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ON A $M^{[X]}/G/1$ QUEUEING SYSTEM WITH GENERALIZED COXIAN-2 SERVICE AND OPTIONAL GENERALIZED COXIAN-2 VACATION

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Abstract. We study the steady state behaviour of a batch arrival single server queue in which the first service with general service times G_1 is compulsory and the second service with general service times G_2 is optional. We term such a two phase service as generalized Coxian-2 service. Just after completion of a service the server may take a vacation of random length of time with general vacation times V_1. After completion of the first phase of vacation the server may or may not take the second optional vacation with general vacation times V_2. We term this two phase vacation as optional generalized Coxian-2 sever vacation. We obtain steady state probability generating functions for the queue size at a random epoch of time in explicit and closed forms. Some particular cases of interest including some known results have been derived.

1. Introduction

Many authors including [8], [13], [5], [7], [3, 4], [6], [14], [15], [16], [17], [2], [1], [18] and [10, 11, 12] have studied queues with server vacations, assuming various vacation policies including Bernoulli schedules. In the present paper, we study steady state behaviour of an $M^{[X]}/G/1$ queue with Bernoulli schedules and Coxian-2 server vacations, using the supplementary variable technique. The mathematical model of our study is briefly described by the following underlying assumptions:

- Customers arrive at the system in batches of variable size in accordance with a compound Poisson process. Let $\lambda c_i dt$ (for $i = 1, 2, 3, ...$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. The arriving batches wait in the queue in the order of their arrival. It is further assumed that customers with each

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batch are pre-ordered for the purpose of service.

- There is a single server who provides generalized Coxian-2 service which means essential first phase of service followed by optional second phase of service. The first phase of service is provided to all customers one by one on a first-come, first-served basis. Let S_1 and S_2 be the service times for phase 1 and phase 2, respectively. Let $A_1(s_1)$ and $a_1(s_1)$ respectively be the distribution function and the density function of the first phase service time and let $\mu_1(x)dx$ be the conditional probability of completion of first phase service, given that the elapsed time is x, so that

$$\mu_1(x) = \frac{a_1(x)}{1 - A_1(x)}, \tag{1.1}$$

and, therefore,

$$a_1(s_1) = \mu_1(s_1) \exp\left(- \int_0^{s_1} \mu_1(x)\, dx \right). \tag{1.2}$$

- After completion of the first phase of service, the server provides second phase of service which is optional. A customer may take second phase of service with probability α or may leave the system with probability $1 - \alpha$. Let $A_2(s_2)$ and $a_2(s_2)$ respectively be the distribution function and the density function of the second phase service time and let $\mu_2(x)dx$ be the conditional probability of completion of second phase service, given that the elapsed time is x, so that

$$\mu_2(x) = \frac{a_2(x)}{1 - A_2(x)}, \tag{1.3}$$

and, therefore,

$$a_2(s_2) = \mu_2(s_2) \exp\left(- \int_0^{s_2} \mu_2(x)\, dx \right). \tag{1.4}$$

- As soon as the service of a customer is completed, then with probability p the server may opt to take a vacation, or else with probability $1-p$ he may continue staying in the system. In queueing literature this phenomenon is termed as Bernoulli schedules. Most of the papers dealing with the Bernoulli schedules assume that on any service completion epoch the server may take a vacation with probability p or may not take a vacation with probability $1 - p$. They further assume that whenever the server becomes idle on completing a service (i.e., he servers the last customer present in the queue), he must necessarily take a vacation at such an epoch. However, in the present paper, we assume that on any service completion epoch the server may take a vacation with probability p or may not take a vacation with probability $1 - p$, irrespective of whether there are customers waiting in queue or not. We may term this phenomenon as uniform Bernoulli schedules.
- Whenever the server decides to take a vacation, his vacation period follows a generalized Coxian-2 distribution which implies that this period is bifurcated into two parts; phase 1 vacation followed by optional phase 2

vacation. Let V_1 and V_2 denote the long vacation and short vacation times, respectively. Let $B_1(v_1)$ and $b_1(v_1)$ respectively be the distribution function and the density function of the first phase vacation time and let $\mathcal{V}_1(x)dx$ be the conditional probability of completion of first phase vacation, given that the elapsed vacation time is x, so that

$$\mathcal{V}_1(x) = \frac{b_1(x)}{1 - B_1(x)}\,, \tag{1.5}$$

and, therefore,

$$b_1(v_1) = \mathcal{V}_1(v_1) \exp\left(- \int_0^{v_1} \mathcal{V}_1(x)\; dx \right)\,. \tag{1.6}$$

- After completion of the first phase of vacation, the server may take second phase of vacation with probability β or may return to the system with probability $1 - \beta$. Let$B_2(v_2)$ and $b_2(v_2)$ respectively be the distribution function and the density function of the second phase vacation time and let $\mathcal{V}_2(x)dx$ be the conditional probability of completion of second phase vacation, given that the elapsed time is x, so that

$$\mathcal{V}_2(x) = \frac{b_2(x)}{1 - B_2(x)}\,, \tag{1.7}$$

and, therefore,

$$b_2(v_2) = \mathcal{V}_2(v_2) \exp\left(- \int_0^{v_2} \mathcal{V}_2(x)\; dx \right)\,. \tag{1.8}$$

- On completion of a vacation the server instantly takes up a customer (at the head of the queue) for service if there are customers waiting in the queue. However, if on returning back the server finds the queue empty, the server remains idle until a new customer arrives in the system.
- Various stochastic processes involved in the system are independent of each other.

2. Definitions and Notations

We assume that $W_n^{(j)}(x,t)$, $j = 1,2$, is the probability that at time t, there are $n(\geq 0)$ customers in the queue excluding one customer in j-th phase service with elapsed service time x. Accordingly, $W_n^{(j)}(t) = \int_{x=0}^{\infty} W_n^{(j)}(x,t)\; dx$ denotes the probability that at time t, there are n customers in the queue excluding one customer in the j-th phase service irrespective of the value of x. Next, we define $V_n^{(j)}(x,t)$, $j = 1,2$, to be the probability that at time t, there are $n(\geq 0)$ customers in the queue and the server is on ,-th phase vacation with elapsed vacation time x. Accordingly, $V_n^{(j)}(t) = \int_{x=0}^{\infty} V_n^{(j)}(x,t)\; dx$ denotes the probability that at time t, there are n customers in the queue and the server is on j-th phase vacation irrespective of the value of x. Further, let $P_n(t) = \sum_{j=1}^{2} W_n^{(j)}(t) + \sum_{j=1}^{2} V_n^{(j)}(t)$ denote the probability that at time t there are $n(\geq 0)$ customers in the queue irrespective of whether the server is providing service or is on vacation. Finally, let $Q(t)$ be the probability that at time t, there is no customer in the system and the server is idle.

Further, for $j = 1, 2$, let the following denote the corresponding steady state probabilities:

$$\lim_{t\to\infty} W_n^{(j)}(x,t) = W_n^{(j)}(x)\,, \ \lim_{t\to\infty} W_n^{(j)}(t) = W_n^{(j)}\,,$$
$$\lim_{t\to\infty} V_n^{(j)}(x,t) = V_n^{(j)}(x)\,, \ \lim_{t\to\infty} V_n^{(k)}(t) = V_n^{(k)}\,,$$
$$\lim_{t\to\infty} P_n(t) = \sum_{j=1}^{2} \lim_{t\to\infty} W_n^{(j)}(t) + \sum_{j=1}^{2} \lim_{t\to\infty} V_n^{(j)}(t) = P_n$$
$$\lim_{t\to\infty} Q(t) = Q\,.$$

Next, we define the following probability generating functions (pgf's) for $|z| \le 1$ and $j = 1, 2$:

$$\begin{aligned} W^{(j)}(x,z) &= \sum_{n=0}^{\infty} z^n W_n^{(j)}(x)\,, \ W^{(j)}(z) = \sum_{n=0}^{\infty} z^n W_n^{(j)}\,, \\ V^{(j)}(x,z) &= \sum_{n=0}^{\infty} z^n V_n^{(j)}(x)\,, \ V^{(j)}(z) = \sum_{n=0}^{\infty} z^n V_n^{(j)}\,, \\ P(z) &= \sum_{n=0}^{\infty} z^n P_n = \sum_{n=0}^{\infty} z^n \left(\sum_{j=1}^{2} W_n^{(j)} + \sum_{j=1}^{2} V_n^{(j)} \right)\,, \\ C(z) &= \sum_{i=1}^{\infty} z^i c_i\,. \end{aligned} \tag{2.1}$$

Further, we define the Laplace-Steiltjes transform of the j-th phase service time:

$$\bar{A}^{(j)}[\lambda - \lambda C(z)] = \int_0^{\infty} e^{[\lambda - \lambda C(z)]x} dA^{(j)}(x)\,, \ j = 1, 2\,, \tag{2.2}$$

and the Laplace-Steiltjes transform of the k-th phase vacation time:

$$\bar{B}^{(j)}[\lambda - \lambda C(z)] = \int_0^{\infty} e^{[\lambda - \lambda C(z)]x} dB^{(j)}(x)\,, \ j = 1, 2\,. \tag{2.3}$$

3. Steady State Equations Governing the System

The usual probability arguments lead to the following steady state equations.

$$\frac{d}{dx} W_n^{(1)}(x) + (\lambda + \mu_1(x)) W_n^{(1)}(x) = \lambda \sum_{i=1}^{n} c_i W_{n-i}^{(1)}(x)\,, \ n \ge 1\,, \tag{3.1}$$

$$\frac{d}{dx} W_0^{(1)}(x) + (\lambda + \mu_1(x)) W_0^{(1)}(x) = 0\,, \tag{3.2}$$

$$\frac{d}{dx} W_n^{(2)}(x) + (\lambda + \mu_2(x)) W_n^{(2)}(x) = \lambda \sum_{i=1}^{n} c_i W_{n-i}^{(2)}(x)\,, \ n \ge 1\,, \tag{3.3}$$

$$\frac{d}{dx} W_0^{(2)}(x) + (\lambda + \mu_2(x)) W_0^{(2)}(x) = 0\,, \tag{3.4}$$

$$\frac{d}{dx}V_n^{(1)}(x)+(\lambda+\mathcal{V}_1(x))V_n^{(1)}(x)=\lambda\sum_{i=1}^{n}c_iV_{n-i}^{(1)}(x)\,,\ n\geq 1\,, \tag{3.5}$$

$$\frac{d}{dx}V_0^{(1)}(x)+(\lambda+\mathcal{V}_1(x))V_0^{(1)}(x)=0\,, \tag{3.6}$$

$$\frac{d}{dx}V_n^{(2)}(x)+(\lambda+\mathcal{V}_2(x))V_n^{(2)}(x)=\lambda\sum_{i=1}^{n}c_iV_{n-i}^{(2)}(x)\,,\ n\geq 1\,, \tag{3.7}$$

$$\frac{d}{dx}V_0^{(2)}(x)+(\lambda+\mathcal{V}_2(x))V_0^{(2)}(x)=0\,, \tag{3.8}$$

$$\begin{aligned}\lambda Q=(1-\beta)\int_0^\infty V_0^{(1)}(x)\mathcal{V}_1(x)\ dx+\int_0^\infty V_0^{(2)}(x)\mathcal{V}_2(x)\ dx\\ +(1-p)(1-\alpha)\int_0^\infty W_0^{(1)}(x)\mu_1(x)\ dx+(1-p)\int_0^\infty W_0^{(2)}(x)\mu_2(x)\ dx\,.\end{aligned} \tag{3.9}$$

Equations (3-1) through (3.8) are to be solved subject to the following boundary conditions, where $n\geq 0$:

$$\begin{aligned}W_n^{(1)}(0)=(1-p)(1-\alpha)\int_0^\infty W_{n+1}^{(1)}(x)\mu_1(x)\ dx\\ +(1-p)\int_0^\infty W_{n+1}^{(2)}(x)\mu_2(x)\ dx+(1-\beta)\int_0^\infty V_{n+1}^{(1)}(x)\mathcal{V}_1(x)\ dx\\ +\int_0^\infty V_{n+1}^{(2)}(x)\mathcal{V}_2(x)\ dx+\lambda c_{n+1}Q\,,\end{aligned} \tag{3.10}$$

$$W_n^{(2)}(0)=\alpha\int_0^\infty W_n^{(1)}(x)\mu_1(x)\ dx\,, \tag{3.11}$$

$$V_n^{(1)}(0)=p(1-\alpha)\int_0^\infty W_n^{(1)}(x)\mu_1(x)\ dx+p\int_0^\infty W_n^{(2)}(x)\mu_2(x)\ dx\,, \tag{3.12}$$

$$V_n^{(2)}(0)=\beta\int_0^\infty V_n^{(1)}(x)\mathcal{V}_1(x)\ dx\,. \tag{3.13}$$

4. Steady State Solution: Queue Size at a random Epoch

Theorem 4.1. *Under the model assumptions described above, the steady state probability generating function of the queue size at a random epoch is given by*

(4.1)

$$P(z)=\frac{\left[\begin{array}{c}\left(\bar{A}^{(1)}[\lambda-\lambda C(z)]-1\right)+\alpha\left(\bar{A}^{(2)}[\lambda-\lambda C(z)]-1\right)\\ +p\left(\bar{B}^{(1)}[\lambda-\lambda C(z)]-1\right)+p\beta\left(\bar{B}^{(2)}[\lambda-\lambda C(z)]-1\right)\end{array}\right]Q}{H(z)}\,,$$

Where

$$
\begin{aligned}
H(z) = z - (1-p)(1-\alpha)\bar{A}^{(1)}[\lambda - \lambda C(z)] \\
-(1-p)\alpha\bar{A}^{(1)}[\lambda - \lambda C(z)]\bar{A}^{(2)}[\lambda - \lambda C(z)] \\
-(1-\beta)p(1-\alpha)\bar{A}^{(1)}[\lambda - \lambda C(z)]\bar{B}^{(1)}[\lambda - \lambda C(z)] \\
-(1-\beta)p\alpha\bar{A}^{(1)}[\lambda - \lambda C(z)]\bar{A}^{(2)}[\lambda - \lambda C(z)]\bar{B}^{(1)}[\lambda - \lambda C(z)] \\
-\beta p(1-\alpha)\bar{A}^{(1)}[\lambda - \lambda C(z)]\bar{B}^{(1)}[\lambda - \lambda C(z)]\bar{B}^{(2)}[\lambda - \lambda C(z)] \\
-\beta p\alpha\bar{A}^{(1)}[\lambda - \lambda C(z)]\bar{A}^{(2)}[\lambda - \lambda C(z)]\bar{B}^{(1)}[\lambda - \lambda C(z)]\bar{B}^{(2)}[\lambda - \lambda C(z)]\,,
\end{aligned} \tag{4.2}
$$

$$
Q = 1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))\,, \tag{4.3}
$$

$E(I)$ is the mean size of the arriving batch, $E(S_1)$ is the mean service time of the first phase service, $E(S_2)$ is the mean service time of the second phase service, $E(V_1)$ is the mean vacation time of the first phase vacation and $E(V_2)$ is the mean vacation time of the second phase vacation.

Proof. Multiplying equation (3.1) by z^n, summing over n and adding the result to (3.2) and using (2.1) we get

$$
\frac{d}{dx}W^{(1)}(x,z) + (\lambda + \mu_1(x) - \lambda C(z))W^{(1)}(x,z) = 0\,. \tag{4.4}
$$

Similar operation on equations (3.3) and (3.4), (3.5) and (3.6), and (3.7) and (3.8) yield

$$
\frac{d}{dx}W^{(2)}(x,z) + (\lambda + \mu_2(x) - \lambda C(z))W^{(2)}(x,z) = 0\,, \tag{4.5}
$$

$$
\frac{d}{dx}V^{(1)}(x,z) + (\lambda + \mathcal{V}_1(x) - \lambda C(z))V^{(1)}(x,z) = 0\,, \tag{4.6}
$$

$$
\frac{d}{dx}V^{(2)}(x,z) + (\lambda + \mathcal{V}_2(x) - \lambda C(z))V^{(2)}(x,z) = 0\,. \tag{4.7}
$$

Next, we perform the similar operations on the boundary conditions (3.10), (3.11), (3.12), (3.13) and make use of equation (3.9). Thus we get

$$
\begin{aligned}
zW^{(1)}(0,z) = (1-p)(1-\alpha)\int_0^\infty W^{(1)}(x,z)\mu_1(x)\ dx \\
+(1-p)\int_0^\infty W^{(2)}(x,z)\mu_2(x)\ dx + (1-\beta)\int_0^\infty V^{(1)}(x,z)\mathcal{V}_1(x)\ dx \\
+\int_0^\infty V^{(2)}(x,z)\mathcal{V}_2(x)\ dx + (\lambda C(z) - \lambda)Q\,,
\end{aligned} \tag{4.8}
$$

$$
W^{(2)}(0,z) = \alpha\int_0^\infty W^{(1)}(x,z)\mu_1(x)\ dx\,, \tag{4.9}
$$

$$
V^{(1)}(0,z) = p(1-\alpha)\int_0^\infty W^{(1)}(x,z)\mu_1(x)\ dx + p\int_0^\infty W^{(2)}(x,z)\mu_2(x)\ dx\,, \tag{4.10}
$$

$$
V^{(2)}(0,z) = \beta\int_0^\infty V^{(1)}(x,z)\mathcal{V}_1(x)\ dx\,. \tag{4.11}
$$

Now we integrate equations (4.4) to (4.7) between the limits 0 and x and obtain

$$W^{(1)}(x,z) = W^{(1)}(0,z)\exp\left(-(\lambda-\lambda C(z))x - \int_0^x \mu_1(t)\,dt\right), \tag{4.12}$$

$$W^{(2)}(x,z) = W^{(2)}(0,z)\exp\left(-(\lambda-\lambda C(z))x - \int_0^x \mu_2(t)\,dt\right), \tag{4.13}$$

$$V^{(1)}(x,z) = V^{(1)}(0,z)\exp\left(-(\lambda-\lambda C(z))x - \int_0^x \mathcal{V}_1(t)\,dt\right), \tag{4.14}$$

$$V^{(2)}(x,z) = V^{(2)}(0,z)\exp\left(-(\lambda-\lambda C(z))x - \int_0^x \mathcal{V}_2(t)\,dt\right), \tag{4.15}$$

where $W^{(1)}(0,z)$, $W^{(2)}(0,z)$, $V^{(1)}(0,z)$ and $V^{(2)}(0,z)$ are given above in equations (4.8), (4.9), (4.10) and (4.11) respectively.

Next we again integrate equations (4.12) to (4.15) w.r.t. x by parts and obtain

$$W^{(1)}(z) = W^{(1)}(0,z)\left(\frac{1-\bar{A}^{(1)}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right), \tag{4.16}$$

$$W^{(2)}(z) = W^{(2)}(0,z)\left(\frac{1-\bar{A}^{(21)}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right), \tag{4.17}$$

$$V^{(1)}(z) = V^{(1)}(0,z)\left(\frac{1-\bar{B}^{(1)}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right), \tag{4.18}$$

$$V^{(2)}(z) = V^{(2)}(0,z)\left(\frac{1-\bar{B}^{(21)}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right). \tag{4.19}$$

Now we shall determine the integrals $\int_0^\infty W^{(1)}(x,z)\mu_1(x)\,dx$, $\int_0^\infty W^{(2)}(x,z)\mu_2(x)\,dx$, $\int_0^\infty V^{(1)}(x,z)\mathcal{V}_1(x)\,dx$ and $\int_0^\infty V^{(2)}(x,z)\mathcal{V}_2(x)\,dx$ appearing in the right sides of equations (4.8) to (4.11). For this purpose we multiply equations (4.4) to (4.7) by $\mu_1(x)$, $\mu_2(x)$, $\mathcal{V}_1(x)$ and $\mathcal{V}_2(x)$ respectively and integrate each w.r.t. x. Thus we obtain

$$\int_0^\infty W^{(1)}(x,z)\mu_1(x)\,dx = W^{(1)}(0,z)\bar{A}^{(1)}[\lambda-\lambda C(z)], \tag{4.20}$$

$$\int_0^\infty W^{(2)}(x,z)\mu_2(x)\,dx = W^{(2)}(0,z)\bar{A}^{(2)}[\lambda-\lambda C(z)], \tag{4.21}$$

$$\int_0^\infty V^{(1)}(x,z)\mathcal{V}_1(x)\,dx = V^{(1)}(0,z)\bar{B}^{(1)}[\lambda-\lambda C(z)], \tag{4.22}$$

$$\int_0^\infty V^{(2)}(x,z)\mathcal{V}_2(x)\,dx = V^{(2)}(0,z)\bar{B}^{(2)}[\lambda-\lambda C(z)]. \tag{4.23}$$

Utilizing equations (4.20) to (4.23) into equations (4.8) to (4.11) we get on simplifying

$$W^{(1)}(z) = \frac{\bar{A}^{(1)}[\lambda-\lambda C(z)]-1}{H(z)}, \tag{4.24}$$

$$W^{(2)}(z) = \frac{\bar{A}^{(2)}[\lambda - \lambda C(z)] - 1}{H(z)}, \tag{4.25}$$

$$V^{(1)}(z) = p\frac{\bar{B}^{(1)}[\lambda - \lambda C(z)] - 1}{H(z)}, \tag{4.26}$$

$$V^{(2)}(z) = p\beta\frac{\bar{B}^{(2)}[\lambda - \lambda C(z)] - 1}{H(z)}. \tag{4.27}$$

Further, adding equations (4.24), (4.25), (4.26) and (4.27), we get

$$P(z) = \frac{\left[\begin{array}{c} \left(\bar{A}^{(1)}[\lambda - \lambda C(z)] - 1\right) + \alpha\left(\bar{A}^{(2)}[\lambda - \lambda C(z)] - 1\right) \\ +p\left(\bar{B}^{(1)}[\lambda - \lambda C(z)] - 1\right) + p\beta\left(\bar{B}^{(2)}[\lambda - \lambda C(z)] - 1\right) \end{array}\right] Q}{H(z)}, \tag{4.28}$$

where $H(z)$ is given by (4.2).

Now in order to determine the only unknown constant Q, we employ the normalizing condition

$$Q + P(1) = 1. \tag{4.29}$$

We note that each factor in the right side of each of the equations (4.24) to (4.27) is indeterminate of the zero by zero form at $z = 1$. Therefor, employing L'Hopital's rule, we obtain

$$W^{(1)}(1) = \lim_{z\to 1} W^{(1)}(z) = \lim_{z\to 1}\left(\frac{\bar{A}^{(1)}[\lambda - \lambda C(z)] - 1}{H(z)}\right) Q = \frac{\lambda E(I)E(S_1)Q}{H(z)},$$

which is further simplified as

$$W^{(1)}(z) = \frac{\lambda E(I)E(S_1)Q}{1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))}. \tag{4.30}$$

Similarly equations (4.25) to (4.27) yield

$$\begin{aligned} W^{(2)}(1) &= \lim_{z\to 1} W^{(2)}(z) = \lim_{z\to 1} \alpha\left(\frac{\bar{A}^{(2)}[\lambda - \lambda C(z)] - 1}{H(z)}\right) Q \\ &= \frac{\lambda\alpha E(I)E(S_2)Q}{1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))}, \end{aligned} \tag{4.31}$$

$$\begin{aligned} V^{(1)}(1) &= \lim_{z\to 1} V^{(1)}(z) = \lim_{z\to 1} p\left(\frac{\bar{B}^{(1)}[\lambda - \lambda C(z)] - 1}{H(z)}\right) Q \\ &= \frac{\lambda p E(I)E(V_1)Q}{1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} V^{(2)}(1) &= \lim_{z\to 1} V^{(2)}(z) = \lim_{z\to 1} p\beta\left(\frac{\bar{B}^{(2)}[\lambda - \lambda C(z)] - 1}{H(z)}\right) Q \\ &= \frac{\lambda p\beta E(I)E(V_2)Q}{1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))}, \end{aligned} \tag{4.33}$$

$$P(1) = \frac{\lambda E(I)Q(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))}{1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2))}. \tag{4.34}$$

Using (4.34) in the normalising equation (4.29), we obtain

$$Q = 1 - \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2)) \,. \tag{4.35}$$

Further, we obtain the utilization factor of the system as

$$\rho = 1 - Q = \lambda E(I)(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2)) \,. \tag{4.36}$$

□

5. Steady State Solution: Mean Queue Size at a Random Epoch

Theorem 5.1. *Under the model assumptions described above, the steady state mean queue size at a random epoch is given by*

$$L_q = \lim_{z\to 1}\left(\frac{J_1 + \alpha J_2 + pK_1 + p\beta K_2}{(1-M)^2} + \frac{M[\alpha J_1 J_2 + 2p(\alpha K_1 J_2 + \alpha\beta J_2 K_2 + \beta K_1 K_2 + \beta J_1 K_2)]}{(1-M)^2} \right) . \tag{5.1}$$

where

$$\begin{aligned} J_1 &= \lambda E[I(I-1)]E(S_1) + (\lambda E(I))^2 E(S_1^2) \,, \\ J_2 &= \lambda E[I(I-1)]E(S_2) + (\lambda E(I))^2 E(S_2^2) \,, \\ K_1 &= \lambda E[I(I-1)]E(V_1) + (\lambda E(I))^2 E(V_1^2) \,, \\ K_2 &= \lambda E[I(I-1)]E(V_2) + (\lambda E(I))^2 E(V_2^2) \,, \\ M &= \lambda E[I]Q(E(S_1) + \alpha E(S_2) + pE(V_1) + p\beta E(V_2)) \,. \end{aligned} \tag{5.2}$$

Proof. Let L_q denote the steady state mean queue size at a random epoch. Then using (4.27) we get

$$L_q = \frac{d}{dz}P(z)\Big|_{z=1} = \frac{d}{dz}\left(\frac{N(z)}{H(z)}\right)\Big|_{z=1} ,$$

where

$$N(z) = \left[\begin{array}{c} \left(\bar{A}^{(1)}[\lambda - \lambda C(z)] - 1\right) + \alpha\left(\bar{A}^{(2)}[\lambda - \lambda C(z)] - 1\right) \\ +p\left(\bar{B}^{(1)}[\lambda - \lambda C(z)] - 1\right) + p\beta\left(\bar{B}^{(2)}[\lambda - \lambda C(z)] - 1\right) \end{array} \right] Q \,, \tag{5.3}$$

and $H(z)$ is given by (4.2) and Q is given by (4.3).

Since both $N(z)$ and $H(z)$ are zero at $z = 1$, we employ the following formula which uses L'Hopital's rule twice:

$$L_q = \lim_{z\to 1} \frac{H'(z)N''(z) - N'(z)H''(z)}{2(H'(z))^2} \,, \tag{5.4}$$

where primes and double primes in (5.4) denote the first and second derivative respectively at $z = 1$. Carrying out the derivatives and simplifying a lot of cumbersome algebra we obtain the results in (5.1) and (5.2). □

6. Particular Cases

Case 1: Two-phase general heterogeneous service and two-phase general vacation

The results of this case can be obtained from the above main results by putting $\alpha = 1$ and $\beta = 1$.

Case 2: Two-phase general heterogeneous service and one phase general vacation

The results of this case can be obtained from the above main results by putting $\alpha = 1$ and $\beta = 0$.

Case 3: One phase general service and two-phase general vacation

The results of this case can be obtained from the above main results by putting $\alpha = 0$ and $\beta = 1$.

Case 4: One phase general service and one phase general vacation

The results of this case can be obtained from the above main results by putting $\alpha = 0$ and $\beta = 0$.

Case 5: Two-phase general heterogeneous service and no vacation

The results of this case can be obtained from the above main results by putting $\alpha = 1$ and $p = 0$.

Case 6: One phase general service and no vacation

The results of this case can be obtained from the above main results by putting $\alpha = 0$ and $p = 0$.

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