Mathematical Sciences and Applications E-Notes
4 (1) 125-135 (2016) ©MSAEN

Transmuted New Modified Weibull Distribution

R. Vishnu Vardhan, S. Balaswamy

(Communicated by K. Şanlı KULA)

Abstract
In statistical and reliability theory, the transmuted distributions are the present day researcher's interest because these distributions will fit the data in a better manner by involving a new parameter namely transmuted parameter. This paper aims to produce another transmuted distribution based on the new modified weibull distribution using the quadratic rank transmutation map. Further, the properties such as moments, moment generating function, Estimation of parameters, order statistics are derived for the proposed distribution along with the hazard and survival functions.

Keywords: *New Modified Weibull Distribution, Transmuted Distributions, Maximum Likelihood Estimation, Reliability Function, Moment Generating Function.*

AMS Subject Classification (2010): *Primary: 90B25 ; Secondary: 62N05.*

1. Introduction and Preliminaries

In Statistical and Reliability theory, life distributions plays a major role in explaining the nature and behavior of the data along with its properties. Of these life distributions, weibull distribution has gained attention from researchers of various subject domains such as reliability engineering, medicine, hydrology and many more. In recent years, generalisations of the basic form of weibull distribution are made and one such generalisation is the new modified weibull distribution [2]. In this paper, a generalisation of new modified weibull distribution is proposed by making use of an interesting idea, known in the literature as transmutation. The transmutation of new modified weibull distribution is derived using the quadratic rank transformation map [15]. The main focus of the rank transmutation map (RTM) is to meet the needs of parametric families of distributions and it can be used to investigate a novel technique for introducing skewness or kurtosis into a symmetric or other distribution by considering wider statistical applications. That is the RTM is a tool for the discovery of new families of non - Gaussian distributions. We use it to modulate a given base distribution for the purposes of modifying the moments, in particular the skew and kurtosis. An important example will be to take the base distribution to be normal, but there is wide latitude in the choice of the base distribution. An attraction of the approach is that if the CDF and inverse CDF (or quantile function) are tractable for the base distribution, they will remain so for the transmuted distribution [15].

Received : *15–September–2015,* ***Accepted :*** *05–March–2016*

A random variable 'X', is said to have transmuted distribution if its cumulative distribution function (cdf) and probability density function (pdf) are given by (1.1) and (1.2).

$$G(x) = (1+\delta)F(x) - \delta F(x)^2 \tag{1.1}$$

$$g(x) = f(x)\left[(1+\delta) - 2\delta F(x)\right] \tag{1.2}$$

where f(x) and F(x) are the pdf and cdf of the subject distribution and δ is the transmuted parameter which lies between -1 and 1. The effect of the quadratic rank transmutation map (QRTM) is to introduce skew to a symmetric base distribution. There is no specific requirement that the base distribution F be symmetric. However, if the F distribution is symmetric about the origin, in the sense that $F(x) = 1 - F(-x)$, we have the result that the distribution of the square of the transmuted random variable is identical to that of the distribution of the square of the original random variable [15]. For more information and basic properties of the quadratic rank transmutation, one can refer Shaw and Buckley [15].

In literature, so many new distributions are continuously been derived by many researchers in the context of transmutation. Here, we outline some existing transmuted distributions with subject distribution as Weibull in the literature. Aryal and Tsokos [9] proposed the transmuted weibull distribution in the case of two parameter and they studied the mathematical properties of the distribution and also provided the maximum likelihood estimation procedure. Elbatal and Aryal [4] proposed the transmuted additive weibull distribution based on the quadratic rank transmutation map and they have studied the subject distribution extensively by providing the properties and the estimation of parameters.

The transmuted inverse weibull distribution is proposed by Elbatal [10] and given many reduction forms for subject distribution by varying the parameter values. Shuaib Khan and Robert King [12] derived the transmuted modified weibull distribution and the least squares procedure is used to estimate the parameter and also given the explicit expressions for the quantiles of the distribution. Ebraheim [1] has been introduced the Exponentiated Transmuted Weibull Distribution (ETWD). The ETW distribution has the advantage of being capable of modeling various shapes of aging and failure criteria. Furthermore, Ebraheim showed that eleven lifetime distributions such as the Weibull, exponentiated Weibull, Rayleigh and exponential distributions, among others follow as special cases.

Apart from Weibull distribution as a subject distribution, there are many new transmuted distributions have been proposed by many authors in the literature. Some of which are presented here. Faton Merovci [5] developed a transmuted exponentiated exponential distribution using the quadratic rank transmutation map to generalize the exponentiated exponential distribution. Further, the properties and the estimation of parameters are discussed for the proposed distribution. Faton Merovci et. al. [8] is proposed a transmuted generalized inverse Weibull distribution and studied its properties. Faton Merovci [6] is proposed a transmuted Rayleigh distribution and provided a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. Further, Manisha and Montip [11] introduced the beta transmuted Weibull distribution, which contains a number of distributions as special cases. The distribution and moments of order statistics along with estimation of the model parameters are studied. Faton Merovci [7] derived the transmuted generalized Rayleigh distribution using quadratic rank transmutation map and the properties of the proposed distribution are derived and investigated. Recently, Shuaib Khan and Robert King [13] has been introduced the transmuted modified Inverse Rayleigh distribution by using quadratic rank transmutation map, which extends the modified Inverse Rayleigh distribution. Further, they derived the quantile, moments, moment generating function, entropy, mean deviation, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation.

In this paper, we derive a new transmuted distribution based on new modified weibull distribution which is proposed by Almalki and Yuan [2].

2. Proposed Distribution

Almalki and Yuan [2] introduced a new modified Weibull (NMW) distribution. It generalizes several commonly used distributions in reliability and lifetime data analysis, including the modified weibull distribution, the additive weibull distribution, the S-Z modified weibull distribution of Sarhan [14] , the weibull distribution, the exponential distribution, the Rayleigh distribution, the extreme value distribution and the linear failure rate (LFR) distribution by Bain [3].
The cumulative distribution function and probability density function of the new modified weibull distribution is given in (2.1) and (2.2)

$$F(x) = 1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}} \qquad x \geq 0 \tag{2.1}$$

$$f(x) = \left[\alpha\theta x^{\theta-1} + \beta(\gamma + \lambda x)x^{\gamma-1}\,\mathrm{e}^{\lambda x}\right]\,\mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}} \tag{2.2}$$

where α, β, θ, γ and λ are non-negative with θ and γ being shape parameters and α and β being scale parameters and λ is the acceleration parameter.

A random variable X is said to have the transmuted new modified weibull distribution (TNMWD) with parameter α, β, θ, γ, λ & $-1 \leq \delta \leq 1$ and its cumulative distribution function is defined using equations (1.1) and (2.1) as follows,

$$G(x) = \left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\left[1 + \delta - \delta\left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\right] \tag{2.3}$$

The probability density function of the transmuted new modified weibull distribution is obtained by substituting (2.2) in (1.2),

$$g(x) = \left[\alpha\theta x^{\theta-1} + \beta(\gamma + \lambda x)x^{\gamma-1}\,\mathrm{e}^{\lambda x}\right]\left[\mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\left[1 + \delta - 2\delta\left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\right] \tag{2.4}$$

where θ and γ are shape parameters; α and β are scale parameters; λ is the acceleration parameter and δ is the transmutation parameter. The new modified weibull distribution is a special case for $\delta = 0$. This TNMWD is a generalised distribution which fits and analyze more complex data and this distribution has the flexibility to have many other distributions as the parameters of the subject distribution varies. The following are some sub models for the TNMWD (Table 1).

The probability density and cumulative distribution functions of TNMWD will exhibit different behavior depending on the values of the parameters when chosen to be positive (Figures 1 & 2).

3. Reliability Analysis

In this section, the survival and hazard functions for TNMWD are presented.
Survival function
The survival function for the transmuted new modified weibull distribution is defined in equation (3.1),

$$S(x) = 1 - \left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\left[1 + \delta - \delta\left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\right] \tag{3.1}$$

Hazard Function
The hazard rate function of the TNMWD is defined as follows,

$$\begin{aligned} h(x) &= \frac{f(x)}{1 - F(x)} \\ &= \frac{\left[\alpha\theta x^{\theta-1} + \beta(\gamma + \lambda x)x^{\gamma-1}\,\mathrm{e}^{\lambda x}\right]\left[\mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\left[1 + \delta - 2\delta\left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\right]}{1 - \left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\left[1 + \delta - \delta\left[1 - \mathrm{e}^{-\alpha x^{\theta} - \beta x^{\gamma} \mathrm{e}^{\lambda x}}\right]\right]} \end{aligned} \tag{3.2}$$

Table 1. Sub models of Transmuted New Modified Weibull Distribution

Model	δ	λ	β	γ	θ	α	Probability Density Function
NMWD	0	-	-	-	-	-	$\left[\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}\mathrm{e}^{\lambda x}\right]\ \mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}$
MWD	0	0	-	-	-	-	$\left[\alpha\theta x^{\theta-1}+\beta\gamma x^{\gamma-1}\right]\ \mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}}$
WD	0	0	0	0	-	-	$\left[\alpha\theta x^{\theta-1}\right]\ \mathrm{e}^{-\alpha x^{\theta}}$
RD	0	0	0	0	2	-	$[2\alpha x]\ \mathrm{e}^{-\alpha x^{2}}$
ED	0	0	0	0	1	-	$\alpha\mathrm{e}^{-\alpha x}$
Ext.V.D	0	-	1	0	0	0	$\left[\beta\lambda\mathrm{e}^{\lambda x}\right]\mathrm{e}^{-\mathrm{e}^{\lambda x}}$
SZ.MWD	0	0	-	-	1	-	$\left[\alpha+\beta\gamma x^{\gamma-1}\right]\mathrm{e}^{-\alpha x-\beta x^{\gamma}}$
TRD	-	0	0	0	2	$\frac{1}{2\sigma^2}$	$\frac{x}{\sigma^2}\mathrm{e}^{\frac{-x^2}{2\sigma^2}}\left[1-\delta+2\delta\mathrm{e}^{\frac{-x^2}{2\sigma^2}}\right]$
RNMWD	0	-	-	$\frac{1}{2}$	$\frac{1}{2}$	-	$\frac{1}{2\sqrt{x}}\left[\alpha+\beta(1+2\lambda x)\mathrm{e}^{\lambda x}\right]\mathrm{e}^{-\alpha\sqrt{x}-\beta\sqrt{x}\ \mathrm{e}^{\lambda x}}$
LF Rate D	0	0	-	2	1	-	$[\alpha+2\beta x]\,\mathrm{e}^{-\alpha x-\beta x^{2}}$
AWD	0	0	-	-	-	-	$\left[\alpha\theta x^{\theta-1}+\beta\gamma x^{\gamma-1}\right]\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}}$

T=Transmuted, N=New, M=Modified, W=Weibull, Ext=Extreme, V=Value,
R=Rayleigh, E=Exponential, L=Linear, F=Failure, A=Additive, D=Distribution

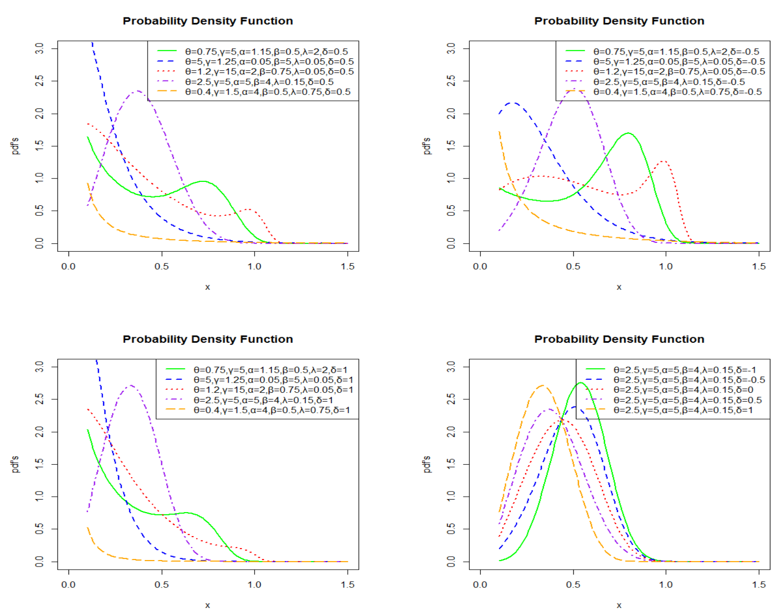

Figure 1. Probability Density Function of the Transmuted New Modified Weibull Distribution

The hazard function can have many different shapes, including bathtub (Figure 3). It is desirable for a bathtub shaped hazard function to have a long useful life period, with relatively constant failure rate in the middle. A few distributions have this property, so does the Transmuted New Modified Weibull Distribution as shown in Figure 3.

4. Statistical Properties

In this section, statistical properties of TNMWD including moments and moment generating function are discussed.

Moments

It is necessary to provide the mean and variance through the method of moments when a new distribution is proposed. Therefore, we derive the explicit expressions for the r^{th} order moments associated with (3.1) which is as follows,

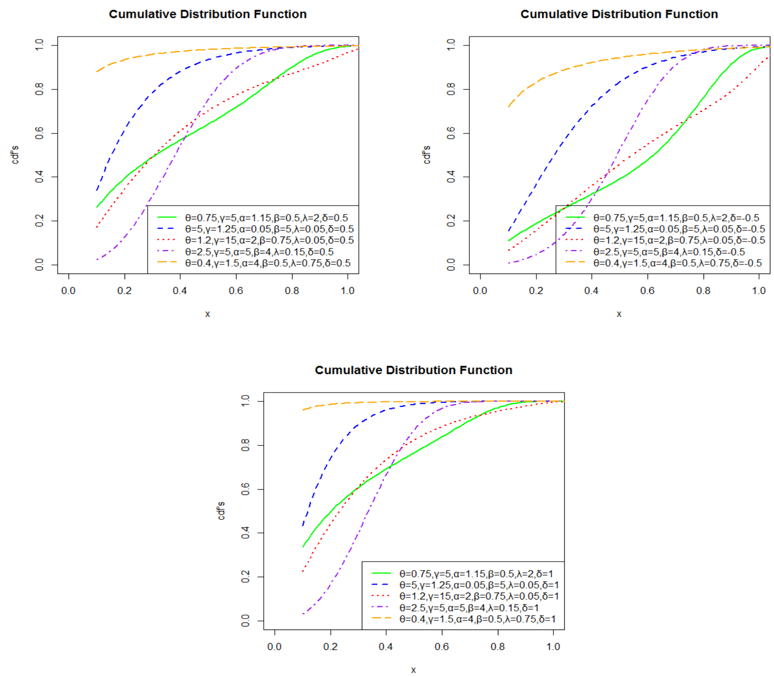

Figure 2. Cumulative Distribution Function of the Transmuted New Modified Weibull Distribution

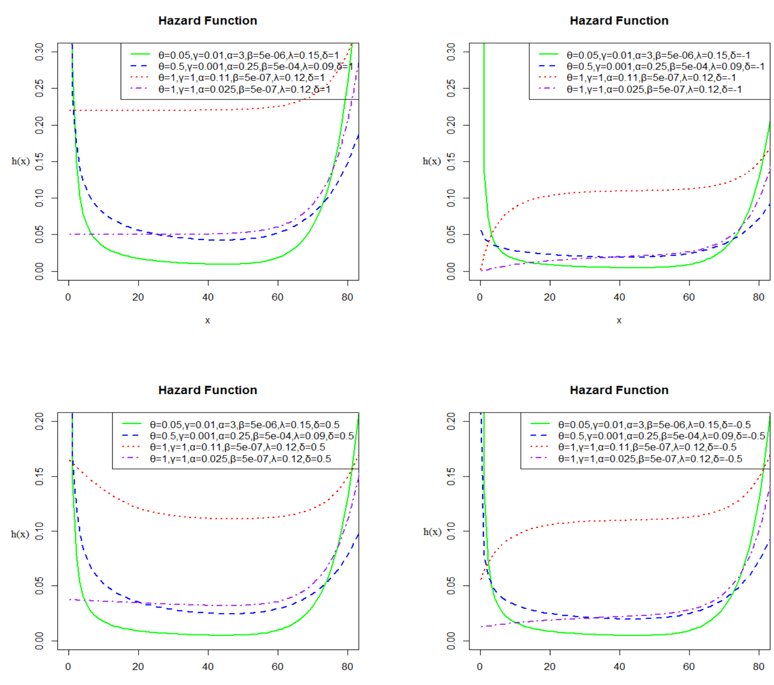

Figure 3. Hazard Function of the Transmuted New Modified Weibull Distribution

$$E(x^r) = \int_0^\infty x^r \mathrm{d}F(x)$$

$$E(x^r) = r\int_0^\infty x^{r-1}\left\{1-\left[1-\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\right]\left[1+\delta-\delta\left[1-\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\right]\right]\right\}\mathrm{d}x$$

$$E(x^r) = r\int_0^\infty x^{r-1}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x \;-\; r\delta\int_0^\infty x^{r-1}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x$$

Consider, $r\int_0^\infty x^{r-1}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x$

$$r\int_0^\infty x^{r-1}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}r\int_0^\infty x^{n\gamma+m+r-1}\mathrm{e}^{-\alpha x^\theta}\mathrm{d}x$$
$$=\frac{r}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\alpha^{-(n\gamma+m+r)/\theta}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right) \tag{4.2}$$

for $r=1,2,\ldots$, where $\Gamma(\cdot)$ is the gamma function.
In a similar manner, the explicit expression for $r\delta\int_0^\infty x^{r-1}\mathrm{e}^{-2\alpha x^\theta-2\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x$ is given as follows,

$$r\delta\int_0^\infty x^{r-1}\mathrm{e}^{-2\alpha x^\theta-2\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x=\frac{r\delta}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\beta)^n(\lambda n)^m}{n!m!}(2\alpha)^{-(n\gamma+m+r)/\theta}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right) \tag{4.3}$$

on substituting equations (4.2) & (4.3) in equation (4.1), the r^{th} order moments are given below,

$$\begin{aligned}E(x^r)=&\frac{r}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\alpha^{-(n\gamma+m+r)/\theta}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right)\\&-\frac{r\delta}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\alpha^{-(n\gamma+m+r)/\theta}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right)\\&+\frac{r\delta}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\beta)^n(\lambda n)^m}{n!m!}(2\alpha)^{-(n\gamma+m+r)/\theta}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right)\end{aligned} \tag{4.4}$$

When $\delta=0$ in equation (4.4), we have the r^{th} order moments for the new modified weibull distribution of Almalki and Yuan [2] which is as follows,

$$E(x^r)=\frac{r}{\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m}{n!m!}\alpha^{-(n\gamma+m+r)/\theta}\Gamma\left(\frac{n\gamma+m+r}{\theta}\right)$$

Moment Generating Function
In this subsection we derived the moment generating function (mgf) of transmuted new modified weibull distribution as follows.
If X follows transmuted new modified weibull distribution with $|\delta|\le 1$, then the moment generating function of X is given by,

$$\begin{aligned}M_X(t)=&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m t^{k+1}}{n!m!k!\theta}\alpha^{-(n\gamma+m+k+1)/\theta}\Gamma\left(\frac{n\gamma+m+k+1}{\theta}\right)\\&-\delta\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m t^{k+1}}{n!m!k!\theta}\alpha^{-(n\gamma+m+k+1)/\theta}\Gamma\left(\frac{n\gamma+m+k+1}{\theta}\right)\\&+\delta\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-2\beta)^n(\lambda n)^m t^{k+1}}{n!m!k!\theta}(2\alpha)^{-(n\gamma+m+k+1)/\theta}\Gamma\left(\frac{n\gamma+m+k+1}{\theta}\right)\end{aligned} \tag{4.5}$$

The moment generating function for the transmuted new modified weibull distribution is derived as,

$$
\begin{aligned}
M_X(t) &= \int_0^\infty \mathrm{e}^{tx}\mathrm{d}F(x) \\
M_X(t) &= \int_0^\infty t\mathrm{e}^{tx}\left\{1-\left[1-\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\right]\left[1+\delta-\delta\left[1-\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\right]\right]\right\}\mathrm{d}x \\
M_X(t) &= \int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x \;-\; \delta\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x \\
&\quad + \delta\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-2\alpha x^\theta-2\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x
\end{aligned} \tag{4.6}
$$

Consider, $\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x$

$$
\begin{aligned}
\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-\alpha x^\theta-\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x &= \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(-\beta)^n(\lambda n)^m}{n!m!}\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-\alpha x^\theta}x^{n\gamma+m}\mathrm{d}x \\
&= \sum_{n=0}^\infty\sum_{m=0}^\infty\sum_{k=0}^\infty \frac{(-\beta)^n(\lambda n)^m t^{k+1}}{n!m!k!\theta}\alpha^{-(n\gamma+m+k+1)/\theta}\Gamma\left(\frac{n\gamma+m+k+1}{\theta}\right)
\end{aligned} \tag{4.7}
$$

In a similar manner, the explicit expression for $\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-2\alpha x^\theta-2\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x$ is given as follows,

$$
\delta\int_0^\infty t\mathrm{e}^{tx}\mathrm{e}^{-2\alpha x^\theta-2\beta x^\gamma \mathrm{e}^{\lambda x}}\mathrm{d}x = \delta\sum_{n=0}^\infty\sum_{m=0}^\infty\sum_{k=0}^\infty \frac{(-2\beta)^n(\lambda n)^m t^{k+1}}{n!m!k!\theta}(2\alpha)^{-(n\gamma+m+k+1)/\theta}\Gamma\left(\frac{n\gamma+m+k+1}{\theta}\right) \tag{4.8}
$$

on substituting equations (4.7) & (4.8) in equation (4.6), the moment generating function for the transmuted new modified weibull distribution is as same as given in equation (4.5).

5. Maximum Likelihood Estimation

Let $x_1,\ x_2\ \ldots, x_n$ be a sample of size 'n' from transmuted new modified weibull distribution. Then the likelihood function 'L' is given by,

$$
L = \prod_{i=1}^n \left[\alpha\theta {x_i}^{\theta-1} + \beta(\gamma+\lambda x_i){x_i}^{\gamma-1}\,\mathrm{e}^{\lambda x_i}\right]\left[\mathrm{e}^{-\alpha {x_i}^\theta-\beta {x_i}^\gamma \mathrm{e}^{\lambda x_i}}\right]\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha {x_i}^\theta-\beta {x_i}^\gamma \mathrm{e}^{\lambda x_i}}\right]\right]
$$

The log-likelihood function for the TNMW distribution is,

$$
\begin{aligned}
\log L = \sum_{i=1}^n \log\left[\alpha\theta {x_i}^{\theta-1} + \beta(\gamma+\lambda x_i){x_i}^{\gamma-1}\,\mathrm{e}^{\lambda x_i}\right] - \alpha\sum_{i=1}^n {x_i}^\theta - \beta\sum_{i=1}^n {x_i}^\gamma \mathrm{e}^{\lambda x_i} \\
+ \sum_{i=1}^n \log\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha {x_i}^\theta-\beta {x_i}^\gamma \mathrm{e}^{\lambda x_i}}\right]\right]
\end{aligned} \tag{5.1}
$$

The partial differentiation with respect to the parameters of log-likelihood function are given as follows,

$$\frac{\partial \log L}{\partial \delta} = \sum_{i=1}^{n} \frac{1}{\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\right]\right]} \left\{1-2\left[1-\mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\right]\right\}$$

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\left[\alpha\theta x_i{}^{\theta-1}+\beta(\gamma+\lambda x_i)x_i{}^{\gamma-1}\,\mathrm{e}^{\lambda x_i}\right]} \left\{\theta x_i{}^{\theta-1}\right\} - \sum_{i=1}^{n} x_i{}^{\theta}$$
$$+ \sum_{i=1}^{n} \frac{1}{\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\right]\right]} \left\{2\delta \mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\left(-x_i{}^{\theta}\right)\right\}$$

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^{n} \frac{1}{\left[\alpha\theta x_i{}^{\theta-1}+\beta(\gamma+\lambda x_i)x_i{}^{\gamma-1}\,\mathrm{e}^{\lambda x_i}\right]} \left\{\alpha x_i{}^{\theta-1}+\alpha\theta x_i{}^{\theta-1}\log(x_i)\right\} - \alpha\sum_{i=1}^{n} x_i{}^{\theta}\log(x_i)$$
$$+ \sum_{i=1}^{n} \frac{1}{\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\right]\right]} \left\{2\delta \mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\left(-\alpha x_i{}^{\theta}\log(x_i)\right)\right\}$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{\left[\alpha\theta x_i{}^{\theta-1}+\beta(\gamma+\lambda x_i)x_i{}^{\gamma-1}\,\mathrm{e}^{\lambda x_i}\right]} \left\{(\gamma+\lambda x_i)x_i{}^{\gamma-1}\mathrm{e}^{\lambda x_i}\right\} - \sum_{i=1}^{n} x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}$$
$$+ \sum_{i=1}^{n} \frac{1}{\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\right]\right]} \left\{2\delta \mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\left(-x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}\right)\right\}$$

$$\frac{\partial \log L}{\partial \gamma} = \sum_{i=1}^{n} \frac{1}{\left[\alpha\theta x_i{}^{\theta-1}+\beta(\gamma+\lambda x_i)x_i{}^{\gamma-1}\,\mathrm{e}^{\lambda x_i}\right]} \left\{\beta x_i{}^{\gamma-1}\mathrm{e}^{\lambda x_i}+\beta\gamma x_i{}^{\gamma-1}\mathrm{e}^{\lambda x_i}\log(x_i)\right\}$$
$$-\beta\sum_{i=1}^{n} x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}\log(x_i)$$
$$+ \sum_{i=1}^{n} \frac{1}{\left[1+\delta-2\delta\left[1-\mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\right]\right]} \left\{2\delta \mathrm{e}^{-\alpha x_i{}^{\theta}-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}}\left(-\beta x_i{}^{\gamma}\mathrm{e}^{\lambda x_i}\log(x_i)\right)\right\}$$

The above partial derivatives are to be solved by equating to zero using the numerical methods. Hence, the Newton-Raphson's algorithm is useful to obtain the maximum likelihood estimates $\hat{\theta}$, $\hat{\gamma}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\delta}$ by directly maximizing the log-likelihood function (5.1). For the transmuted new modified Weibull distribution pdf, all the second order derivatives exist. Thus we have the inverse dispersion matrix as

$$\begin{pmatrix} \hat{\theta} \\ \hat{\gamma} \\ \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \\ \hat{\delta} \end{pmatrix} \sim N\left[\begin{pmatrix} \theta \\ \gamma \\ \alpha \\ \beta \\ \lambda \\ \delta \end{pmatrix},\ \Sigma\right]$$

with

$$\Sigma = -E\begin{bmatrix} V_{\theta\theta} & V_{\theta\gamma} & V_{\theta\alpha} & V_{\theta\beta} & V_{\theta\lambda} & V_{\theta\delta} \\ V_{\gamma\theta} & V_{\gamma\gamma} & V_{\gamma\alpha} & V_{\gamma\beta} & V_{\gamma\lambda} & V_{\gamma\delta} \\ V_{\alpha\theta} & V_{\alpha\gamma} & V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha\lambda} & V_{\alpha\delta} \\ V_{\beta\theta} & V_{\beta\gamma} & V_{\beta\alpha} & V_{\beta\beta} & V_{\beta\lambda} & V_{\beta\delta} \\ V_{\lambda\theta} & V_{\lambda\gamma} & V_{\lambda\alpha} & V_{\lambda\beta} & V_{\lambda\lambda} & V_{\lambda\delta} \\ V_{\delta\theta} & V_{\delta\gamma} & V_{\delta\alpha} & V_{\delta\beta} & V_{\delta\lambda} & V_{\delta\delta} \end{bmatrix}^{-1}$$

Here $V_{ii},\ i=\theta,\ \gamma,\ \alpha,\ \beta,\ \lambda,\ \delta$ denotes the second order derivative of log-likelihood function with respect to the parameters.

By solving this inverse dispersion matrix, the solutions will yield the asymptotic variance and co-variances of these maximum likelihood estimators of the parameters. Therefore, the $100(1-\zeta)\%$ confidence intervals for the parameters are approximately given as follows,

$$\hat{\theta} \pm Z_{\frac{\zeta}{2}}\sqrt{\hat{V}_{\theta\theta}}; \quad \hat{\gamma} \pm Z_{\frac{\zeta}{2}}\sqrt{\hat{V}_{\gamma\gamma}}; \quad \hat{\alpha} \pm Z_{\frac{\zeta}{2}}\sqrt{\hat{V}_{\alpha\alpha}}$$
$$\hat{\beta} \pm Z_{\frac{\zeta}{2}}\sqrt{\hat{V}_{\beta\beta}}; \quad \hat{\lambda} \pm Z_{\frac{\zeta}{2}}\sqrt{\hat{V}_{\lambda\lambda}}; \quad \hat{\delta} \pm Z_{\frac{\zeta}{2}}\sqrt{\hat{V}_{\delta\delta}}$$

where $Z_{\frac{\zeta}{2}}$ is the upper ζ^{th} percentile of the standard normal distribution.

6. Random Number Generation

The random numbers from the transmuted new modified weibull distribution can be done using the method of inversion as follows,

$$\left[1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]\left[1+\delta-\delta\left[1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]\right]=u \tag{6.1}$$

where $u \sim U(0,1)$. Further, the above equation can be reduced to the following expression

$$\alpha x^{\theta}+\beta x^{\gamma}\mathrm{e}^{\lambda x}+\log\left(\frac{(\delta-1)+\sqrt{(1-\delta)^2-4\delta(u-1)}}{2\delta}\right)=0 \tag{6.2}$$

The above equation has no closed form solution in x. Therefore, the random numbers can be generated by numerical methods only.

7. Order Statistics

If $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ denotes the order statsitics of a random sample $X_1, X_2, \ldots X_n$ from a continuous population with the cdf $G_X(x)$ and pdf $g_x(x)$ then the pdf of $X_{(j)}$ is given by

$$g_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}g_X(x)\left[G(x)\right]^{j-1}\left[1-G(x)\right]^{n-j}$$

$$\begin{aligned}g_{X_{(j)}}(x)=&\frac{n!}{(j-1)!(n-j)!}\left[\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}\mathrm{e}^{\lambda x}\right]\\&\left[\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]\left[1+\delta-2\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]\\&\left[1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]^{j-1}\left[1+\delta-\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]^{j-1}\\&\left\{1-\left[1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]\left[1+\delta-\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]\right\}^{n-j}\end{aligned}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$\begin{aligned}g_{X_{(n)}}(x)=&n\left[\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}\mathrm{e}^{\lambda x}\right]\left(\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\\&\left[1+\delta-2\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]\left[1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]^{n-1}\\&\left[1+\delta-\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]^{n-1}\end{aligned}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$\begin{aligned}g_{X_{(1)}}(x)=&n\left[\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}\mathrm{e}^{\lambda x}\right]\left(\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\\&\left[1+\delta-2\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]\\&\left\{1-\left[1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right]\left[1+\delta-\delta\left(1-\mathrm{e}^{-\alpha x^{\theta}-\beta x^{\gamma}\mathrm{e}^{\lambda x}}\right)\right]\right\}^{n-1}\end{aligned}$$

8. Conclusion

In Statistical and Reliability theory, life distributions plays a major role in explaining the nature and behavior of the data along with its properties. Of these life distributions, weibull distribution has gained attention from researchers of various subject domains. In this paper, we have introduced a new generalisation of new modified weibull distribution called the transmuted new modified weibull distribution. The subject distribution is derived using quadratic rank transformation map. The main focus of the quadratic rank transmutation map is to meet the needs of parametric families of distributions and it can be used to investigate a novel technique for introducing skewness or kurtosis into a symmetric or other distribution by considering wider statistical applications. Statistical properties such as moments and moment generating functions are presented. Further, estimation of parameters of TNMWD are obtained using maximum likelihood estimation procedure. The behavior of TNMWD is studied using the hazard and survival functions. It is shown that the proposed distribution possesses a bath-tub shaped curve, which serves the practical needs of the experimenter to make use of this TNMWD for studying the monotone hazard rates and fitness of the parameters.

References

[1] Abd El Hady N. Ebraheim., Exponentiated Transmuted Weibull Distribution - A Generalization of the Weibull Distribution. *International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering*, 8 (2014), no.6, 901-909.

[2] Almalki Saad J. and Yuan Jingsong. A new modified weibull distribution. *Reliability Engineering and System Safety* 111 (2013), 164-170.

[3] Bain L.J. Analysis for the linear failure-rate life-testing distribution. *Technometrics* 16 (1974), no. 4, 551-559.

[4] Elbatal Ibrahim and Aryal Gokarna., On the transmuted additive weibull distribution. *Austrian Journal of Statistics* 42 (2013), no. 2, 117-132.

[5] Faton Merovci., Transmuted Exponentiated Exponential Distribution. *Mathematical Sciences And Applications E-Notes*, 1 (2013), no. 2, 112-122.

[6] Faton Merovci., Transmuted Rayleigh Distribution. *Austrian Journal of Statistics*, 42 (2013), no. 1, 21-31.

[7] Faton Merovci., Transmuted Generalized Rayleigh Distribution. *Journal of Statistics Applications and Probability*, 3 (2014), no. 1, 9-20.

[8] Faton Merovci, Ibrahim Elbatal, and Alaa Ahmed., Transmuted Generalized Inverse Weibull Distribution. 2013, arXvi:1309.3268v1 [stat.ME].

[9] Gokarna R. Aryal and Chris P. Tsokos., Transmuted weibull distribution: A generalization of the weibull probability distribution. *European Journal of Pure and Applied Mathematics* 4 (2011), no. 2, 89-102.

[10] Ibrahim Elbatal., Transmuted modified inverse weibull distribution: A generalization of the modified inverse weibull probability distribution. *International Journal of Mathematical Archive* 4 (2013), no. 8, 117-129.

[11] Manisha Pal and Montip Tiensuwan., The Beta Transmuted Weibull Distribution. *Austrian Journal of Statistics*, 43 (2014), no. 2, 133-149.

[12] Muhammad Shuaib Khan and Robert King. Transmuted modified weibull distribution: A generalization of the modified weibull probability distribution. *European Journal of Pure and Applied Mathematics* 6 (2013), no. 1, 66-88.

[13] Muhammad Shuaib Khan and Robert King. Transmuted Modified Inverse Rayleigh Distribution. *Austrian Journal of Statistics* 44 (2015), 17-29.

[14] Sarhan A.M. and Zaindin M., Modified weibull distribution. *Austrian Journal of Statistics*, 11 (2009), 123-136.

[15] Shaw W.T. and Buckley I.R.C. The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. 2007, arXiv preprint, page arXiv:0901.0434.

Affiliations

R. Vishnu Vardhan
Address: Department of Statistics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry - 605 014
e-mail: rvvcrr@gmail.com