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T_3 and T_4-Objects at p in the Category of Cauchy Spaces

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Abstract
There are various generalization of the usual topological T_3 and T_4- axioms to topological categories defined in [2] and [8]. [8] is shown that they lead to different T_3 and T_4 concepts, in general. In this paper, an explicit characterizations of each of the separation properties T_3 and T_4 at a point p and the generalized separation properties is given in the topological category of Cauchy spaces. Moreover, specific relationships that arise among the various T_i, $i = 0, 1, 2, 3, 4$, $PreT_2$, and T_2 structures at p and the generalized separation properties are examined in this category. Finally, we investigate the relationships between the generalized separation properties and the separation properties at a point p in this category.

Keywords: *Topological category; Cauchy space; Cauchy map; separation.*

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1. Introduction

The theory of Cauchy spaces was initiated by H. J. Kowalsky [19]. But H. Keller [18] was the first to give the axiomatic definition of Cauchy spaces in its currently used form, which is given briefly in the preliminaries section. In that paper, the relation between Cauchy spaces, uniform convergence spaces, and convergence spaces was developed. Several generalizations of Cauchy spaces have been studied, e.g. filtermerotopic spaces (shortly: filter spaces) by M. Katětov [17] in 1965 and semiuniform convergence spaces by Preuss [28] in 1995. The category of Cauchy spaces is also known to be a bireflective, finally dense subcategory of the category of filter spaces, **FIL** [27].

Baran [2] defined separation properties at a point, p i.e., locally (see [3], [5], [9] and [12]), then generalized this to point free definitions by using the generic element, [16] p. 39, method of topos theory for an arbitrary topological category over sets. One reason for doing this is that, in general, objects in a topos may not have points, however they always have a generic point. The other reason is that the notions of "closedness" and "strong closedness" on arbitrary topological catogories is defined in terms of T_0 and T_1 at a point, p. 335 [2]. The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [2], [4], [7] and it is shown in [9], [10], [12] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [15] in some well-known topological categories.

Some basic concepts in general topology are the notions of separation (T_0, T_1, T_2, T_3, T_4) which appear in many important theorems such as the Urysohn Metrization theorem, the Urysohn Lemma, the Tietze extension theorem, among others. In view of this, it is useful to be able to extend these various notions to arbitrary topological categories [2].

The main goal of this paper is

1. to give the characterization of each of the separation properties T_i, $i = 3, 4$ at a point p in the topological category of Cauchy spaces,

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2. to examine how these generalizations are related,

3. to show that a generalized separation property implies that separation property at p in the topological category of Cauchy spaces.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful, amnestic and transportable), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure [1], [5], [11], [24], or [26].

Recall in [1] or [26], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in$Ob $\mathcal{E}$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

A filter on a set X is a collection of subsets of X, containing X, which is closed under finite intersection and formation of supersets (it may contain $\emptyset$). Let $\mathbf{F}(X)$ denote the set of filters on X. If $\alpha,\ \beta\ \in \mathbf{F}(X)$, then $\beta \geq \alpha$ if and only if for each U$\in \alpha$, $\exists$V$\in \beta$ such that $V \subseteq U$, that is equivalent to $\beta \supset \alpha$. This defines a partial order relation on $\mathbf{F}(X)$. $\dot{x} = [\{x\}]$ is the filter generated by the singleton set $\{x\}$ where $[\cdot]$ means generated filter and $\alpha \cap \beta = [\{ \text{U} \cup \text{V} \mid \text{U} \in \alpha, \text{V} \in \beta \}]$. If U$\cap$V$\neq \emptyset$, for all U$\in \alpha$ and V$\in \beta$, then $\alpha \vee \beta$ is the filter $[\{\text{U} \cap \text{V} \mid \text{U} \in \alpha, \text{V} \in \beta \}]$. If $\exists$U$\in \alpha$ and V$\in \beta$ such that U$\cap$V=$\emptyset$, then we say that $\alpha \vee \beta$ fails to exist.

Let A be a set and q be a function on A that assigns to each point x of A a set of filters (proper or not, where a filter δ is proper iff δ does not contain the empty set, $\emptyset$, i.e., $\delta \neq [\emptyset]$) (the filters converging to x) is called a *convergence structure on* A ((A,q) a *convergence space* (in [26], it is called a convergence space)) iff it satisfies the following three conditions ([25] p. 1374 or [26] p. 142):

1. $[x] = [\{x\}] \in q(x)$ for each $x \in A$ (where $[F] = \{B \subset A\ :\ F \subset B\}$).
2. $\beta \supset \alpha \in q(x)$ implies $\beta \in q(x)$ for any filter β on A.
3. $\alpha \in q(x) \Rightarrow \alpha \cap [x] \in q(x)$.

A map $f : (A,q) \to (B,s)$ between two convergence spaces is called *continuous* iff $\alpha \in q(x)$ implies $f(\alpha) \in s(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D)\ : D \in \alpha\}$). The category of convergence spaces and continuous maps is denoted by $\mathbf{Con}$ (in [26] $\mathbf{Conv}$).

For filters α and β we denote by $\alpha \cup \beta$ the smallest filter containing both α and β.

Definition 2.1. (cf. [18]) Let A be a set and $K \subset \mathbf{F}(A)$ be subject to the following axioms:

1. $[x] = [\{x\}] \in K$ for each $x \in A$ (where $[x] = \{B \subset A\ :\ x \in B\}$);
2. $\alpha \in K$ and $\beta \geq \alpha$ implies $\beta \in K$ (i.e., $\beta \supset \alpha \in K$ implies $\beta \in K$ for any filter β on A);
3. if $\alpha,\ \beta \in K$ and $\alpha \vee \beta$ exists (i.e., $\alpha \cup \beta$ is proper), then $\alpha \cap \beta \in K$.

Then K is a precauchy (Cauchy) structure if it obeys 1-2 (resp. 1-3) and the pair (A,K) is called a precauchy space (Cauchy space), resp. Members of K are called Cauchy filters. A map $f : (A,K) \to (B,L)$ between Cauchy spaces is said to be Cauchy continuous (Cauchy map) iff $\alpha \in K$ implies $f(\alpha) \in L$ (where $f(\alpha)$ denotes the filter generated by $\{f(D)\ : D \in \alpha\}$). The concrete category whose objects are the precauchy (Cauchy) spaces and whose morphisms are the Cauchy continuous maps is denoted by $\mathbf{PCHY}$ ($\mathbf{CHY}$), respectively.

2.2. A source $\{f_i : (A,K) \to (A_i,K_i)\,,\ i \in I\}$ in $\mathbf{CHY}$ is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [22], [27] or [29].

2.3. An epimorphism $f : (A,K) \to (B,L)$ in $\mathbf{CHY}$ (equivalently, f is surjective) is a final lift iff $\alpha \in L$ implies that there exists a finite sequence $\alpha_1, ...,\ \alpha_n$ of Cauchy filters in K such that every member of α_i intersects every member of α_{i+1} for all $i < n$ and such that $\overset{n}{\underset{i=1}{\cap}} f(\alpha_i) \subset \alpha$ [22], [27] or [29].

2.4. Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p ([2] p. 334), i.e., two disjoint copies of B identified at p, i.e., the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in $\mathbf{Set}$). An epi sink $\{i_1,\ i_2 : (B,K) \to (B \vee_p B, L)\}$, where $i_1,\ i_2$ are the canonical injections, in $\mathbf{CHY}$ is a final lift if and only if the following statement holds. For any filter α on the wedge $B \vee_p B$, where either $\alpha \supset i_k(\alpha_1)$ for some $k = 1,2$ and some $\alpha_1 \in K$, or $\alpha \in L$, we have that there exist Cauchy filters $\alpha_1, \alpha_2 \in K$ such that every member of α_1 intersects every member of α_2 (i.e., $\alpha_1 \cup \alpha_2$ is proper) and $\alpha \supset i_1\alpha_1 \cap i_2\alpha_2$. This is a special case of 2.3.

2.5. The discrete structure (A, K) on A in **CHY** is given by $K = \{[a] \mid a \in A\} \cup \{[\emptyset]\}$ [22] or [27].
2.6. The indiscrete structure (A, K) on A in **CHY** is given by $K = F(A)$ [22] or [27].
CHY is a normalized topological category. The category of Cauchy spaces is cartesian closed, and contains the category of uniform spaces as a full subcategory [27].

3. T_3 and T_4-Objects at p

In this section, we give explicit characterizations of the generalized separation properties at p for the topological category of Cauchy spaces, **CHY**.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p ([2] p. 334), i.e., two disjoint copies of B identified at p, i.e., the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second component, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal p-axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$.

The fold map at p, $\bigtriangledown_p : B \vee_p B \to B$ is given by $\bigtriangledown_p(x_i) = x$ for $i = 1, 2$ [2], [4].

Note that the maps S_p and $\bigtriangledown_p$ are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\bigtriangledown_p i_j = id$, $j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at p.

Remark 3.1. We define $p_1,\ p_2$ by $1 + p$, $p + 1 : B \vee_p B \to B$, respectively where $1 : B \to B$ is the identity map, $p : B \to B$ is constant map at p (i.e., having value p). Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \nabla_p$, where $\pi_i : B^2 \to B$ is the i-th projection, $i = 1,\ 2$. When showing A_p and S_p are initial it is sufficient to show that (p_1 and p_2) and (p_1 and ∇_p) are initial lifts, respectively [2], [4].

The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, ..., p, x, p, ...)$ (infinite principal p-axis map), and $\bigtriangledown_p^\infty : \vee_p^\infty B \longrightarrow B$ by $\bigtriangledown_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), [2], [4].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place [10].

Definition 3.1. (cf. [2], [4] or [10]) Let $\mathcal{U} : \mathcal{E} \longrightarrow \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $*$ [2]. Let p be a point in B.

1. X is $\overline{T}_0$ at p iff the initial lift of the $\mathcal{U}$-source $\{A_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.

2. X is T_0' at p iff the initial lift of the $\mathcal{U}$-source $\{id : B \vee_p B \to \mathcal{U}(X \vee_p X) = B \vee_p B$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $X \vee_p X$ is the wedge in $\mathcal{E}$ i.e., the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ where i_1, i_2 denote the canonical injections.

3. X is T_1 at p iff the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.

4. *p is closed iff the initial lift of the $\mathcal{U}-$*source $\{A_p^\infty : \vee_p^\infty B \longrightarrow \mathcal{U}(X^\infty) = B^\infty$ *and* $\nabla_p^\infty : \vee_p^\infty B \longrightarrow \mathcal{U}\mathcal{D}(B) = B\}$ *is discrete.*

5. *$F \subset X$ is closed iff $\{*\}$*, the image of F, *is closed in X/F or $F = \emptyset$.*

6. *$F \subset X$ is strongly closed iff X/F is T_1 at $\{*\}$ or $F = \emptyset$.*

7. *If $B = F = \emptyset$, then we define F to be both closed and strongly closed.*

8. X is $Pre\overline{T}_2$ at p iff the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2\}$ and the initial lift of the $\mathcal{U}$-source $\{A_p : B \vee_p B \to \mathcal{U}(X^2) = B^2\}$ agree.

9. X is $PreT_2'$ at p iff the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ agree.

10. X is $\overline{T}_2$ at p iff X is $\overline{T}_0$ at p and $Pre\overline{T}_2$ at p.

11. X is T_2' at p iff X is T_0' at p and $PreT_2'$ at p.

Remark 3.2. 1. Note that for the category ***Top*** of topological spaces we have:

(i) $\overline{T}_0$ at p is equivalent to T_0' at p and they both reduce to the following (called T_0 at p in [6]): for each point x distinct from p, there exists a neighborhood of x missing p or there exists a neighborhood of p missing x [2].

(ii) The notion of closedness coincides with the usual closedness [2] and M is strongly closed iff M is closed and for each $x \notin M$ there exists a neighborhood of M missing x, [2]. If a topological space is T_1, then the notions of closedness and strong closedness coincide [2].

(iii) $Pre\overline{T}_2$ at p is equivalent to $PreT_2'$ at p and they both reduce to the following (called $PreT_2$ at p in [6]): for each point x distinct from p, if the set $\{x, p\}$ is not indiscrete, then there exist disjoint neighborhoods of p and x [2].

(iv) $\overline{T}_2$ at p is equivalent to T_2' at p and they both reduce to (called T_2 at p in [6]): for each point x distinct from p, there exist disjoint neighborhoods of x and p [2].

2. In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other [4]. Even if $X \in \mathcal{E}$ is T_1, where $\mathcal{E}$ is a topological category, then these notions are still independent of each other [8], [9].

Definition 3.2. (cf. [2], [4] or [10])

1. X is $S\overline{T}_3$ at p iff X is T_1 at p and X/F is $Pre\overline{T}_2$ at p for all strongly closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.
2. X is ST_3' at p iff X is T_1 at p and X/F is $PreT_2'$ at p for all strongly closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.
3. X is $\overline{T}_3$ at p iff X is T_1 at p and X/F is $Pre\overline{T}_2$ at p for all closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.
4. X is T_3' at p iff X is T_1 at p and X/F is $PreT_2'$ at p for all closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.

Definition 3.3. (cf. [2], [4] or [10])

1. X is $S\overline{T}_4$ at p iff X is T_1 at p and X/F is $S\overline{T}_3$ at $*$ for all strongly closed F in $\mathcal{U}(X)$ containing p.
2. X is ST_4' at p iff X is T_1 at p and X/F is ST_3' at $*$ for all strongly closed F in $\mathcal{U}(X)$ containing p.
3. X is $\overline{T}_4$ at p iff X is T_1 at p and X/F is $\overline{T}_3$ at $*$ for all closed F in $\mathcal{U}(X)$ containing p.
4. X is T_4' at p iff X is T_1 at p and X/F is T_3' at $*$ for all closed F in $\mathcal{U}(X)$ containing p.

Remark 3.3. Note that for the category ***Top*** of topological spaces we have:

(1) $\overline{T}_3$ at p and T_3' at p are equivalent and both reduce to: for each $x \neq p$ in $\mathcal{U}(X)$, there exists a neighborhood of x missing p and a neighborhood of p missing x, and for any nonempty closed set F missing p, there exist disjoint open sets containing F and p, respectively [2].

(2) $S\overline{T}_3$ at p and ST_3' at p are equivalent and both reduce to: for each $x \neq p$ in $\mathcal{U}(X)$, there exists a neighborhood of x missing p and a neighborhood of p missing x, and for any nonempty closed set F missing p for which each point x is not in F there exists a neighborhood of F missing x (i.e. F is strongly closed set), there exist disjoint open sets containing F and p, respectively [2].

(3) $\overline{T}_4$ at p and T_4' at p are equivalent and both reduce to: for each $x \neq p$ in $\mathcal{U}(X)$, there exists a neighborhood of x missing p and a neighborhood of p missing x, and for any disjoint strongly closed subset F and nonempty closed subset F' of $\mathcal{U}(X)$ with $p \in F$, there exist disjoint open sets containing F and F', respectively [2].

(4) $S\overline{T}_4$ at p and ST_4' at p are equivalent and they both reduce to: for each $x \neq p$ there exists a neighborhood of x missing p and a neighborhood of p missing x, and for any disjoint strongly closed subsets F and F' of $\mathcal{U}(X)$ with $p \in F$ and $F' \neq \emptyset$, there exist disjoint open sets containing F and F', respectively [2], [8], [9].

Theorem 3.1. *Let (A, K) be a Cauchy space and $p \in A$. (A, K) is $\overline{T}_0$ at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$ [21].*

Theorem 3.2. *All (A, K) in* **CHY** *are T_0' at p [21].*

Theorem 3.3. *Let (A, K) be a Cauchy space and $p \in A$. (A, K) is T_1 at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$ [20].*

Theorem 3.4. *All (A,K) in* $\mathbf{CHY}$ *are* $Pre\overline{T}_2$ *at* p *[21].*

Theorem 3.5. *Let (A,K) be in* $\mathbf{CHY}$. *(A,K) is $PreT_2'$ at p iff $\alpha \cap [p] \in K$ for any $\alpha \in K$ [21].*

Theorem 3.6. *(A,K) in* $\mathbf{CHY}$ *is $\overline{T}_2$ at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$ [21].*

Theorem 3.7. *(A,K) in* $\mathbf{CHY}$ *is T_2' at p iff for any $\alpha \in K$, we have $\alpha \cap [p] \in K$ [21].*

Theorem 3.8. *$\{p\}$ in A is closed for (A,K) in CHY iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$ [20].*

Theorem 3.9. *Let (A,K) be in CHY. $\emptyset \neq F \subset A$ is closed iff for each $a \in A$ with $a \notin F$ and for all $\alpha \in K$, $\alpha \cup [F]$ is improper or $\alpha \nsubseteq [a]$ [20].*

Theorem 3.10. *Let (A,K) be in CHY. $\emptyset \neq F \subset A$ is strongly closed iff for each $a \in A$ with $a \notin F$ and for all $\alpha \in K$, $\alpha \cup [F]$ is improper or $\alpha \nsubseteq [a]$ [20].*

Theorem 3.11. *A Cauchy space (A,K) is $S\overline{T}_3$ at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.4. □

Theorem 3.12. *A Cauchy space (A,K) is ST_3' at p iff A is a point or the empty set.*

Proof. Suppose (A,K) is ST_3' at p and Card $A > 1$. Since (A,K) is T_1 at p, by Theorem 3.3, for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$. If α is in K, $q(\alpha) \in L$, where L is the structure on A/F induced by q. Since $(A/F, L)$ is $PreT_2'$, by Theorem 3.5, $\alpha \cap [p] \in L$ for any $\alpha \in L$. F is closed iff by Definition 3.1, $*$ is closed in A/F iff by Theorem 3.8, for each $\alpha \in L$ such that $\alpha \neq [*]$, there exists $U \in \alpha$ such that $* \notin U$ iff for each $a \neq *$ in A/F, $[\{a,*\}] \notin L$. It follows easily that $p \neq *$ and $[p] \cap [*]$=$[\{p,*\}] \notin L$. This contradicts the fact that $(A/F, L)$ is $PreT_2'$ at p.

Conversely, $A = \{x\}$, i.e., a singleton, then clearly, (A,K) is ST_3'. □

Theorem 3.13. *A Cauchy space (A,K) is $\overline{T}_3$ at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.4. □

Theorem 3.14. *A Cauchy space (A,K) is T_3' at p iff A is a point or the empty set.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.5. □

Theorem 3.15. *A Cauchy space (A,K) is $S\overline{T}_4$ at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.11. □

Theorem 3.16. *A Cauchy space (A,K) is ST_4' at p iff A is a point or the empty set.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.12. □

Theorem 3.17. *A Cauchy space (A,K) is $\overline{T}_4$ at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.13. □

Theorem 3.18. *A Cauchy space (A,K) is T_4' at p iff A is a point or the empty set.*

Proof. It follows from Definition 3.2, Theorems 3.3 and Theorems 3.14. □

Remark 3.4. By Theorem 3.11, Theorem 3.12, Theorem 3.13, Theorem 3.14, Theorem 3.15, Theorem 3.16, Theorem 3.17 and Theorem 3.18, (A,K) is $S\overline{T}_3$ at p or $\overline{T}_3$ at p or $S\overline{T}_4$ at p or $\overline{T}_4$ at p if (A,K) is ST_3' at p or T_3' at p or ST_4' at p or T_4' at p. However, the converse is not true generally. For example, let $A = \{a,b\}$, $p = a$ and $K = \{[a], [b], [\emptyset]\}$. Then (A,K) is $S\overline{T}_3$ at p or $\overline{T}_3$ at p or $S\overline{T}_4$ at p or $\overline{T}_4$ at p but it is not (A,K) is ST_3' at p or T_3' at p or ST_4' at p or T_4' at p since Card$A > 1$.

4. Separation properties

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$(or $(x, y)_2$) if (x, y) is in the first (or second) component of $B^2 \vee_\Delta B^2$, respectively. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$ [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$. Note that $\pi_1 S = \pi_{11} = \pi_1 A$, $\pi_2 S = \pi_{21} = \pi_2 A$, $\pi_3 A = \pi_{12}$, and $\pi_3 S = \pi_{22}$, where $\pi_k : B^3 \to B$ the k-th projection $k = 1,\ 2,\ 3$ and $\pi_{ij} = \pi_i + \pi_j : B^2 \vee_\Delta B^2 \to B$, for $i, j \in \{1, 2\}$ [2].

Definition 4.1. (cf. [2] and [11]) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

1. X is $\overline{T_0}$ iff the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
2. X is T_0' iff the initial lift of the $\mathcal{U}$-source $\{id : B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ and $\mathcal{D}(B^2)$ is the discrete structure on B^2. Here, i_1 and i_2 are the canonical injections.
3. X is T_0 iff X does not contain an indiscrete subspace with (at least) two points [31] or [23].
4. X is T_1 iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete.
5. X is $Pre\overline{T_2}$ iff the initial lifts of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ coincide.
6. X is $PreT_2'$ iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ coincide, where i_1 and i_2 are the canonical injections.
7. X is $\overline{T_2}$ iff X is $\overline{T_0}$ and $Pre\overline{T_2}$ [2].
8. X is T_2' iff X is T_0' and $PreT_2'$ [2].
9. X is ST_2 iff Δ, the diagonal, is strongly closed in X^2 [4].
10. X is ΔT_2 iff Δ, the diagonal, is closed in X^2 [4].

Remark 4.1. 1. Note that for the category $\boldsymbol{Top}$ of topological spaces, $\overline{T_0}$, T_0', T_0, or T_1, or $Pre\overline{T_2}$, $PreT_2'$, or $\overline{T_2}$, T_2', ST_2, ΔT_2 reduce to the usual T_0, or T_1, or $PreT_2$ (where a topological space is called $PreT_2$ if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods), or T_2 separation axioms, respectively [2].
2. Let (X, τ) be a topological space. By Theorem 1.5 (5) of [6], (X, τ) is T_i, $i = 0, 1, 2$ iff (X, τ) is T_i at p for all p in X, $i = 0, 1, 2$.
3. For an arbitrary topological category,
(i) By Theorem 3.2 of [13] or Theorem 2.7(1) of [14], $\overline{T_0}$ implies T_0' but the converse of implication is generally not true. Moreover, there are no further implications between $\overline{T_0}$ and T_0 (see [13] 3.4(1) and (2)) and between T_0' and T_0 (see [13] 3.4(1) and (3)).
(ii) By Theorem 3.1(1) of [7], if X is $PreT_2'$, then X is $Pre\overline{T_2}$. But the converse of implication is generally not true.
4. By Theorem 2.6 and Corollary 2.7 of [5], if $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ is normalized, then $\overline{T_0}$, T_1, $Pre\overline{T_2}$, and $\overline{T_2}$ imply $\overline{T_0}$ at p, T_1 at p, $Pre\overline{T_2}$ at p, and $\overline{T_2}$ at p, respectively.

Definition 4.2. A Cauchy space (A, K) is said to be $\mathbf{T}_2$ if and only if $x = y$, whenever $[x] \cap [y] \in K$ [30].

Theorem 4.1. *(A, K) in* **CHY** *is $\overline{T}_0$ iff it is T_0 iff it is T_1 iff for each distinct pair x and y in A, we have $[x] \cap [y] \notin K$ [20].*

Theorem 4.2. *All objects (A, K) in* **CHY** *are T_0' [20].*

Remark 4.2. If a Cauchy space (A, K) is $\overline{T}_0$ or T_0 (T_1) then it is T_0'. However, the converse is not true generally. For example, let $A = \{x, y\}$ and $K = \{[x],\ [y],\ [\{x,\ y\}],\ [\emptyset]\}$. Then (A, K) is T_0' but it is not $\overline{T}_0$ or T_0 (T_1) [20].

Theorem 4.3. *All objects (A, K) in $\mathbf{CHY}$ are $Pre\overline{T}_2$ [20].*

Theorem 4.4. *Let (A, K) be in $\mathbf{CHY}$. (A, K) is $PreT_2'$ iff for each pair of distinct points x and y in A, we have $[x] \cap [y] \in K$(equivalently, for each finite subset F of A, we have $[F] \in K$) [20].*

Remark 4.3. If a Cauchy space (A, K) is $PreT_2'$ then it is $Pre\overline{T}_2$. However, the converse is not true, in general. For example, let $A = \{x, y\}$ and $K = \{[x], [y], [\emptyset]\}$. Then (A, K) is $Pre\overline{T}_2$ but it is not $PreT_2'$ [20].

Theorem 4.5. *Let (A, K) be a Cauchy space. (A, K) is $\overline{T}_2$ iff for each distinct pair x and y in A, we have $[x] \cap [y] \notin K$ [20].*

Theorem 4.6. *Let (A, K) be a Cauchy space. (A, K) is T_2' iff for each distinct points x and y in A, we have $[x] \cap [y] \in K$(equivalently, for each finite subset F of A, we have $[F] \in K$) [20].*

Remark 4.4. (A, K) be in $\mathbf{CHY}$. By Theorem 4.5 and Theorem 4.6, the following are equivalent:
(a) (A, K) is $\overline{T_2}$ and T_2'.
(b) A is a point or the empty set [20].

Corollary 4.1. *Let (A, K) be in $\mathbf{CHY}$. (A, K) is ST_2 iff it is ΔT_2 iff for each pair of distinct points x and y in A and for any $\alpha,\ \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$ [20].*

We give explicit relationships between the generalized separation properties T_i, $i = 0, 1, 2$, $PreT_2$, T_2, the usual $\mathbf{T}_2$, ST_2 or ΔT_2 and the separation properties at a point p in the topological category of Cauchy spaces.

Remark 4.5. Let (A, K) be a Cauchy space and $p \in A$. Since there is only one Cauchy structure on the empty set and on a point, $\mathcal{U} : \mathbf{CHY} \to \boldsymbol{Set}$ is normalized.

1. By Theorem 3.1, Theorem 4.1, Definition 4.2 and Corollary 4.1, (A, K) is $\overline{T}_0$ at p if (A, K) is $\overline{T}_0$ or $\mathbf{T}_2$ or T_1 or T_0 or ST_2 or ΔT_2.
2. By Theorem 3.3, Theorem 4.1, Definition 4.2 and Corollary 4.1, (A, K) is T_1 at p if (A, K) is T_1 or $\mathbf{T}_2$ or T_0 or $\overline{T}_0$ or ST_2 or ΔT_2.
3. By Theorem 3.2 and Theorem 4.2, (A, K) is T_0' at p for all $p \in A$ iff (A, K) is T_0'.
4. By Theorem 3.2, Definition 4.2 and Corollary 4.1, (A, K) is T_0' at p if (A, K) is $\mathbf{T}_2$ or $\overline{T}_0$ or T_0 or T_1 or ST_2 or ΔT_2.
5. By Theorem 3.4 and Theorem 4.3, (A, K) is $Pre\overline{T}_2$ at p for all $p \in A$ iff (A, K) is $Pre\overline{T}_2$.
6. By Theorem 3.4, Definition 4.2 and Corollary 4.1, (A, K) is $Pre\overline{T}_2$ at p if (A, K) is $\mathbf{T}_2$ or $\overline{T}_0$ or T_0 or T_1 or ST_2 or ΔT_2.
7. By Theorem 3.5 and Theorem 4.4(Theorem 3.7 and Theorem 4.6), (A, K) is $PreT_2'$ (T_2') at p for all $p \in A$ iff (A, K) is $PreT_2'$ (T_2').
8. By Theorem 3.5 (Theorem 3.7) and Definition 4.2, the following are equivalent:
(a) (A, K) is $PreT_2'$ (T_2') at p for all $p \in A$ and $\mathbf{T}_2$.
(b) A is a point or the empty set.
9. By Theorem 3.11, Theorem 3.13, Theorem 3.15, Theorem 3.17, Definition 4.2 and Corollary 4.1, (A, K) is $\overline{T}_3$ at p or (A, K) is $S\overline{T}_4$ at p or (A, K) is $\overline{T}_4$ at p if (A, K) is $\overline{T}_0$ or $\mathbf{T}_2$ or T_1 or T_0 or ST_2 or ΔT_2.
10. By Theorem 3.12, Theorem 3.14, Theorem 3.16, Theorem 3.18, (A, K) is ST_3' at p or (A, K) is T_3' at p or (A, K) is ST_4' at p or (A, K) is T_4' at p iff A is a point or the empty set.

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