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On (p,q) and (q,k)-extensions of a double-inequality bounding a ratio of Gamma functions

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Abstract
In this paper, the authors present the (p,q) and (q,k)-extensions of a double inequality involving a ratio of Gamma functions. The method is based on some monotonicity properties of certain functions associated with the (p,q) and (q,k)-extensions of the Gamma function.

Keywords: *Gamma function, psi function, inequality, (p,q)-extension, (q,k)-extension.*

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1. Introduction

The classical Euler's Gamma function, $\Gamma(x)$ and the classical psi or digamma function, $\psi(x)$ are usually defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\dots(x+n)}$$

and

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The p-extension (also known as p-analogue, p-deformation or p-generalization) of the Gamma function, $\Gamma_p(x)$ is defined (see [2]) for $p \in \mathbb{N}$ and $x > 0$ as

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\dots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\dots(1+\frac{x}{p})}$$

where $\lim_{p\to\infty}\Gamma_p(x) = \Gamma(x)$.

Also, the q-extension of the Gamma function, $\Gamma_q(x)$ is defined (see [5]) for $q \in (0,1)$ and $x > 0$ as

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}} = (1-q)^{1-x}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{n-1+x}}$$

where $\lim_{q\to1}\Gamma_q(x) = \Gamma(x)$.

The k-extension of the Gamma function, $\Gamma_k(x)$ is similarly defined (see [3]) for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}^-$ as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

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where $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$ is the k-Pochhammer symbol and $\lim_{k\to 1}\Gamma_k(x) = \Gamma(x)$.

Krasniqi and Merovci [6] defined the (p,q)-extension of the Gamma for $p \in \mathbb{N}$, $q \in (0,1)$ and $x > 0$ as

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q[x+1]_q \dots [x+p]_q}$$

where $[p]_q = \frac{1-q^p}{1-q}$, and $\Gamma_{p,q}(x) \to \Gamma(x)$ as $p \to \infty$ and $q \to 1$. It satisfies the following identities.

$$\Gamma_{p,q}(x+1) = [x]_q \Gamma_{p,q}(x)$$
$$\Gamma_{p,q}(1) = 1.$$

The (p,q)-extension of the psi fuction is similarly defined as

$$\psi_{p,q}(x) = \frac{d}{dx}\ln\Gamma_{p,q}(x) = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}.$$

It satisfies the series representation:

$$\psi_{p,q}(x) = \ln[p]_q + (\ln q)\sum_{n=0}^{p}\frac{q^{n+x}}{1-q^{n+x}}$$

where $\psi_{p,q}(x) \to \psi(x)$ as $p \to \infty$ and $q \to 1$.

Díaz and Teruel [4] further defined the (q,k)-extension of the Gamma for $q \in (0,1)$, $k > 0$ and $x > 0$ as

$$\Gamma_{q,k}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}(1-q)^{\frac{x}{k}-1}}$$

where $(x+y)_{q,k}^n := \prod_{i=0}^{n-1}(x+q^{ik}y)$, $(1+x)_{q,k}^{\infty} := \prod_{i=0}^{\infty}(1+q^{ik}x)$, and $(1+x)_{q,k}^t := \frac{(1+x)_{q,k}^{\infty}}{(1+q^{kt}x)_{q,k}^{\infty}}$ for $x, y, t \in \mathbb{R}$ and $n \in \mathbb{N}$. It also satisfies the following identities.

$$\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x)$$
$$\Gamma_{q,k}(k) = 1$$

Similarly, the (q,k)-extension of the psi function is defined as

$$\psi_{q,k}(x) = \frac{d}{dx}\ln\Gamma_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)}$$

satisfying the series representation (see also [9] and the references therein):

$$\psi_{q,k}(x) = -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{nk+x}}{1-q^{nk+x}}$$

where $\psi_{q,k}(x) \to \psi(x)$ as $q \to 1$ and $k \to 1$.

In 2009, Vinh and Ngoc [10] by using the Dirichlet's integral approach proved the inequality

$$\frac{\prod_{i=1}^n \Gamma(1+\alpha_i)}{\Gamma(\beta + \sum_{i=1}^n \alpha_i)} \leq \frac{\prod_{i=1}^n \Gamma(1+\alpha_i x)}{\Gamma(\beta + \sum_{i=1}^n \alpha_i x)} \leq \frac{1}{\Gamma(\beta)} \tag{1.1}$$

where $x \in [0,1]$, $\beta \geq 1$, $\alpha_i > 0$, $n \in \mathbb{N}$. This provides a generalization of the previous results of Alsina and Tomás [1].

In the papers [8] and [12], the authors by using different procedures, proved a k-extension of (1.1) together with other results. Also in [9], the authors established amongst other results, a (q,k)-extension of the inequality (1.1).

Also, at the latter part of 2009, Ngoc, Vinh and Hien [11] further proved the following generalization of (1.1).

$$\frac{\prod_{i=1}^{n}\Gamma(b_i+\alpha_i)^{\mu_i}}{\Gamma(\beta+\sum_{i=1}^{n}\alpha_i)^{\lambda}} \leq \frac{\prod_{i=1}^{n}\Gamma(b_i+\alpha_i x)^{\mu_i}}{\Gamma(\beta+\sum_{i=1}^{n}\alpha_i x)^{\lambda}} \leq \frac{\prod_{i=1}^{n}\Gamma(b_i)^{\mu_i}}{\Gamma(\beta)^{\lambda}} \tag{1.2}$$

for $x \in [0,1]$ where b_i, α_i, β are real numbers, and μ_i, λ are positive real numbers such that $\lambda \geq \mu_i$, $\alpha_i > 0$, $\beta+\sum_{i=1}^{n}\alpha_i x \geq b_i+\alpha_i x > 0$ and $\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) > 0$, $i=1,\ldots,n$, $n \in \mathbb{N}$.

In this paper, our main interest is to establish the (p,q) and (q,k)-extensions of (1.2) by using techniques similar to those of [11].

2. Lemmas

In order to establish our results, we need the following Lemmas.

Lemma 2.1 ([6], [7]). *Let $0 < x \leq y$, $p \in \mathbb{N}$ and $q \in (0,1)$. Then,*

$$\psi_{p,q}(x) \leq \psi_{p,q}(y). \tag{2.1}$$

Lemma 2.2. *Let b_i, μ_i, α_i, β, λ and x be positive real numbers such that $\lambda \geq \mu_i$ and $\beta \geq b_i$. If $\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) > 0$ where $p \in \mathbb{N}$ and $q \in (0,1)$ then,*

$$\mu_i\psi_{p,q}(b_i+\alpha_i x) - \lambda\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) \leq 0.$$

Proof. Since $b_i+\alpha_i x \leq \beta+\sum_{i=1}^{n}\alpha_i x$ then by Lemma 2.1 we have $\psi_{p,q}(b_i+\alpha_i x) \leq \psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x)$. Then, $\lambda \geq \mu_i > 0$ and $\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) > 0$ implies

$$\lambda\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) \geq \mu_i\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) \geq \mu_i\psi_{p,q}(b_i+\alpha_i x)$$

Hence,

$$\mu_i\psi_{p,q}(b_i+\alpha_i x) - \lambda\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) \leq 0.$$

□

Lemma 2.3 ([7], [9]). *Let $0 < x \leq y$, $q \in (0,1)$ and $k > 0$. Then,*

$$\psi_{q,k}(x) \leq \psi_{q,k}(y). \tag{2.2}$$

Lemma 2.4. *Let b_i, μ_i, α_i, β, λ and x be positive real numbers such that $\lambda \geq \mu_i$ and $\beta \geq b_i$. If $\psi_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x) > 0$ where $q \in (0,1)$ and $k > 0$ then,*

$$\mu_i\psi_{q,k}(b_i+\alpha_i x) - \lambda\psi_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x) \leq 0.$$

Proof. By using Lemma 2.3, the proof is identical to that of Lemma 2.2.

□

3. Main Results

We now present our results.

Theorem 3.1. *Let b_i, μ_i, β, λ be positive real numbers, and α_i be real numbers such that $\lambda \geq \mu_i$ and $\beta \geq b_i$, $i=1,\ldots,n$, $n \in \mathbb{N}$. If $\psi_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x) > 0$ and $\alpha_i > 0$, then the inequality*

$$\frac{\prod_{i=1}^{n}\Gamma_{p,q}(b_i+\alpha_i)^{\mu_i}}{\Gamma_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i)^{\lambda}} \leq \frac{\prod_{i=1}^{n}\Gamma_{p,q}(b_i+\alpha_i x)^{\mu_i}}{\Gamma_{p,q}(\beta+\sum_{i=1}^{n}\alpha_i x)^{\lambda}} \leq \frac{\prod_{i=1}^{n}\Gamma_{p,q}(b_i)^{\mu_i}}{\Gamma_{p,q}(\beta)^{\lambda}} \tag{3.1}$$

holds for $x \in [0,1]$, $p \in \mathbb{N}$ and $q \in (0,1)$.

Proof. Define a function G for $x \in [0, \infty)$, $p \in \mathbb{N}$ and $q \in (0,1)$ by

$$G(x) = \frac{\prod_{i=1}^{n} \Gamma_{p,q}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}}.$$

Let $g(x) = \ln G(x)$. Then,

$$\begin{aligned} g(x) &= \ln \frac{\prod_{i=1}^{n} \Gamma_{p,q}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}} \\ &= \mu_i \ln \prod_{i=1}^{n} \Gamma_{p,q}(b_i + \alpha_i x) - \lambda \ln \Gamma_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x) \end{aligned}$$

implying that,

$$\begin{aligned} g'(x) &= \sum_{i=1}^{n} \mu_i \alpha_i \frac{\Gamma'_{p,q}(b_i + \alpha_i x)}{\Gamma_{p,q}(b_i + \alpha_i x)} - \lambda \left(\sum_{i=1}^{n} \alpha_i \right) \frac{\Gamma'_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x)}{\Gamma_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x)} \\ &= \sum_{i=1}^{n} \mu_i \alpha_i \psi_{p,q}(b_i + \alpha_i x) - \lambda \left(\sum_{i=1}^{n} \alpha_i \right) \psi_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x) \\ &= \sum_{i=1}^{n} \alpha_i \left[\mu_i \psi_{p,q}(b_i + \alpha_i x) - \lambda \psi_{p,q}(\beta + \sum_{j=1}^{n} \alpha_j x) \right] \leq 0. \end{aligned}$$

This is as a result of Lemma 2.2. That implies g is decreasing on $x \in [0, \infty)$. As a result, G is decreasing on $x \in [0, \infty)$ and for $x \in [0, 1]$ we have

$$G(1) \leq G(x) \leq G(0)$$

yielding the result as in (3.1) . □

Corollary 3.1. *If $x \in (1, \infty)$ in Theorem 3.1, then the inequality*

$$\frac{\prod_{i=1}^{n} \Gamma_{p,q}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}} < \frac{\prod_{i=1}^{n} \Gamma_{p,q}(b_i + \alpha_i)^{\mu_i}}{\Gamma_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i)^{\lambda}} \tag{3.2}$$

is satisfied.

Proof. For $x \in (1, \infty)$, we have $G(x) < G(1)$ ending the proof. □

Theorem 3.2. *Let b_i, μ_i, β, λ be positive real numbers, and α_i be real numbers such that $\lambda \geq \mu_i$ and $\beta \geq b_i$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. If $\psi_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x) > 0$ and $\alpha_i > 0$, then the inequality*

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i)^{\mu_i}}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i)^{\mu_i}}{\Gamma_{q,k}(\beta)^{\lambda}} \tag{3.3}$$

holds for $x \in [0, 1]$, $q \in (0, 1)$ and $k > 0$.

Proof. Similarly, define a function H for $x \in [0, \infty)$, $q \in (0, 1)$ and $k > 0$ by

$$H(x) = \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}}.$$

Let $h(x) = \ln H(x)$. Then, by following the steps of Theorem 3.1, in conjunction with Lemma 2.4, we arrive at

$$h'(x) = \sum_{i=1}^{n} \alpha_i \left[\mu_i \psi_{q,k}(b_i + \alpha_i x) - \lambda \psi_{q,k}(\beta + \sum_{j=1}^{n} \alpha_j x) \right] \leq 0$$

implying that h is decreasing on $x \in [0, \infty)$. Consequently H is decreasing on $x \in [0, \infty)$ and for $x \in [0, 1]$ we have

$$H(1) \leq H(x) \leq H(0)$$

yielding the result (3.3) . □

Corollary 3.2. *If $x \in (1, \infty)$ in Theorem 3.2, then the inequality*

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}} < \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i)^{\mu_i}}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)^{\lambda}} \tag{3.4}$$

is satisfied.

Proof. For $x \in (1, \infty)$, we have $H(x) < H(1)$ yielding the result. □

4. Concluding Remarks

In this section, we make the following remarks concerning our results.

Remark 4.1. If in Theorem 3.1, $\alpha_i < 0$ such that $0 < b_i + \alpha_i x \leq \beta + \sum_{i=1}^{n} \alpha_i x$ and $\psi_{p,q}(\beta + \sum_{i=1}^{n} \alpha_i x) > 0$, then the inequalities (3.1) and (3.2) are reversed.

Remark 4.2. Also, if in Theorem 3.2, $\alpha_i < 0$ such that $0 < b_i + \alpha_i x \leq \beta + \sum_{i=1}^{n} \alpha_i x$ and $\psi_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x) > 0$, then the inequalities (3.3) and (3.4) are reversed.

Remark 4.3. If we allow $p \to \infty$ in Theorem 3.1, or we set $k = 1$ in Theorem 3.2, then we obtain a q-extension of (1.2).

Remark 4.4. If we allow $q \to 1$ in Theorem 3.1, then we obtain a p-extension of (1.2).

Remark 4.5. If we allow $q \to 1$ in Theorem 3.2, then we obtain a k-extension of (1.2).

Remark 4.6. If we allow $p \to \infty$ as $q \to 1$ in Theorem 3.1, or we allow $q \to 1$ as $k \to 1$ in Theorem 3.2, then we obtain (1.2).

Remark 4.7. If we set $\mu_i = 1$, $b_i = k$ for $i = 1, \ldots, n$, and $\lambda = 1$ in Theorem 3.2, then we obtain the (q, k)-extension of (1.1) as established in [9].

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References

[1] C. Alsina and M. S. Tomás, A geometrical proof of a new inequality for the gamma function, *J. Ineq. Pure Appl. Math.*, 6(2)(2005), Art. 48.

[2] T. M. Apostol, Introduction to Analytic Number Theory, *Springer-Verlag*, 1976.

[3] R. Díaz and E. Pariguan, On hypergeometric functions and Pachhammer k-symbol, *Divulgaciones Matemtícas*, 15(2)(2007), 179-192.

[4] R. Díaz and C. Teruel, q, k-generalized gamma and beta functions, *J. Nonlin. Math. Phys.*, 12(2005), 118-134.

[5] F. H. Jackson, On a q-Definite Integrals, *Quarterly Journal of Pure and Applied Mathematics*, 41(1910), 193-203.

[6] V. Krasniqi and F. Merovci, Some Completely Monotonic Properties for the (p, q)-Gamma Function, *Mathematica Balkanica, New Series*, 26(2012), Fasc. 1-2.

[7] K. Nantomah, Some Inequalities for the Ratios of Generalized Digamma Functions, *Advances in Inequalities and Applications*, 2014(2014), Article ID 28.

[8] K. Nantomah and M. M. Iddrisu, The k-analogue of some inequalities for the Gamma function, *Electron. J. Math. Anal. Appl.*, 2(2)(2014), 172-177.

[9] K. Nantomah, E. Prempeh and S. B. Twum, The (q, k)-extension of some Gamma function inequalities, *Konuralp Journal of Mathematics*, 4(1)(2016), 148-154.

[10] N. V. Vinh and N. P. N. Ngoc, An inequality for the Gamma Function, *International Mathematical Forum*, 4(28)(2009), 1379-1382.

[11] N. P. N. Ngoc, N. V. Vinh and P. T. T. Hien, Generalization of Some Inequalities for the Gamma Function, *Int. J. Open Problems Compt. Math.*, 2(4)(2009), 532-535.

[12] J. Zhang and H. Shi, Two double inequalities for k-gamma and k-Riemann zeta functions, *Journal of Inequalities and Applications*, 2014, 2014:191.

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