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Hadamard-type and Bullen-type inequalities for Lipschitzian functions via fractional integrals

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Abstract
In this paper, the author establishes some Hadamard-type and Bullen-type inequalities for Lipschitzian functions via Riemann Liouville fractional integral.

Keywords: *Lipschitzian functions; Hadamard inequality; Bullen inequality; Riemann–Liouville fractional integral.*

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1. Introduction

Following inequality is well known in the literature as Hermite-Hadamard's inequality:

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [8], Tseng et al. established the following Hadamard-type inequality which refines the inequality (1.1).

Theorem 1.2. *Suppose that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$. Then we have the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

The third inequality in (1.2) is known in the literature as Bullen's inequality.
In what follows we recall the following definition.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called an M-Lipschitzian function on the interval I of real numbers with $M \geq 0$, if

$$|f(x) - f(y)| \leq M\,|x - y|$$

for all $x, y \in I$.

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For some recent results connected with Hermite-Hadamard type integral inequalities for Lipschitzian functions, see [2–4, 9, 10].

In [9], Tseng et al. established some Hadamard-type and Bullen-type inequalities for Lipschitzian functions as follows.

Theorem 1.3. *Let I be an interval in $\mathbb{R}$, $a \leq A \leq B \leq b$ in I, $V = (1-\alpha)a + \alpha b$, $\alpha \in [0,1]$ and let $f : I \to \mathbb{R}$ be an L-Lipschitzian function with $L \geq 0$. Then we have the inequality*

$$\left| \alpha f(A) + (1-\alpha) f(B) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{L V_\alpha(A,B)}{b-a}, \tag{1.3}$$

where

$$V_\alpha(A,B) = \begin{cases} (A-a)^2 - (A-V)^2 + (B-V)^2 + (b-B)^2, \\ \qquad a \leq V \leq A \leq B \leq b, \\ (A-a)^2 + (V-A)^2 + (B-V)^2 + (b-B)^2, \\ \qquad a \leq A \leq V \leq B \leq b, \\ (A-a)^2 + (V-A)^2 + (b-B)^2 - (V-B)^2, \\ \qquad a \leq A \leq B \leq V \leq b \end{cases}.$$

Theorem 1.4. *Let I be an interval in $\mathbb{R}$, $a \leq A \leq B \leq C \leq b$ in I, $V_1 = (1-\alpha)a + \alpha b$, $V_2 = \gamma a + (\alpha+\beta) b$, $\alpha, \beta, \gamma \in [0,1]$, $\alpha + \beta + \gamma = 1$, and let $f : I \to \mathbb{R}$ be an L-Lipschitzian function with $L \geq 0$. Then we have the inequality*

$$\left| \alpha f(A) + \beta f(B) + \gamma f(C) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{L V_{\alpha,\beta,\gamma}(A,B,C)}{b-a}, \tag{1.4}$$

where $V_{\alpha,\beta,\gamma}$ is defined as in [9, Section 3].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$ (see [5]).

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities, see [1, 6, 7, 11].

The aim of this paper is to establish some Hadamard-type and Bullen-type inequalities for Lipschitzian functions via Riemann–Liouville fractional integral. The results obtained herein is a generalization of the results obtained in Theorem 3 and Theorem 4 via fractional integrals.

2. Hadamard-type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let I be an interval in $\mathbb{R}$, $a \leq x \leq y \leq b$ in I and let $f : I \to \mathbb{R}$ be an M-Lipschitzian function. In the next theorem, let $\lambda \in [0,1]$, $V = (1-\lambda)a + \lambda b$, and $V_{\alpha,\lambda}$, $\alpha > 0$, as follows:

(1) If $a \leq V \leq x \leq y \leq b$, then

$$V_{\alpha,\lambda}(x,y) = (V-a)^{\alpha}\left[\frac{x-a}{\alpha} - \frac{V-a}{\alpha+1}\right] + \frac{2\,(b-y)^{\alpha+1}}{\alpha\,(\alpha+1)} + (b-V)^{\alpha}\left[\frac{b-V}{\alpha+1} - \frac{b-y}{\alpha}\right]$$

(2) If $a \leq x \leq V \leq y \leq b$, then

$$\begin{aligned} V_{\alpha,\lambda}(x,y) &= \frac{2\,(x-a)^{\alpha+1}}{\alpha\,(\alpha+1)} + (V-a)^{\alpha}\left[\frac{V-a}{\alpha+1} - \frac{x-a}{\alpha}\right] \\ &\quad + \frac{2\,(b-y)^{\alpha+1}}{\alpha\,(\alpha+1)} + (b-V)^{\alpha}\left[\frac{b-V}{\alpha+1} - \frac{b-y}{\alpha}\right]. \end{aligned}$$

(3) If $a \leq x \leq y \leq V \leq b$, then

$$V_{\alpha,\lambda}(x,y) = \frac{2\,(x-a)^{\alpha+1}}{\alpha\,(\alpha+1)} + (V-a)^{\alpha}\left[\frac{V-a}{\alpha+1} - \frac{x-a}{\alpha}\right] + (b-V)^{\alpha}\left[\frac{b-y}{\alpha} - \frac{b-V}{\alpha+1}\right].$$

Theorem 2.1. *Let $x, y, \alpha, \lambda, V, V_{\alpha,\lambda}$ and the function f be defined as above. Then we have the inequality for fractional integrals*

$$\left|\lambda^{\alpha} f(x) + (1-\lambda)^{\alpha} f(y) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V-} f(a) + J^{\alpha}_{V+} f(b)\right]\right| \leq \frac{\alpha M V_{\alpha,\lambda}(x,y)}{(b-a)^{\alpha}}. \tag{2.1}$$

Proof. Using the hypothesis of f, we have the following inequality

$$\begin{aligned} &\left|\lambda^{\alpha} f(x) + (1-\lambda)^{\alpha} f(y) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V-} f(a) + J^{\alpha}_{V+} f(b)\right]\right| \\ &= \frac{\alpha}{(b-a)^{\alpha}}\left|\int_a^V [f(x) - f(t)]\,(t-a)^{\alpha-1}\,dt + \int_V^b [f(y) - f(t)]\,(b-t)^{\alpha-1}\,dt\right| \\ &\leq \frac{\alpha}{(b-a)^{\alpha}}\left[\int_a^V |f(x) - f(t)|\,(t-a)^{\alpha-1}\,dt + \int_V^b |f(y) - f(t)|\,(b-t)^{\alpha-1}\,dt\right] \\ &\leq \frac{\alpha M}{(b-a)^{\alpha}}\left[\int_a^V |x-t|\,(t-a)^{\alpha-1}\,dt + \int_V^b |y-t|\,(b-t)^{\alpha-1}\,dt\right]. \end{aligned} \tag{2.2}$$

Now using simple calculations, we obtain the following identities $\int_a^V |x-t|\,(t-a)^{\alpha-1}\,dt$ and $\int_V^b |y-t|\,(b-t)^{\alpha-1}\,dt$.

(1) If $a \leq V \leq x \leq y \leq b$, then

$$\int_a^V |x-t|\,(t-a)^{\alpha-1}\,dt = (V-a)^{\alpha}\left[\frac{x-a}{\alpha} - \frac{V-a}{\alpha+1}\right]$$

and

$$\int_V^b |y-t|\,(b-t)^{\alpha-1}\,dt = \frac{2\,(b-y)^{\alpha+1}}{\alpha\,(\alpha+1)} + (b-V)^{\alpha}\left[\frac{b-V}{\alpha+1} - \frac{b-y}{\alpha}\right].$$

(2) If $a \leq x \leq V \leq y \leq b$, then

$$\int_a^V |x-t|\,(t-a)^{\alpha-1}\,dt = \frac{2\,(x-a)^{\alpha+1}}{\alpha\,(\alpha+1)} + (V-a)^{\alpha}\left[\frac{V-a}{\alpha+1} - \frac{x-a}{\alpha}\right]$$

and

$$\int_V^b |y-t|\,(b-t)^{\alpha-1}\,dt = \frac{2\,(b-y)^{\alpha+1}}{\alpha\,(\alpha+1)} + (b-V)^{\alpha}\left[\frac{b-V}{\alpha+1} - \frac{b-y}{\alpha}\right].$$

(3) If $a \le x \le y \le V \le b$, then

$$\int_a^V |x-t|\,(t-a)^{\alpha-1}\,dt = \frac{2\,(x-a)^{\alpha+1}}{\alpha\,(\alpha+1)} + (V-a)^{\alpha}\left[\frac{V-a}{\alpha+1} - \frac{x-a}{\alpha}\right]$$

and

$$\int_V^b |y-t|\,(b-t)^{\alpha-1}\,dt = (b-V)^{\alpha}\left[\frac{b-y}{\alpha} - \frac{b-V}{\alpha+1}\right].$$

Using the inequality (2.2) and the above identities $\int_a^V |x-t|\,(t-a)^{\alpha-1}\,dt$ and $\int_V^b |y-t|\,(b-t)^{\alpha-1}\,dt$, we derive the inequality (2.1). This completes the proof. □

Under the assumptions of Theorem 2.1, we have the following corollaries and remarks:

Remark 2.1. In Theorem 2.1, if we take $\alpha = 1$, then the inequality (2.1) reduces the inequality (1.3) in Theorem 1.3.

Corollary 2.1. *1. In Theorem 2.1, let $\delta \in \left[\frac{1}{2}, 1\right]$, $x = \delta a + (1-\delta)b$ and $y = (1-\delta)a + \delta b$. Then, we have the inequality*

$$\left|\lambda^{\alpha} f(\delta a + (1-\delta)b) + (1-\lambda)^{\alpha} f((1-\delta)a + \delta b) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V-}f(a) + J^{\alpha}_{V+}f(b)\right]\right| \le \frac{M L\,(\alpha,\lambda,\delta)\,(b-a)}{\alpha+1} \tag{2.3}$$

where

$$L\,(\alpha,\lambda,\delta) = \begin{cases} \lambda^{\alpha}\left[(1-\delta)\,(1+\alpha) - \lambda\alpha\right] + 2\,(1-\delta)^{\alpha+1} + (1-\lambda)^{\alpha}\left[(1-\lambda)\,\alpha - (1-\delta)\,(1+\alpha)\right], \\ \qquad \lambda \le 1-\delta \\ 4\,(1-\delta)^{\alpha+1} + \lambda^{\alpha}\left[\lambda\alpha - (1-\delta)\,(1+\alpha)\right] + (1-\lambda)^{\alpha}\left[(1-\lambda)\,\alpha - (1-\delta)\,(1+\alpha)\right], \\ \qquad 1-\delta \le \lambda \le \delta \\ 2\,(1-\delta)^{\alpha+1} + \lambda^{\alpha}\left[\lambda\alpha - (1-\delta)\,(1+\alpha)\right] + (1-\lambda)^{\alpha}\left[(1-\delta)\,(1+\alpha) - (1-\lambda)\,\alpha\right], \\ \qquad \delta \le \lambda. \end{cases}$$

2. In Theorem 2.1, if we take $x = y = V$, then we have the inequality

$$\left|\left[\lambda^{\alpha} + (1-\lambda)^{\alpha}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V-}f(a) + J^{\alpha}_{V+}f(b)\right]\right| \le M\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{(\alpha+1)\,(b-a)^{\alpha}}. \tag{2.4}$$

Corollary 2.2. *We have the following weighted Hadamard-type inequalities for Lipschitzian functions via Rieamnn-liouville fractional integrals*

(1) In the inequality (2.3), if we take $\delta = 1$, then we have

$$\left|\lambda^{\alpha} f(a) + (1-\lambda)^{\alpha} f(b) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V-}f(a) + J^{\alpha}_{V+}f(b)\right]\right| \le \alpha M\,(b-a)\,\frac{\lambda^{\alpha+1} + (1-\lambda)^{\alpha+1}}{\alpha+1},$$

in this inequality, specially if we choose $\lambda = \frac{x-a}{b-a}$ *for* $x \in [a,b]$, *then*

$$\begin{aligned} & \left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{(b-a)^{\alpha}} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) \right] \right| \\ \leq \; & \alpha M \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha}}, \end{aligned}$$

(2) *In the inequality (2.4), if we take* $x = \delta a + (1-\delta) b,\ \delta \in [0,1]$, *then*

$$\begin{aligned} & \left| [\lambda^{\alpha} + (1-\lambda)^{\alpha}] f(\delta a + (1-\delta)b) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{V-}^{\alpha} f(a) + J_{V+}^{\alpha} f(b) \right] \right| \\ \leq \; & M(b-a) \frac{\delta^{\alpha+1} + (1-\delta)^{\alpha+1}}{(\alpha+1)}, \end{aligned}$$

in this inequality, specially if we choose $\lambda = \frac{1}{2}$, *then*

$$\begin{aligned} & \left| f(\delta a + (1-\delta)b) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) \right] \right| \\ \leq \; & 2^{\alpha-1} M(b-a) \frac{\delta^{\alpha+1} + (1-\delta)^{\alpha+1}}{(\alpha+1)}, \end{aligned}$$

(3) *In the inequality (2.4), if we take* $\lambda = \frac{1}{2}$ *and* $\delta = \frac{3}{4}$ *then*

$$\begin{aligned} & \left| \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) \right] \right| \\ \leq \; & M(b-a) \frac{1 + 2^{\alpha-1}(\alpha-1)}{2^{\alpha+1}(\alpha+1)} \end{aligned}$$

3. Bullen -type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let I be an interval in $\mathbb{R}$, $a \leq x \leq y \leq z \leq b$ in I and $f : I \to \mathbb{R}$ be an M-lipschitzian function. In the next theorem, let $\lambda + \eta + \mu = 1$, $\lambda, \eta, \mu \in [0,1]$, $V_1 = (1-\lambda)a + \lambda b$, $V_2 = \mu a + (\lambda + \eta) b$, and define $V_{\alpha,\lambda,\eta,\mu}$, $\alpha > 0$, as follows:

1. If $V_1 \leq V_2 \leq x \leq y \leq z$ or $V_1 \leq x \leq V_2 \leq y \leq z$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) = \; & (V_1-a)^{\alpha} \left[\frac{x-a}{\alpha} - \frac{V_1-a}{\alpha+1} \right] + (V_2-V_1)^{\alpha} \left[\frac{y-V_2}{\alpha} + \frac{V_2-V_1}{\alpha+1} \right] \\ & + \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha} \left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha} \right]. \end{aligned}$$

2. If $V_1 \leq x \leq y \leq V_2 \leq z$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) = \; & (V_1-a)^{\alpha} \left[\frac{x-a}{\alpha} - \frac{V_1-a}{\alpha+1} \right] + \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)} + (V_2-V_1)^{\alpha} \left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha} \right] \\ & + \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha} \left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha} \right]. \end{aligned}$$

3. If $V_1 \leq x \leq y \leq z \leq V_2$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) = \; & (V_1-a)^{\alpha} \left[\frac{x-a}{\alpha} - \frac{V_1-a}{\alpha+1} \right] + \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)} \\ & + (V_2-V_1)^{\alpha} \left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha} \right] + (b-V_2)^{\alpha} \left[\frac{b-z}{\alpha} - \frac{b-V_2}{\alpha+1} \right]. \end{aligned}$$

4. If $x \leq V_1 \leq V_2 \leq y \leq z$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right] + (V_2-V_1)^{\alpha}\left[\frac{y-V_2}{\alpha} + \frac{V_2-V_1}{\alpha+1}\right] \\ &+ \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha}\right]. \end{aligned}$$

5. If $x \leq V_1 \leq y \leq V_2 \leq z$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right] + \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)} \\ &+ (V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha}\right] + \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha}\right]. \end{aligned}$$

6. If $x \leq V_1 \leq y \leq z \leq V_2$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right] + \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)} \\ &+ (V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha}\right] + (b-V_2)^{\alpha}\left[\frac{b-z}{\alpha} - \frac{b-V_2}{\alpha+1}\right]. \end{aligned}$$

7. If $x \leq y \leq V_1 \leq V_2 \leq z$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right] + (V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha}\right] \\ &+ \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha}\right]. \end{aligned}$$

8. If $x \leq y \leq V_1 \leq z \leq V_2$ or $x \leq y \leq z \leq V_1 \leq V_2$, then

$$\begin{aligned} V_{\alpha,\lambda,\eta,\mu}(x,y,z) &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right] \\ &+ (V_2-V_1)^{\alpha}\left[\frac{V_2-y}{\alpha} - \frac{V_2-V_1}{\alpha+1}\right] + (b-V_2)^{\alpha}\left[\frac{b-z}{\alpha} - \frac{b-V_2}{\alpha+1}\right]. \end{aligned}$$

Theorem 3.1. *Let $x, y, z, \lambda, \eta, \mu, V_1, V_2, V_{\alpha,\lambda,\eta,\mu}$ and the function f be defined as above. Then we have the inequality*

$$\begin{aligned} &\left|\lambda^{\alpha} f(x) + \eta^{\alpha} f(y) + \mu^{\alpha} f(z) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V_1-}f(a) + J^{\alpha}_{V_1+}f(V_2) + J^{\alpha}_{V_2+}f(b)\right]\right| \\ &\leq \frac{\alpha M V_{\alpha,\lambda,\eta,\mu}(x,y,z)}{(b-a)^{\alpha}} \end{aligned} \tag{3.1}$$

Proof. Using the hypothesis of f, we have the inequality

$$\begin{aligned} &\left|\lambda^{\alpha} f(x) + \eta^{\alpha} f(y) + \mu^{\alpha} f(z) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V_1-}f(a) + J^{\alpha}_{V_1+}f(V_2) + J^{\alpha}_{V_2+}f(b)\right]\right| \\ &= \frac{\alpha}{(b-a)^{\alpha}}\left|\int_a^{V_1}[f(x)-f(t)](t-a)^{\alpha-1}dt + \int_{V_1}^{V_2}[f(y)-f(t)](V_2-t)^{\alpha-1}dt + \int_{V_2}^{b}[f(z)-f(t)](b-t)^{\alpha-1}dt\right| \\ &\leq \frac{\alpha}{(b-a)^{\alpha}}\left[\int_a^{V_1}|f(x)-f(t)|(t-a)^{\alpha-1}dt + \int_{V_1}^{V_2}|f(y)-f(t)|(V_2-t)^{\alpha-1}dt + \int_{V_2}^{b}|f(z)-f(t)|(b-t)^{\alpha-1}dt\right] \end{aligned}$$

$$\leq \frac{\alpha M}{(b-a)^{\alpha}}\left[\int_{a}^{V_1}|x-t|(t-a)^{\alpha-1}dt+\int_{V_1}^{V_2}|y-t|(V_2-t)^{\alpha-1}dt+\int_{V_2}^{b}|z-t|(b-t)^{\alpha-1}dt\right]. \tag{3.2}$$

Now, using simple calculations, we obtain the following identities $\int_a^{V_1}|x-t|(t-a)^{\alpha-1}dt$, $\int_{V_1}^{V_2}|y-t|(V_2-t)^{\alpha-1}dt$ and $\int_{V_2}^{b}|z-t|(b-t)^{\alpha-1}dt$.

1. If $V_1 \leq V_2 \leq x \leq y \leq z$ or $V_1 \leq x \leq V_2 \leq y \leq z$, then we have

$$\begin{aligned}\int_{a}^{V_1}|x-t|(t-a)^{\alpha-1}dt &= (V_1-a)^{\alpha}\left[\frac{x-a}{\alpha}-\frac{V_1-a}{\alpha+1}\right],\\ \int_{V_1}^{V_2}|y-t|(V_2-t)^{\alpha-1}dt &= (V_2-V_1)^{\alpha}\left[\frac{y-V_2}{\alpha}+\frac{V_2-V_1}{\alpha+1}\right]\end{aligned}$$

and

$$\int_{V_2}^{b}|z-t|(b-t)^{\alpha-1}dt=\frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)}+(b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1}-\frac{b-z}{\alpha}\right].$$

2. If $V_1 \leq x \leq y \leq V_2 \leq z$, then we have

$$\begin{aligned}\int_{a}^{V_1}|x-t|(t-a)^{\alpha-1}dt &= (V_1-a)^{\alpha}\left[\frac{x-a}{\alpha}-\frac{V_1-a}{\alpha+1}\right],\\ \int_{V_1}^{V_2}|y-t|(V_2-t)^{\alpha-1}dt &= \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)}+(V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1}-\frac{V_2-y}{\alpha}\right]\end{aligned}$$

and

$$\int_{V_2}^{b}|z-t|(b-t)^{\alpha-1}dt=\frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)}+(b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1}-\frac{b-z}{\alpha}\right].$$

3. If $V_1 \leq x \leq y \leq z \leq V_2$, then we have

$$\begin{aligned}\int_{a}^{V_1}|x-t|(t-a)^{\alpha-1}dt &= (V_1-a)^{\alpha}\left[\frac{x-a}{\alpha}-\frac{V_1-a}{\alpha+1}\right],\\ \int_{V_1}^{V_2}|y-t|(V_2-t)^{\alpha-1}dt &= \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)}+(V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1}-\frac{V_2-y}{\alpha}\right]\end{aligned}$$

and

$$\int_{V_2}^{b}|z-t|(b-t)^{\alpha-1}dt=(b-V_2)^{\alpha}\left[\frac{b-z}{\alpha}-\frac{b-V_2}{\alpha+1}\right].$$

4. If $x \leq V_1 \leq V_2 \leq y \leq z$, then we have

$$\begin{aligned}\int_{a}^{V_1}|x-t|(t-a)^{\alpha-1}dt &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)}+(V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1}-\frac{x-a}{\alpha}\right],\\ \int_{V_1}^{V_2}|y-t|(V_2-t)^{\alpha-1}dt &= (V_2-V_1)^{\alpha}\left[\frac{y-V_2}{\alpha}+\frac{V_2-V_1}{\alpha+1}\right],\end{aligned}$$

and

$$\int_{V_2}^{b} |z-t|(b-t)^{\alpha-1}\,dt = \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha}\right].$$

5. If $x \le V_1 \le y \le V_2 \le z$, then we have

$$\begin{aligned}\int_{a}^{V_1} |x-t|(t-a)^{\alpha-1}\,dt &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right],\\ \int_{V_1}^{V_2} |y-t|(V_2-t)^{\alpha-1}\,dt &= \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)} + (V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha}\right]\end{aligned}$$

and

$$\int_{V_2}^{b} |z-t|(b-t)^{\alpha-1}\,dt = \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha}\right].$$

6. If $x \le V_1 \le y \le z \le V_2$, then we have

$$\begin{aligned}\int_{a}^{V_1} |x-t|(t-a)^{\alpha-1}\,dt &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right],\\ \int_{V_1}^{V_2} |y-t|(V_2-t)^{\alpha-1}\,dt &= \frac{2(V_2-y)^{\alpha+1}}{\alpha(\alpha+1)} + (V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha}\right]\end{aligned}$$

and

$$\int_{V_2}^{b} |z-t|(b-t)^{\alpha-1}\,dt = (b-V_2)^{\alpha}\left[\frac{b-z}{\alpha} - \frac{b-V_2}{\alpha+1}\right].$$

7. If $x \le y \le V_1 \le V_2 \le z$, then we have

$$\begin{aligned}\int_{a}^{V_1} |x-t|(t-a)^{\alpha-1}\,dt &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right],\\ \int_{V_1}^{V_2} |y-t|(V_2-t)^{\alpha-1}\,dt &= (V_2-V_1)^{\alpha}\left[\frac{V_2-V_1}{\alpha+1} - \frac{V_2-y}{\alpha}\right],\end{aligned}$$

and

$$\int_{V_2}^{b} |z-t|(b-t)^{\alpha-1}\,dt = \frac{2(b-z)^{\alpha+1}}{\alpha(\alpha+1)} + (b-V_2)^{\alpha}\left[\frac{b-V_2}{\alpha+1} - \frac{b-z}{\alpha}\right].$$

8. If $x \le y \le V_1 \le z \le V_2$ or $x \le y \le z \le V_1 \le V_2$, then we have

$$\begin{aligned}\int_{a}^{V_1} |x-t|(t-a)^{\alpha-1}\,dt &= \frac{2(x-a)^{\alpha+1}}{\alpha(\alpha+1)} + (V_1-a)^{\alpha}\left[\frac{V_1-a}{\alpha+1} - \frac{x-a}{\alpha}\right],\\ \int_{V_1}^{V_2} |y-t|(V_2-t)^{\alpha-1}\,dt &= (V_2-V_1)^{\alpha}\left[\frac{V_2-y}{\alpha} - \frac{V_2-V_1}{\alpha+1}\right],\end{aligned}$$

and

$$\int_{V_2}^{b} |z-t| (b-t)^{\alpha-1} dt = (b-V_2)^{\alpha} \left[\frac{b-z}{\alpha} - \frac{b-V_2}{\alpha+1} \right].$$

Using the inequality (3.2) and the above identities $\int_a^{V_1} |x-t| (t-a)^{\alpha-1} dt$, $\int_{V_1}^{V_2} |y-t| (V_2-t)^{\alpha-1} dt$ and $\int_{V_2}^{b} |z-t| (b-t)^{\alpha-1} dt$, we derive the inequality (3.1). This completes the proof.

□

Under the assumptions of Theorem 3.1, we have the following corollaries and remarks:

Remark 3.1. In Theorem 3.1, if we take $\alpha = 1$, then then the inequality (3.1) reduces the inequality (1.4) in Theorem 1.4.

Corollary 3.1. *In Theorem 3.1, let $\delta \in \left[\frac{1}{2}, 1\right]$, $x = \delta a + (1-\delta)b$, $y = \frac{a+b}{2}$ and $z = (1-\delta)a + \delta b$. Then, we have the inequality*

$$\left| \lambda^{\alpha} f(\delta a + (1-\delta)b) + \eta^{\alpha} f(\frac{a+b}{2}) + \mu^{\alpha} f((1-\delta)a + \delta b) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{V_1-} f(a) + J^{\alpha}_{V_1+} f(V_2) + J^{\alpha}_{V_2+} f(b) \right] \right| \leq \frac{MN(\alpha,\lambda,\eta,\delta)(b-a)}{\alpha+1}$$

where $N(\alpha,\lambda,\eta,\delta)$ is defined as follows:

1. If $\lambda+\eta \leq 1-\delta$ or $\lambda \leq 1-\delta \leq \lambda+\eta \leq \frac{1}{2}$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= \lambda^{\alpha}\left[(1-\delta)(\alpha+1)-\alpha\lambda\right] + \eta^{\alpha}\left[\left(\frac{1}{2}-\lambda-\eta\right)(\alpha+1)+\alpha\eta\right] \\ &\quad +2(1-\delta)^{\alpha+1} + (1-\lambda-\eta)^{\alpha}\left[\alpha(1-\lambda-\eta)-(\alpha+1)(1-\delta)\right]. \end{aligned}$$

2. If $\lambda \leq 1-\delta \leq \frac{1}{2} \leq \lambda+\eta \leq \delta$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= \lambda^{\alpha}\left[(1-\delta)(\alpha+1)-\alpha\lambda\right] + 2\left(\lambda+\eta-\frac{1}{2}\right)^{\alpha+1} + \eta^{\alpha}\left[\alpha\eta-(\alpha+1)\left(\lambda+\eta-\frac{1}{2}\right)\right] \\ &\quad +2(1-\delta)^{\alpha+1} + (1-\lambda-\eta)^{\alpha}\left[\alpha(1-\lambda-\eta)-(\alpha+1)(1-\delta)\right]. \end{aligned}$$

3. If $\lambda \leq 1-\delta \leq \frac{1}{2} \leq \delta \leq \lambda+\eta$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= \lambda^{\alpha}\left[(1-\delta)(\alpha+1)-\alpha\lambda\right] + 2\left(\lambda+\eta-\frac{1}{2}\right)^{\alpha+1} \\ &\quad +\eta^{\alpha}\left[\alpha\eta-(\alpha+1)\left(\lambda+\eta-\frac{1}{2}\right)\right] + (1-\lambda-\eta)^{\alpha}\left[(\alpha+1)(1-\delta)-\alpha(1-\lambda-\eta)\right]. \end{aligned}$$

4. If $1-\delta \leq \lambda \leq \lambda+\eta \leq \frac{1}{2}$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= 4(1-\delta)^{\alpha+1} + \lambda^{\alpha}\left[\alpha\lambda-(1-\delta)(\alpha+1)\right] + \eta^{\alpha}\left[\alpha\eta+(\alpha+1)\left(\frac{1}{2}-\lambda-\eta\right)\right] \\ &\quad +(1-\lambda-\eta)^{\alpha}\left[\alpha(1-\lambda-\eta)-(\alpha+1)(1-\delta)\right]. \end{aligned}$$

5. If $1-\delta \leq \lambda \leq \frac{1}{2} \leq \lambda+\eta \leq \delta$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= 4(1-\delta)^{\alpha+1} + \lambda^{\alpha}\left[\alpha\lambda-(1-\delta)(\alpha+1)\right] + 2\left(\lambda+\eta-\frac{1}{2}\right)^{\alpha+1} \\ &\quad +\eta^{\alpha}\left[\alpha\eta-(\alpha+1)\left(\lambda+\eta-\frac{1}{2}\right)\right] + (1-\lambda-\eta)^{\alpha}\left[\alpha(1-\lambda-\eta)-(\alpha+1)(1-\delta)\right]. \end{aligned}$$

6. If $1-\delta \leq \lambda \leq \frac{1}{2} \leq \delta \leq \lambda+\eta$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= 2(1-\delta)^{\alpha+1}+\lambda^{\alpha}\left[\alpha\lambda-(1-\delta)(\alpha+1)\right]+2\left(\lambda+\eta-\frac{1}{2}\right)^{\alpha+1} \\ &\quad +\eta^{\alpha}\left[\alpha\eta-(\alpha+1)\left(\lambda+\eta-\frac{1}{2}\right)\right]+(1-\lambda-\eta)^{\alpha}\left[(\alpha+1)(1-\delta)-\alpha(1-\lambda-\eta)\right]. \end{aligned}$$

7. If $\frac{1}{2} \leq \lambda \leq \lambda+\eta \leq \delta$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= 4(1-\delta)^{\alpha+1}+\lambda^{\alpha}\left[\alpha\lambda-(1-\delta)(\alpha+1)\right]+\eta^{\alpha}\left[\alpha\eta-(\alpha+1)\left(\lambda+\eta-\frac{1}{2}\right)\right] \\ &\quad +(1-\lambda-\eta)^{\alpha}\left[\alpha(1-\lambda-\eta)-(\alpha+1)(1-\delta)\right]. \end{aligned}$$

8. If $\frac{1}{2} \leq \lambda \leq \delta \leq \lambda+\eta$ or $\delta \leq \lambda$, then

$$\begin{aligned} N(\alpha,\lambda,\eta,\delta) &= 2(1-\delta)^{\alpha+1}+\lambda^{\alpha}\left[\alpha\lambda-(1-\delta)(\alpha+1)\right] \\ &\quad +\eta^{\alpha}\left[(\alpha+1)\left(\lambda+\eta-\frac{1}{2}\right)-\alpha\eta\right]+(1-\lambda-\eta)^{\alpha}\left[(\alpha+1)(1-\delta)-\alpha(1-\lambda-\eta)\right]. \end{aligned}$$

Corollary 3.2. *In Corollary 3.1, if we take $\delta=1$, $\lambda=\mu=\frac{\theta}{2}$ and $\eta=1-\theta$ with $\theta\in[0,1]$, then we have the following weighted Bullen-type inequality for M-Lipschitzian functions via fractional integrals*

$$\begin{aligned} &\left|\left(\frac{\theta}{2}\right)^{\alpha}(f(a)+f(b)+(1-\theta)^{\alpha}f(\frac{a+b}{2})+-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{V_1-}f(a)+J^{\alpha}_{V_1+}f(V_2)+J^{\alpha}_{V_2+}f(b)\right]\right| \\ &\leq \frac{M\left[2\alpha\left(\frac{\theta}{2}\right)^{\alpha+1}+(1-\theta)^{\alpha+1}\frac{\alpha-1}{2}+2\left(\frac{1-\theta}{2}\right)^{\alpha+1}\right](b-a)}{\alpha+1}. \end{aligned} \tag{3.3}$$

Remark 3.2. In the inequality (3.3), if we take $\theta=\frac{1}{2}$, then the inequality (3.3) reduces to the following Bullen-type inequality for M-Lipschitzian functions via fractional integrals

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{3a+b}{4}\right)-}f(a)+J^{\alpha}_{\left(\frac{3a+b}{4}\right)+}f(\frac{a+3b}{4})+J^{\alpha}_{\left(\frac{a+3b}{4}\right)+}f(b)\right]\right| \\ &\leq \frac{M(b-a)}{2^{\alpha+2}(\alpha+1)}\left[\alpha+1+2^{\alpha-1}(\alpha-1)\right]. \end{aligned}$$

Remark 3.3. In the inequality (3.3), if we take $\theta=\frac{1}{3}$, then the inequality (3.3) reduces to the following Simpson-type inequality for M-Lipschitzian functions via fractional integrals

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{5a+b}{6}\right)-}f(a)+J^{\alpha}_{\left(\frac{5a+b}{6}\right)+}f(\frac{a+5b}{6})+J^{\alpha}_{\left(\frac{a+5b}{6}\right)+}f(b)\right]\right| \\ &\leq \frac{M(b-a)}{18(\alpha+1)}\left[\alpha+2^{2\alpha}(\alpha-1)3+2^{\alpha+1}\right]. \end{aligned}$$

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