MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
4 (1) 24-30 (2016) ©MSAEN

A Presentation of The Free Lie Algebra $F/\gamma_3(F)^{'}$

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(Communicated by Murat TOSUN)

Abstract
Let F be a free Lie algebra generated by the free generators x and y. By using the technique of Gröbner-Shirshov bases we show that the Lie algebra $F/\gamma_3(F)^{'}$ has the presentation $\langle x, y \mid \Delta \rangle$, where Δ is the minimal Gröbner basis of the algebra $\gamma_3(F)^{'}$.

Keywords: *Free Lie algebra, Presentation, Gröbner basis.*

AMS Subject Classification (2010): *Primary: 17B01.*

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1. Introduction

Gröbner-Shirshov method for Lie algebras invented by A.I.Shirshov in 1962 [11]. He defined a notion of composition of two Lie polynomials relative to an associative word (It was called lately by S- polynomial for commutative polynomials by B. Buchberger [8] and [9]). It leads to the algorithm for construction of a Gröbner-Shirshov basis of the Lie ideal generated by some set S. Shirshov has proved a lemma, now known as the Composition- Diamond Lemma. Several years later L.A. Bokut formulated this lemma in the modern form [2]. Shirshov's Composition-Diamond Lemma for associative algebras was formulated by L.A.Bokut [3] in 1976 and G.Bergman [1] in 1978.

The technique of Gröbner-Shirsov bases is very useful in the study of presentations of Lie algebras, associative algebras, groups, etc., by generators and defining relations (see [4], [5], [6], [7]).

In this work, we give a presentation of a free abelian-by-nilpotent Lie algebra.

We found Gröbner-Shirsov basis of the free Lie algebra $\gamma_3(F)^{'}$ and then we construct a presentation for the Lie algebra $F/\gamma_3(F)^{'}$ defined by generators and defining relations, where F is a free Lie algebra of rank 2 over a field of characteristic zero.

In [10] V.Drensky gave a detailed account of the Gröbner-Shirsov basis theory. We give some known definitions and results here referring to [10].

2. Gröbner basis

In this section we give some definitions and basic results about Gröbner-Shirsov basis which are given in [10].

Received : *18–August–2015,* ***Accepted :*** *30–December–2015*

This article is the written version of author's plenary talk delivered on September 03- August 31, 2015 at 4th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2015) in Athens, Greece.

Let a vector space A is called an (associative) algebra and a free associative algebra $K(X)$ is vector space for every set X. The algebra $K(X)$ has the following universal property. For any algebra A and any mapping $h : X \to A$ there exists a unique homomorphism (which we denote also by h) $h : K(X) \to A$ which extends the given mapping $h : X \to A$.

Definition 2.1. Let $A \cong K(X)/U$. Any generating set R of the ideal U is called a set of defining relations of A. We say that A is presented by the generating set X and the set of defining relations R and use the notation $A = K\langle X \mid R\rangle$ for the presentation of A or, allowing some freedom in the notation $A = K\langle X \mid R = 0\rangle$. If both sets X and R are finite, we say that A is finitely presented.

Let us define Hall basis ordering of the free Lie algebra F on X. Let $u = u_1u_2$, $v = v_1v_2$ be elements of H. If $length(u) > length(v)$, put $u > v$. If u and v have the same length, then put $u > v$ if and only if either $u_1 > v_1$ or $u_1 = v_1$ and $u_2 > v_2$. The introduced ordering has the very important property that the set (X) is well ordered. This allows to apply inductive arguments in our considerations.

Definition 2.2. (i) Let $f \in K(X)$,

$$f = \alpha u + \sum_{v<u} \beta_v v,\ u, v \in (X), \alpha, \beta_v \in K, \alpha \neq 0.$$

The word $f^{'} = u$ is called the leading word of f.

(ii) If $B \subset K(X)$ we denote by $B^{'} = \{f^{'} \mid 0 \neq f \in B\}$ the set of leading words of B.

(iii) The word $w \in (X)$ is called normal with respect to $B \subset K(X)$ if w does not contain as a subword a word of B.

Definition 2.3. Let $U \triangleright K(X)$. The set $G \subset U$ is called a Gröbner basis of U(or a complete system of defining relations of the algebra $A = K(X)/U$) if the sets of normal words with respect to G and U coincide. A tirivial example of a Gröbner basis of U is U itself.

Proposition 2.1. *For any $U \triangleright K(X)$, there exists a minimal (with respect to inclusion) Gröbner basis.*

Lemma 2.1. *(The Composition Lemma) Let $L(X)$ be a free Lie algebra and I be its ideal generated by a complete set S. The element $f \in L(X)$, $f \neq 0$, belongs to I only if the leading term $f^{'}$ of f contains a subword $s^{'}$, for some $s \in S$.*

Let us define free generating sets and Hall basis for $\gamma_m(F)$ which will used in this paper. We denote the Lie product on F by (ab), where $a, b \in F$. A word of length n is an ordered $n - tuples$ of the elements of X. We write $\ell(u)$ for the length of the word $u \in F$.

We construct a Hall basis H^{C_m} on C_m for the Lie algebra $\gamma_m(F)$, by forming products of elements of C_m such that C_m is a set of free generators for $\gamma_m(F)$.

Theorem 2.1. [12] *1) The set C_m defined as*

$$C_m = \{x = (a_1a_2) \mid x, a_1, a_2 \in H;\ \ell(x) \geq m; \ell(a_2) < m\}$$

is a set of free generators for $\gamma_m(F)$.
2) The set $C_{m,n}$ defined as

$$\begin{aligned} C_{m,n} &= \{x = (a_1a_2) \mid x \in H^{C_m}; C_m - \ell(x) \geq n; \\ C_n - \ell(a_2) &< n\} \end{aligned}$$

is a set of free generators for $\gamma_n(\gamma_m(L))$ where H^{C_m} is the Hall basis for $\gamma_m(F)$.

We will refer $C_m - \ell$ and $X - \ell$ meaning the number of letters used from C_m or X respectively. We order $C_{m,n}$ as follows: Let $g, h \in C_{m,n}$. If $C_m - \ell(g) < C_m - \ell(h)$, put $g < h$. If $C_m - \ell(g) = C_m - \ell(h)$ and $X - \ell(g) < X - \ell(h)$ then again we put $g < h$. Suppose both $C_m - \ell(g) = C_m - \ell(h)$ and $X - \ell(g) < X - \ell(h)$. Then put $g < h$ if either $g_2 < h_2$ or $g_2 = h_2$ and $g_1 < h_1$, where $g = g_1g_2$ and $h = h_1h_2$.

3. A Presentation of $F/\gamma_3(F)^{'}$

Our strategy is to obtain the minimal Gröbner basis (with respect to the inclusion) for the ideal $\gamma_3(F)^{'}$. After that we give a presentation for the algebra $F/\gamma_3(F)^{'}$.

We consider the Lie algebra defined by the presentation,

$$\langle x, y \mid \gamma_3(F)^{'} \rangle.$$

Then,

$$F/\gamma_3(F)^{'} \cong \langle x, y \mid \gamma_3(F)^{'} \rangle (1)$$

We eliminate certain types of basic words by finding minimal Gröbner basis for $\gamma_3(F)^{'}$ and introduce a refinement of the presentation 1.

Define the subset

$$\Delta = \{(ab), (a(ax)), (b(by))\}.$$

of F where $a = ((xy)x)$, $b = ((xy)y)$.

For the obtain of this presentation we need the following technical propositions.

Proposition 3.1. *The set Δ is the minimal Gröbner basis for $\gamma_3(F)^{'}$.*

Proof of this proposition can be obtained easily by using the following algorithm given in the proof of Proposition(4).

Proposition(4) give us the following algorithm for constructing a minimal Gröbner basis for $\gamma_3(F)^{'}$ which is generated by $C_{3,2}$.

Algorithm: The input is the set $C_{3,2}$. Output is the minimal Gröbner basis of $\gamma_3(F)^{'}$.
Step 1: We start with the words of minimal length in $C_{3,2}$ (i.e. the words u such that $C_3 - \ell(u) = 2$ and $X - \ell(u) = 6$) and we construct a subset G_1 of $C_{3,2}$ such that no word of G_1 is a proper subword of another word of G_1 and the set of normal words with respect to $C_{3,2}$ and G_1 are the same.

Step 2: Construct a subset G_2 of G_1 which contains the words v of $C_3 - \ell(v) \geq 2$ and $X - \ell(v) > 6$ such that no word of G_2 is a proper subword of another word of G_2 and the set of normal words with respect to G_1 and G_2 are the same.

Step 3: G_2 is a minimal Gröbner basis of $\gamma_3(F)^{'}$. Put $\Delta = G_2$.
The following proposition can be obtained easily, by using the Jacobi identity.

Proposition 3.2. *Let a, b be any monomials of F then*

$$((ab)x^s) = \sum_{k=0}^{s} \binom{s}{k} (ax^{s-k})(bx^k)$$

Proof. We prove the Lemma by induction on s. For $s = 1$,

$$\begin{aligned} ((ab)x^1) &= -((bx)a) - ((xa)b) \\ &= (a(bx)) + ((ax)b) \\ &= \sum_{k=0}^{1} \binom{1}{k} (ax^{1-k})(bx^k) \end{aligned}$$

Assume that the assertion it is true for $s-1$, that is;

$$((ab)x^{s-1}) = \sum_{k=0}^{s-1} \binom{s-1}{k} (ax^{s-1-k})(bx^k)$$

Then,

$$\begin{aligned}
((ab)x^s) &= (((ab)x^{s-1})x) \\
&= (\sum_{k=0}^{s-1}\binom{s-1}{k}(ax^{s-1-k})(bx^k))x \\
&= \sum_{k=0}^{s-1}\binom{s-1}{k}(ax^{s-1-k})(bx^{k+1}) + \sum_{k=0}^{s-1}\binom{s-1}{k}(ax^{s-k})(bx^k) \\
&= \sum_{k=0}^{s-1}\binom{s-1}{k}(ax^{s-1-k})(bx^{k+1}) + \sum_{k=0}^{s-1}\binom{s-1}{k+1}(ax^{s-1-k})(bx^{k+1}) \\
&\quad + \binom{s-1}{0}((ax^s)b) \\
&= \sum_{k=0}^{s-1}(\binom{s-1}{k} + \binom{s-1}{k+1})(ax^{s-1-k})(bx^{k+1}) + \binom{s-1}{0}((ax^s)b) \\
&= \sum_{k=0}^{s-1}\binom{s}{k+1}(ax^{s-1-k})(bx^{k+1}) + \binom{s-1}{0}((ax^s)b) \\
&= \sum_{k=0}^{s-1}\binom{s}{k}(ax^{s-k})(bx^k) - \binom{s}{0}((ax^s)b) \\
&\quad + \binom{s}{s}(a(bx^s)) + \binom{s-1}{0}((ax^s)b) \\
&= \sum_{k=0}^{s}\binom{s}{k}(ax^{s-k})(bx^k).
\end{aligned}$$

□

The proof of the following proposition is a consequence of the Composition Lemma and Proposition (4).

Proposition 3.3. *The following monomials belong to the ideal* $\langle\Delta\rangle$.
1) $(a(by^j))$,
2) $(ax^i)(ax^j)$, $i > j$,
3) $(by^i)(by^j)$, $i > j$,
4) $(ax^i)(by^j)$, $i \geq j$,
5) $(ax^i)((by^j)x^s)$,
6) $(by^i)((by^j)x^s)$,
7) $((by^i)x^j)((by^r)x^s)$,
8) $((ax^i)(xy)^r)(ax^j)$,
9) $((ax^i)(xy)^r)(by^j)$,
10) $((ax^i)(xy)^r)((by^j)x^s)$,
11) $((by^i)(xy)^r)(by^j)$,
12) $((by^i)(xy)^r)((by^j)x^s)$,
13) $(((by^i)x^t)(xy)^r)((by^j)x^s)$,
14) $((by^i)(xy)^r)(ax^j)$,
15) $(((by^i)x^s)(xy)^r)(ax^j)$,
16) $((ax^i)(xy)^r)((ax^j)(xy)^m)$,
17) $((ax^i)(xy)^r)((by^j)(xy)^m)$,
18) $((ax^i)(xy)^r)(((by^j)x^s)(xy)^m)$,
19) $((by^i)(xy)^r)((by^j)(xy)^m)$,
20) $((by^i)(xy)^r)(((by^j)x^s)(xy)^m)$,
21) $(((by^i)x^t)(xy)^r)(((by^j)x^s)(xy)^m)$,
22) $(a(ay^j))$,
23) $(b(bx^j))$.

Proof. 1) $(a(by^j))$ can be written as $-((by^j)a)$. Applying the Jacobi identity for $((ba)y^j)$, we obtain the following statement,

$$\begin{aligned}((ba)y^j) &= \sum_{k=0}^{j}\binom{j}{k}(by^{j-k})(ay^k)\\ &= ((by^j)a)+\sum_{k=1}^{j}\binom{j}{k}(by^{j-k})(ay^k).\end{aligned}$$

In this case, $((by^j)a))$ can be written as follows:

$$((by^j)a)) = ((ba)y^j) - \sum_{k=1}^{j}\binom{j}{k}(by^{j-k})(ay^k).$$

Since $((ba)y^j) = (((ba)y^{j-k})y^k)$ and $(by^{j-k})(ay^k) < ((ba)y^j)$ then the leading term of $((by^j)a))$ is $((ba)y^j)$. When we consider the equality $((ba)y^j) = -((ab)y^j)$ we get (ab) is included as a subword in the set Δ. Hence, $((ab)y^j)$ is an element of the ideal $\langle\Delta\rangle$. So, $((by^j)a)) = -\,(a(by^j))$ is an element of the ideal $\langle\Delta\rangle$.
The proof of cases that up to 2-to-7 are obtained similarly to the case 1.
8) Let $(xy) = z$. Applying the Jacobi identity we obtain,

$$\begin{aligned}((ax^i)(ax^j))z^r) &= \sum_{k=0}^{r}\binom{r}{k}((ax^i)z^{r-k})((ax^j)z^k)\\ &= ((ax^i)z^r)(ax^j)+\sum_{k=1}^{r}\binom{r}{k}((ax^i)z^{r-k})((ax^j)z^k).\end{aligned}$$

In this case, $((ax^i)z^r)(ax^j)$ can be written as follows:

$$((ax^i)z^r)(ax^j) = ((ax^i)(ax^j))z^r) - \sum_{k=1}^{r}\binom{r}{k}((ax^i)z^{r-k})((ax^j)z^k).$$

Here, $((ax^i)(ax^j))z^r) = (((ax^i)(ax^j))z^{r-k})z^k)$. Since, $((ax^i)z^{r-k})((ax^j)z^k) < ((ax^i)(ax^j))z^r)$ then the leading term of $((ax^i)(ax^j))z^r)$ is $(ax^i)(ax^j)$. In this case, $(ax^i)(ax^j)$ is included as a subword in the word of $((ax^i)(ax^j))z^r)$. Since, $(ax^i)(ax^j)$ is an element of the ideal of $\langle\Delta\rangle$ from the Case 1 we get$((ax^i)(ax^j))(xy)^r)$ is an element of the ideal $\langle\Delta\rangle$. Hence, $((ax^i)z^r)(ax^j)$ is an element of the ideal $\langle\Delta\rangle$.
The proof of cases that up to 9-to-21 are obtained similarly to the case 8.

22) Applying the Jacobi identity for $(a(ay^j))$ we obtain,

$$\begin{aligned}(a(ay^j)) &= (a(((xy)x)y^j)\\ &= (a(\sum_{k=0}^{j}\binom{j}{k}((xy)y^{j-k})(xy^k)))\\ &= a(((xy)y^j)x)+\sum_{k=1}^{j}\binom{j}{k}((xy)y^{j-k})(xy^k)\\ &= ((xy)y^j)x)((xy)x)+\sum_{k=1}^{j}\binom{j}{k}(((xy)y^{j-k})(xy^k))((xy)x)\end{aligned}$$

The first term is an element of the ideal $\langle\Delta\rangle$ from the Case 5 and the second term is an element of the ideal $\langle\Delta\rangle$ from the Case 14. Hence, $(a(ay^j))$ is an element of the ideal $\langle\Delta\rangle$. The proof of case 23 is obtained similarly to the case 22. $\square$

Theorem 3.1. *Every element of* $\gamma_3(F)^{'}$ *belongs to the ideal* $\langle\Delta\rangle$.

Proof. For every $w \in \gamma_3(F)^{'}$, the form of w is $w = \sum_k \alpha_k((((c_1c_2)c_3)...)c_k)$ where $c_i \in C_{3,2}$ for $i = 1, 2, ..., k$. Every c_i can be written in the form $((((u_{i1}u_{i2})u_{i3})...)u_{is})$ $(s \geq 2)$ such that $u_{ik} \in C_3$. Considering the all possibilities for u_{i1} and u_{i2}, we obtain the following words:

$$\begin{array}{ll} (ax^i)(xy)^j & , i, j \geq 0 \\ ((by^m)x^n)(xy)^p & , m, n, p \geq 0 \end{array}$$

So, $(u_{i1}u_{i2})$ is expressed as follows:

$$(u_{i1}u_{i2}) = ((ax^i)(xy)^j)(((by^m)x^n)(xy)^p)$$

By Proposition(9), $(u_{i1}u_{i2})$ is an element of the ideal $\langle\Delta\rangle$. Thus, $c_i \in \langle\Delta\rangle$ and so $w \in \langle\Delta\rangle$. Hence, every element of $\gamma_3(F)^{'}$ belongs to the ideal $\langle\Delta\rangle$.

Theorem 3.2. *The Lie algera* $F/\gamma_3(F)^{'}$ *admits the following presentation*

$$\langle x, y \mid \Delta\rangle.$$

Proof. Since $\gamma_3(F)^{'}$ contained in the ideal $\langle\Delta\rangle$, we obtain

$$F/\gamma_3(F)^{'} = \langle x, y \mid \gamma_3(F)^{'}\rangle = \langle x, y \mid \langle\Delta\rangle\rangle = \langle x, y \mid \Delta\rangle.$$

□

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