Geometric and Analytic Connections of the Jensen and Hermite-Hadamard Inequality

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(Communicated by Nihal YILMAZ ÖZGÜR)

Abstract
The aim of this paper is to present connections between the Jensen and Hermite-Hadamard inequality. The study includes convex functions of one and several variables. The basis of the research are convex combinations with the common center.

Keywords: convex combination; convex function; simplex; Jensen's inequality.

AMS Subject Classification (2010): Primary: 26B25; Secondary: 52A40; 52B11.

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1. Introduction

In summary form, we present the concept of convexity and affinity by using binomial combinations.

Let $\mathcal{X}$ be a real linear space. Let $a, b \in \mathcal{X}$ be points and let $\alpha, \beta \in \mathbb{R}$ be coefficients. Their binomial combination $\alpha a + \beta b$ is convex if $\alpha, \beta \geq 0$ and if $\alpha + \beta = 1$. If $c = \alpha a + \beta b$, then the point c itself is called the combination center.

A subset of $\mathcal{X}$ is convex if it contains all binomial convex combinations of its points. The convex hull $\mathrm{conv}\mathcal{S}$ of a set $\mathcal{S} \subseteq \mathcal{X}$ is the smallest convex set which contains $\mathcal{S}$, and it consists of all binomial convex combinations of the points of $\mathcal{S}$.

Let $\mathcal{C} \subseteq \mathcal{X}$ be a convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is convex if the inequality

$$f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b) \tag{1.1}$$

holds for all binomial convex combinations $\alpha a + \beta b$ of pairs of points $a, b \in \mathcal{C}$.

Using the adjective affine instead of convex, requiring the coefficients condition $\alpha + \beta = 1$, and requiring the equality in equation (1.1), we get a characterization of the affinity.

Implementing mathematical induction, it can be verified that the above concept applies to n-membered affine or convex combinations for any positive integer $n > 2$.

We present the discrete and integral form of the famous Jensen's inequality using convex and measurable sets.

In 1905, applying mathematical induction to convex combinations, Jensen (see [6]) has obtained the following discrete inequality.

Discrete form of the Jensen inequality. Let $\mathcal{C}$ be a convex set of a real linear space, and let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of points $a_i \in \mathcal{C}$.

Then every convex function $f : \mathcal{C} \to \mathbb{R}$ satisfies the inequality

$$f\left(\sum_{i=1}^{n} \alpha_i a_i\right) \leq \sum_{i=1}^{n} \alpha_i f(a_i). \tag{1.2}$$

In 1906, working on transition to integrals, Jensen (see [7]) has stated the another form.

Integral form of the Jensen inequality. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. Let $\mathcal{A}$ be a measurable set of a space of positive measure μ so that $\mu(\mathcal{A}) > 0$, and let $g : \mathcal{A} \to \mathbb{R}$ be an integrable function so that $g(\mathcal{A}) \subseteq \mathcal{I}$.

Then every convex function $f : \mathcal{I} \to \mathbb{R}$, such that $f(g)$ is integrable, satisfies the inequality

$$f\left(\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} g(x)\,d\mu \right) \leq \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(g(x))\,d\mu. \tag{1.3}$$

Because of its attractiveness, the Jensen and related inequalities were studied during the whole last century. So, there are the Steffensen, Brunk and Olkin inequality. In this century the research goes on and we got the Jensen-Mercer and Mercer-Steffensen inequality. For more information concerning these inequalities, one may refer to the in book [9], and the papers in [1], [11], [12] and [13].

In 1883, studying convex functions, Hermite (see [5]) has attained the important inequality. In 1893, not knowing Hermite's result, Hadamard (see [4]) has gotten its first part, the left inequality in equation (1.4).

Classic form of the Hermite-Hadamard inequality. Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval.

Then every convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2}. \tag{1.4}$$

For information as regards the Hermite-Hadamard inequality, one may refer to book [3], and the papers in [2], [8], [10] and [14].

1.1 Set Barycenter in Euclidean Space

The notion of the set barycenter is important for inequalities involving integrals.

Let $\mathcal{A} \subseteq \mathbb{R}^k$ be a set, let μ be a positive measure on $\mathbb{R}^k$ such that $\mu(\mathcal{A}) > 0$, and let $g : \mathcal{A} \to \mathbb{R}$ be an integrable function.

The barycenter of the set $\mathcal{A}$ respecting the measure μ is the point

$$\left(\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} x_1\,d\mu, \ldots, \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} x_k\,d\mu \right) \tag{1.5}$$

belonging to the convex hull of the set $\mathcal{A}$.

The integral mean of the function g respecting the measure μ is the number

$$\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} g(x_1, \ldots, x_k)\,d\mu \tag{1.6}$$

belonging to the convex hull of the set $g(\mathcal{A})$.

Let $g_1, \ldots, g_k : \mathcal{A} \to \mathbb{R}$ be integrable functions. Let $\mathcal{C} \subseteq \mathbb{R}^k$ be a convex set containing the image set $\{(g_1(x), \ldots, g_k(x)) : x \in \mathcal{A}\}$. Then every convex function $f : \mathcal{C} \to \mathbb{R}$, such that $f(g_1, \ldots, g_k)$ is integrable, satisfies the inequality

$$f\left(\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} g_1\,d\mu, \ldots, \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} g_k\,d\mu \right) \leq \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(g_1, \ldots, g_k)\,d\mu. \tag{1.7}$$

The above inequality presents the integral form of the Jensen inequality for convex functions of several variables. If the function f is affine, then the equality holds in equation (1.7). The same is true for equations (1.2), (1.3) and (1.4).

2. Geometric Presentation of Inequalities

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, let $f : \mathcal{I} \to \mathbb{R}$ be a convex function, and let $c = \sum_{i=1}^n \alpha_i a_i$ be a convex combination of points $a_i \in \mathcal{I}$. To explain the Jensen inequality visually, we take the convex combination $C = \sum_{i=1}^n \alpha_i A_i$ of the points $A_i = (a_i, f(a_i))$ belonging to the graph of f. The center c is in the convex hull of line points a_i,

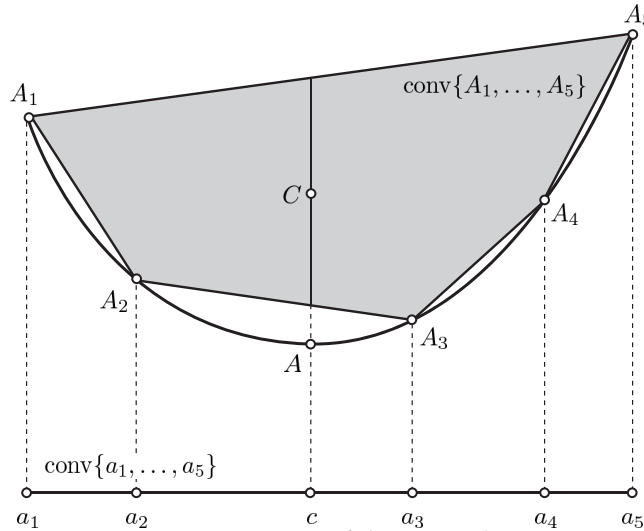$$c \in \text{conv}\{a_1, \ldots, a_n\}, \tag{2.1}$$

Figure 1. Geometric presentation of the inequality in equation (1.2)

as well as the center C is in the convex hull of plane points A_i,

$$C \in \text{conv}\{A_1, \ldots, A_n\}. \tag{2.2}$$

The convex hull in equation (2.2) is the convex polygon inscribed in the function epigraph. So, the polygon point

$$C = \left(\sum_{i=1}^{n} \alpha_i a_i, \sum_{i=1}^{n} \alpha_i f(a_i) \right) = \left(c, \sum_{i=1}^{n} \alpha_i f(a_i) \right) \tag{2.3}$$

is above the graph point

$$A = \left(c, f(c) \right) = \left(c, f \left(\sum_{i=1}^{n} \alpha_i a_i \right) \right), \tag{2.4}$$

which can be seen in Figure 1. Therefore, the relation between ordinates of points A and C,

$$f \left(\sum_{i=1}^{n} \alpha_i a_i \right) \leq \sum_{i=1}^{n} \alpha_i f(a_i),$$

represents the classical discrete form of the Jensen inequality for convex functions of one variable.

The visual explanation of the Hermite-Hadamard inequality for a convex function $f : [a, b] \to \mathbb{R}$ depends on two lines. We use the secant line function $f^{\text{sec}}_{\{a,b\}}$ whose graph passes through the points $(a, f(a))$ and $(b, f(b))$, and for $c \in (a, b)$ a supporting line function $f^{\text{sup}}_{\{c\}}$ whose graph passes through the point $(c, f(c))$.

So we get the curvilinear trapezoid which contains the supporting trapeze, and which is contained in the bounding trapeze. This is clearly seen in Figure 2. By calculating the areas of all three shapes, it follows that

$$(b - a) f^{\text{sup}}_{\{c\}} \left(\frac{a + b}{2} \right) \leq \int_a^b f(x) \, dx \leq (b - a) f^{\text{sec}}_{\{a,b\}} \left(\frac{a + b}{2} \right). \tag{2.5}$$

We could also obtain the above result by integrating the double inequality

$$f^{\text{sup}}_{\{c\}}(x) \leq f(x) \leq f^{\text{sec}}_{\{a,b\}}(x)$$

over the interval $[a, b]$.

Applying the formula $f^{\text{sup}}_{\{c\}}(c) = f(c)$ to the midpoint $c = (a+b)/2$ to the supporting line function (the supporting trapeze area attains a maximum value at the midpoint), we get

$$f^{\text{sup}}_{\{\frac{a+b}{2}\}} \left(\frac{a + b}{2} \right) = f \left(\frac{a + b}{2} \right). \tag{2.6}$$
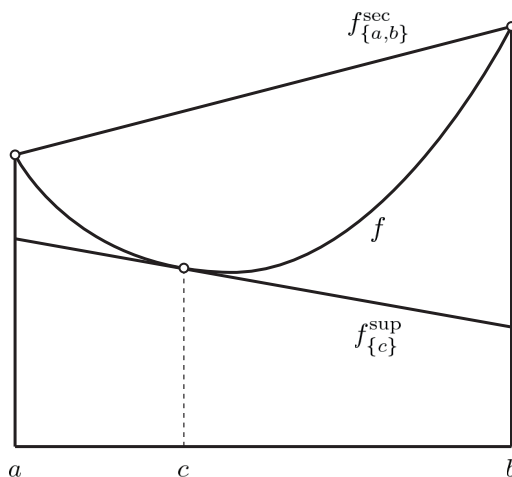
Figure 2. Geometric presentation of the inequality in equation (2.5)

Using the affinity of the secant line function, we obtain

$$f_{\{a,b\}}^{\text{sec}}\left(\frac{a+b}{2}\right) = \frac{1}{2}f_{\{a,b\}}^{\text{sec}}(a) + \frac{1}{2}f_{\{a,b\}}^{\text{sec}}(b) = \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Involving the above equalities to equation (2.5) and dividing with $b - a$, we achieve the classical form of the Hermite-Hadamard inequality,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

3. Preceding Inequality for Functions of One Variable

The aim of this section is give the basic inequality from which can be derived both the Jensen and Hermite-Hadamard inequality.

Through the section we will use a bounded closed interval $[a,b] \subset \mathbb{R}$ with endpoints $a < b$, and a convex function $f : [a,b] \to \mathbb{R}$.

Every number $x \in \mathbb{R}$ can be presented by the unique affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b, \tag{3.1}$$

which is convex if and only if the number x belongs to the interval $[a,b]$. Applying the affinity of $f_{\{a,b\}}^{\text{sec}}$ to the equality in (3.1), we obtain the secant line equation

$$f_{\{a,b\}}^{\text{sec}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \tag{3.2}$$

The function f has both left and right derivative on (a,b). Given a point $c \in (a,b)$ and a slope $\kappa \in [f'(c-), f'(c+)]$, we get the supporting line equation

$$f_{\{c\}}^{\text{sup}}(x) = \kappa(x-c) + f(c). \tag{3.3}$$

Finally, we have the double inequality

$$f_{\{c\}}^{\text{sup}}(x) \le f(x) \le f_{\{a,b\}}^{\text{sec}}(x), \ \ x \in [a,b]. \tag{3.4}$$

Theorem 3.1. *Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval, let $\sum_{i=1}^n \alpha_i a_i$ be a convex combination of points $a_i \in [a,b]$, and let $\alpha a + \beta b$ be the unique convex combination of endpoints a and b such that*

$$\alpha a + \beta b = \sum_{i=1}^n \alpha_i a_i. \tag{3.5}$$

Then every convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\alpha a + \beta b\right) \le \sum_{i=1}^{n} \alpha_i f(a_i) \le \alpha f(a) + \beta f(b). \tag{3.6}$$

Proof. We put $c = \sum_{i=1}^{n} \alpha_i a_i$. Since $c \in [a, b]$, we have two cases depending on the position of the center c.

If c is the endpoint, $c \in \{a, b\}$, then the trivial inequality $f(c) \le f(c) \le f(c)$ represents the inequality in equation (3.6).

If c is the interior point, $c \in (a, b)$, then we use a supporting function $f_{\{c\}}^{\text{sup}}$ and the secant function $f_{\{a,b\}}^{\text{sec}}$ to obtain the series of inequalities

$$
\begin{aligned}
f\left(\alpha a + \beta b\right) &= f_{\{c\}}^{\text{sup}}\left(\alpha a + \beta b\right) = \sum_{i=1}^{n} \alpha_i f_{\{c\}}^{\text{sup}}(a_i) \\
&\le \sum_{i=1}^{n} \alpha_i f(a_i) \\
&\le \sum_{i=1}^{n} \alpha_i f_{\{a,b\}}^{\text{sec}}(a_i) = f_{\{a,b\}}^{\text{sec}}\left(\alpha a + \beta b\right) \\
&= \alpha f(a) + \beta f(b)
\end{aligned}
\tag{3.7}
$$

including the double inequality in equation (3.6). $\square$

According to representing equation (3.1), the coefficients α and β in Theorem 3.1 are determined with fractions

$$\alpha = \frac{b - c}{b - a}, \; \beta = \frac{c - a}{b - a}. \tag{3.8}$$

Corollary 3.1. *Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval, let μ be a positive measure on $\mathbb{R}$ such that $\mu([a, b]) > 0$, and let $\alpha a + \beta b$ be the unique convex combination of endpoints a and b such that*

$$\alpha a + \beta b = \frac{1}{\mu([a, b])} \int_{[a,b]} x \, d\mu. \tag{3.9}$$

Then every convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\alpha a + \beta b\right) \le \frac{1}{\mu([a, b])} \int_{[a,b]} f(x) \, d\mu \le \alpha f(a) + \beta f(b). \tag{3.10}$$

Proof. We take the barycenter $c = \int_{[a,b]} x \, d\mu / \mu([a, b])$, and since $c \in [a, b]$, we have the following two cases.

If the measure μ is concentrated in the endpoint, $c \in \{a, b\}$, then we have the trivial inequality $f(c) \le f(c) \le f(c)$.

If the measure μ is not concentrated in the endpoint, $c \in (a, b)$, then we can use the framework of equation (3.7). Inserting the integral mean of f, and the equalities

$$h(\alpha a + \beta b) = \frac{1}{\mu([a, b])} \int_{[a,b]} h(x) \, d\mu \tag{3.11}$$

with $h = f_{\{c\}}^{\text{sup}}$ and $h = f_{\{a,b\}}^{\text{sec}}$ into equation (3.7), we obtain equation (3.10). $\square$

The inequalities in equations (3.6) and (3.10) cover the following.

Remark 3.1. The left inequality of equation (3.6) with $\sum_{i=1}^{n} \alpha_i a_i$ instead of the endpoints combination $\alpha a + \beta b$ presents the discrete form of the Jensen inequality in equation (1.2), concerning the interval $[a, b]$.

The left inequality of equation (3.10) with $\int_{[a,b]} x \, d\mu / \mu([a, b])$ instead of $\alpha a + \beta b$ presents the integral form of the Jensen inequality in equation (1.3), concerning the identity function $g(x) = x$ on the interval $[a, b]$.

Using the Riemann integral in Corollary 3.1, the condition in equation (3.9) gives the midpoint

$$\frac{1}{2}a + \frac{1}{2}b = \frac{1}{b - a} \int_{a}^{b} x \, dx, \tag{3.12}$$

and its use in equation (3.10) implies the classic form of the Hermite-Hadamard inequality for convex functions of one variable,

$$f\left(\frac{1}{2}a + \frac{1}{2}b\right) \le \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \le \frac{1}{2}f(a) + \frac{1}{2}f(b). \tag{3.13}$$

4. Generalization to Functions of Several Variables

To generalize the inequalities of Section 3, we will use the smallest convex sets such as segments in the line and triangles in the plane. This kind of sets are called simplexes or simplices.

Let $S_1, \ldots, S_{k+1} \in \mathbb{R}^k$ be points. Their convex hull

$$\mathcal{S} = \text{conv}\{S_1, \ldots, S_{k+1}\}$$

is the k-simplex in the space $\mathbb{R}^k$ if the points $S_1 - S_{k+1}, \ldots, S_k - S_{k+1}$ are linearly independent. The points S_j are called simplex vertices.

Through the section we will use a k-simplex $\mathcal{S} \subset \mathbb{R}^k$ with vertices $S_1, \ldots, S_{k+1}$, and a convex function $f : \mathcal{S} \to \mathbb{R}$. The interior of $\mathcal{S}$ will be denoted by $\mathcal{S}^o$.

Every point $P \in \mathbb{R}^k$ can be presented by the unique affine combination

$$P = \sum_{j=1}^{k+1} \lambda_j S_j, \tag{4.1}$$

which is convex if and only if the point P belongs to the k-simplex $\mathcal{S}$. The coefficients λ_j can be calculated by using determinants presented in [11]. We will use the hyperplane passing through the graph points $(S_j, f(S_j))$, and a supporting hyperplane passing through the graph point $(C, f(C))$ where $C \in \mathcal{S}^o$. Using denotation $f^{\text{hyperplane}}_{\{S_1, \ldots, S_{k+1}\}}$ for the hyperplane function, and $f^{\text{sup}}_{\{C\}}$ for the supporting hyperplane function, we have the double inequality

$$f^{\text{sup}}_{\{C\}}(P) \leq f(P) \leq f^{\text{hyperplane}}_{\{S_1, \ldots, S_{k+1}\}}(P), \quad P \in \mathcal{S}. \tag{4.2}$$

Theorem 4.1. *Let $\mathcal{S} = \text{conv}\{S_1, \ldots, S_{k+1}\}$ be a k-simplex in the space $\mathbb{R}^k$, let $\sum_{i=1}^n \alpha_i A_i$ be a convex combination of points $A_i \in \mathcal{S}$, and let $\sum_{j=1}^{k+1} \sigma_j S_j$ be the unique convex combination of simplex vertices S_j such that*

$$\sum_{j=1}^{k+1} \sigma_j S_j = \sum_{i=1}^n \alpha_i A_i. \tag{4.3}$$

Then every convex function $f : \mathcal{S} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{j=1}^{k+1} \sigma_j S_j\right) \leq \sum_{i=1}^n \alpha_i f(A_i) \leq \sum_{j=1}^{k+1} \sigma_j f(S_j). \tag{4.4}$$

Proof. We put $C = \sum_{i=1}^n \alpha_i A_i$. Since $C \in \mathcal{S}$, we have three cases depending on the position of the center C.

If C is the simplex vertex, $C \in \{S_1, \ldots, S_{k+1}\}$, then the trivial inequality $f(C) \leq f(C) \leq f(C)$ represents the inequality in equation (4.4).

If C is the interior point of $\mathcal{S}$, $C \in \mathcal{S}^o$, then following the proof of Theorem 3.1, we get

$$
\begin{aligned}
f\left(\sum_{j=1}^{k+1} \sigma_j S_j\right) &= f^{\text{sup}}_{\{C\}}\left(\sum_{j=1}^{k+1} \sigma_j S_j\right) = \sum_{i=1}^n \alpha_i f^{\text{sup}}_{\{C\}}(A_i) \\
&\leq \sum_{i=1}^n \alpha_i f(A_i) \\
&\leq \sum_{i=1}^n \alpha_i f^{\text{hyperplane}}_{\{S_1, \ldots, S_{k+1}\}}(A_i) = f^{\text{hyperplane}}_{\{S_1, \ldots, S_{k+1}\}}\left(\sum_{j=1}^{k+1} \sigma_j S_j\right) \\
&= \sum_{j=1}^{k+1} \sigma_j f(S_j),
\end{aligned}
\tag{4.5}
$$

which proves the double inequality in equation (4.4).

If C is the interior point of the l-face $\mathcal{F}$ in the subspace $\mathbb{R}^l$ where $1 \leq l \leq k-1$, $C \in \mathcal{F}^o$, then we can apply the previous procedure to the l-simplex $\mathcal{F}$. If the face $\mathcal{F} = \text{conv}\{S_1, \ldots, S_{l+1}\}$, then the coefficients $\sigma_1, \ldots, \sigma_{l+1}$ are positive, and the coefficients $\sigma_{l+2}, \ldots, \sigma_{k+1}$ are equal to zero. $\square$

Corollary 4.1. *Let $\mathcal{S} = \mathrm{conv}\{S_1, \ldots, S_{k+1}\}$ be a k-simplex in the space $\mathbb{R}^k$, let μ be a positive measure on $\mathbb{R}^k$ such that $\mu(\mathcal{S}) > 0$, and let $\sum_{j=1}^{k+1} \sigma_j S_j$ be the unique convex combination of simplex vertices S_j such that*

$$\sum_{j=1}^{k+1} \sigma_j S_j = \left(\frac{1}{\mu(\mathcal{S})} \int_{\mathcal{S}} x_1 \, d\mu, \ldots, \frac{1}{\mu(\mathcal{S})} \int_{\mathcal{S}} x_k \, d\mu \right). \tag{4.6}$$

Then every convex function $f : \mathcal{S} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{j=1}^{k+1} \sigma_j S_j \right) \le \frac{1}{\mu(\mathcal{S})} \int_{\mathcal{S}} f(x_1, \ldots, x_k) \, d\mu \le \sum_{j=1}^{k+1} \sigma_j f(S_j). \tag{4.7}$$

Proof. We can apply the procedure of proving Corollary 3.1 by utilizing the framework of the proof of Theorem 4.1. □

Using the inequalities in equations (4.4) and (4.7), we can obtain the following.

Remark 4.1. The left inequality of equation (4.4) with $\sum_{i=1}^{n} \alpha_i A_i$ instead of the vertices combination $\sum_{j=1}^{k+1} \sigma_j S_j$ presents the discrete form of the Jensen inequality in equation (1.2), concerning the simplex $\mathcal{S}$.

The left inequality of equation (4.7) with $\left(\int_{\mathcal{S}} x_1 \, d\mu/\mu(\mathcal{S}), \ldots, \int_{\mathcal{S}} x_k \, d\mu/\mu(\mathcal{S}) \right)$ instead of $\sum_{j=1}^{k+1} \sigma_j S_j$ gives a simple version of the integral form of the Jensen inequality for convex functions of k variables in equation (1.7), concerning the simplex $\mathcal{S}$.

Using the Riemann integral in Corollary 4.1, the condition in equation (4.6) gives the barycentric point

$$\frac{1}{k+1} \sum_{j=1}^{k+1} S_j = \left(\frac{1}{\mathrm{vol}(\mathcal{S})} \int_{\mathcal{S}} x_1 \, dx_1 \ldots dx_k, \ldots, \frac{1}{\mathrm{vol}(\mathcal{S})} \int_{\mathcal{S}} x_k \, dx_1 \ldots dx_k \right), \tag{4.8}$$

and its use in equation (4.7) gives the Hermite-Hadamard inequality for convex functions of k variables,

$$f\left(\frac{1}{k+1} \sum_{j=1}^{k+1} S_j \right) \le \frac{1}{\mathrm{vol}(\mathcal{S})} \int_{\mathcal{S}} f(x_1, \ldots, x_k) \, dx_1 \ldots dx_k \le \frac{1}{k+1} \sum_{j=1}^{k+1} f(S_j). \tag{4.9}$$

Acknowledgements

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