theorem 2.1. (*See* [2], *Theorem* 2.1) *For any integer* n, *the number* $(k^2+4)F_{k,n}^2+4(-1)^n$ *is a perfect square.*

Now we prove the converse of Theorem 2.1.

Theorem 2.2. *Let* n *be a positive integer. If* $(k^2+4)n^2\mp 4$ *is a perfect square then* n *is a* k*-Fibonacci number.*

Proof. We use Lemma 2.1. Using the equation (2.6), we get

$$y=\frac{kx\mp\sqrt{(4+k^2)x^2\mp 4}}{2}.$$

Assuming y is positive we simplify this to

$$y=\frac{kx+\sqrt{(4+k^2)x^2\mp 4}}{2}. \tag{2.7}$$

Assume that $(k^2+4)n^2\mp 4$ is a perfect square and that $x=n$. Since x is a positive integer then we see that y must also be a positive integer using the following three cases:

Case 1. If n and k are both odd then (k^2+4) and n^2 must be both odd. Hence the product $(k^2+4)n^2$ is an odd number and so $(k^2+4)n^2\mp 4$ is an odd number and perfect square. By (2.7) we get

$$\frac{1}{2}\left(kn+\text{odd value}\right).$$

Since kn is odd, the number ($kn+$ odd value) is divisible by 2 and then $\frac{1}{2}\left(kn+\text{odd value}\right)$ must be an integer. Thus under the assumption n and k are both odd y must be an integer.

Case 2. If n and k are even then (k^2+4) and n^2 must be both even. Similarly as for $(k^2+4)n^2\mp 4$ is an even number and perfect square, using (2.7) we have

$$\frac{1}{2}\left(kn+\text{even value}\right).$$

Again as kn is even, the number $(kn + \text{even value})$ is divisible by 2 and so we obtain $\frac{1}{2}(kn + \text{even value})$ must be an integer. Thus under the assumption n and k are both even y must be an integer.

Case 3. If one of the numbers n and k is odd, $(k^2+4)x^2 \mp 4$ must be an even number. Thus this case is equivalent to the Case 2.

Consequently, since y must be a positive integer, from Lemma 2.2, both $x = n$ and y must be k-Fibonacci numbers. Namely, n is a k-Fibonacci number. □

We note that it was given a different proof of Lemma 2.2 for the case $k = 2\sinh\theta > 1$ is an odd integer $(\theta > 0)$. Now combining Theorem 2.1 and Theorem 2.2 we obtain the following theorem.

Theorem 2.3. *A positive integer n is a k-Fibonacci number if and only if $(k^2+4)n^2 \mp 4$ is a perfect square.*

For $k = 1$, our results coincide with the results obtained in [4]. On the other hand, there is an alternative proof of Theorem 2.1. Let $n = F_{k,m}$. Using (2.2) and the Catalan's identity (2.5), we have

$$F_{k,m-1}F_{k,m+1} = F_{k,m}^2 + (-1)^m$$

and

$$\begin{aligned} L_{k,m}^2 - 4\left[(-1)^m + F_{k,m}^2\right] &= (F_{k,m+1} + F_{k,m-1})^2 - 4[F_{k,m+1}F_{k,m-1}] \\ &= (F_{k,m+1} - F_{k,m-1})^2 \\ &= (kF_{k,m})^2 \\ &= k^2F_{k,m}^2 \\ L_{k,m}^2 &= (k^2+4)F_{k,m}^2 \mp 4. \end{aligned}$$

So, if n is a k-Fibonacci number $F_{k,m}$ then $(k^2+4)n^2 \mp 4$ is a perfect square.

3. When is a number k-Lucas?

Now we continue our work with the following lemmas.

Lemma 3.1. *The equation*

$$y^2 - kxy - x^2 = \mp(k^2+4) \tag{3.1}$$

is satisfied by $(x, y) = (L_{k,n}, L_{k,n+1})$ for all $n \geq 0$, $x, y \in \mathbb{N}$.

Proof. For $n = 0$, using $(L_{k,0}, L_{k,1}) = (2, k)$ we have

$$k^2 - 2k^2 - 4 = -(k^2+4),$$

that is the equation (3.1) is true.

Now, suppose that the equation is true for n. Then by definition we obtain

$$\begin{aligned} & L_{k,n+2}^2 - kL_{k,n+1}L_{k,n+2} - L_{k,n+1}^2 \\ =\ & (kL_{k,n+1} + L_{k,n})^2 - kL_{k,n+1}(kL_{k,n+1} + L_{k,n}) - L_{k,n+1}^2 \\ =\ & 2kL_{k,n+1}L_{k,n} + L_{k,n}^2 - kF_{k,n+1}F_{k,n} - L_{k,n+1}^2 \\ =\ & -(L_{k,n+1}^2 - kL_{k,n+1}L_{k,n} - L_{k,n}^2) \\ =\ & \mp(k^2+4). \end{aligned}$$

□

Lemma 3.2. *If (x, y) is a pair of positive integers satisfying the equation (3.1) then $(x, y) = (L_{k,n}, L_{k,n+1})$ or $(x, y) = (L_{k,n}, L_{k,n-1})$ for some $n \geq 0$.*

Proof. If we take $kx = y$ then we get

$$y^2 - kxy - x^2 = y^2 - y^2 - x^2 = -x^2 = -(k^2 + 4).$$

Hence there is no pair (x, y) which satisfies the equation (3.1). Let $kx < y$. We see that the pair of positive integers $(a, b) = (y - kx, x)$ satisfies the equation (3.1) since we have

$$\begin{aligned} b^2 - kab - a^2 &= x^2 - k(y - kx)x - (y - kx)^2 \\ &= x^2 - kyx - y^2 + 2kyx \\ &= -(y^2 - kxy - x^2) = \mp(k^2 + 4). \end{aligned}$$

By induction, $(a, b) = (L_{k,n}, L_{k,n+1})$ by some n. Thus we get

$$x = b = L_{k,n+1}$$

and

$$y = kx + a = kL_{k,n+1} + L_{k,n}.$$

This means that $(x, y) = (L_{k,n+1}, L_{k,n+2})$.

Now let $kx > y$. It can be easily seen that the pair of positive integers $(a, b) = (x, kx - y)$ satisfies the equation (3.1) since we have

$$\begin{aligned} b^2 - kab - a^2 &= (kx - y)^2 - kx(kx - y) - x^2 \\ &= y^2 - kxy - x^2 = \mp(k^2 + 4). \end{aligned}$$

By induction, $(a, b) = (L_{k,n}, L_{k,n+1})$ by some n. Thus we get

$$x = a = L_{k,n}$$

and

$$y = kx - b = kL_{k,n} - L_{k,n+1} = L_{k,n-1}.$$

This means that $(x, y) = (L_{k,n}, L_{k,n-1})$. □

Theorem 3.1. *A positive integer n is a k-Lucas number if and only if $\left(k^2 + 4\right) n^2 \mp 4 \left(k^2 + 4\right)$ is a perfect square.*

Proof. The proof is similar to the proofs of Theorem 2.1 and Theorem 2.2 by Lemma 3.1 and Lemma 3.2. □

There is an alternative proof of the first part of Theorem 3.1. Let $n = L_{k,2m+1}$. Using (2.1) and (2.4) we find

$$\begin{aligned} \left(k^2 + 4\right) n^2 + 4(k^2 + 4) &= (k^2 + 4) \left(r_1^{2m+1} + r_2^{2m+1}\right)^2 + 4(k^2 + 4) \\ &= (k^2 + 4) \left[r_1^{4m+2} + r_2^{4m+2} + 2(r_1 r_2)^{2m+1} + 4\right] \\ &= (k^2 + 4) \left(\sqrt{k^2 + 4} F_{k,2m+1}\right)^2 \\ &= (k^2 + 4)^2 F_{k,2m+1}^2. \end{aligned}$$

Now let $n = L_{k,2m}$. By a similar way we find

$$\begin{aligned}
\left(k^2+4\right)n^2 - 4(k^2+4) &= (k^2+4)\left(r_1^{2m}+r_2^{2m}\right)^2 - 4(k^2+4) \\
&= (k^2+4)\left[r_1^{4m}+r_2^{4m}+2(r_1r_2)^{2m}-4\right] \\
&= (k^2+4)\left[r_1^{4m}+r_2^{4m}-2(r_1r_2)^{2m}\right] \\
&= (k^2+4)\left(\sqrt{k^2+4}F_{k,2m}\right)^2 \\
&= (k^2+4)^2F_{k,2m}^2.
\end{aligned}$$

4. **Applications**

In Section 2, we have introduced the algorithm as to whether or not a positive integer is k-Fibonacci or k-Lucas. Using these tests we give an algorithm to realize any given positive integer n whether or not a k-Fibonacci or k-Lucas number. This algorithm is created by MATLAB [13].

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TABLE 1. The Algorithm

```
n= input('n ');
k= input('k ');
t1=(k^2+4)*n^2+4;
t2=(k^2+4)*n^2-4;
t3=(k^2+4)*n^2+4*(k^2+4);
t4=(k^2+4)*n^2-4*(k^2+4);
yes_no=0;
if round(sqrt(t1))==sqrt(t1),
yes_no=1;
end
yes_no2=0;
if round(sqrt(t2))==sqrt(t2),
yes_no2=1;
end
yes_no3=0;
if round(sqrt(t3))==sqrt(t3),
yes_no3=1;
end
yes_no4=0;
if round(sqrt(t4))==sqrt(t4),
yes_no4=1;
end
if yes_no==1
fprintf('n is a k-Fibonacci number',t1);
else if yes_no2==1
fprintf('n is a k-Fibonacci number',t2);
else if yes_no3==1
fprintf('n is a k-Lucas number',t3);
else if yes_no4==1
fprintf('n is a k-Lucas number',t4);
else fprintf('n is a neither k-Fibonacci number nor k-Lucas number',t1);
end
end
end
end
```