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Value Distribution and Uniqueness of Entire Functions Related to Difference Polynomial

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Abstract
In this paper, we investigate the distribution of zeros of certain type of difference polynomial. At the same time we also investigate the uniqueness results when two difference products of entire functions share one value counting or ignoring multiplicities by considering that the functions share the value zero, counting multiplicities. The results of the paper improve and generalize some recent concerning results of W.L. Li and X.M. Li [Bull. Malay. Math. Sci. Soc., 39(2016), 499-515].

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1. Introduction, Definitions and Results

In this paper, a meromorphic function means meromorphic in the complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [7], [8] and [17]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f-a$ and $g-a$ coincide in locations and multiplicity, then we say that f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f-a$ and $g-a$ coincide only in their locations, then we say that f and g share the value a IM (ignoring multiplicities). For a positive integer p, $N_p(r,a;f)$ denotes the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \le p$ and p times if $m > p$. A meromorphic function $\alpha(\not\equiv 0,\infty)$ is called a small function with respect to f, if $T(r,\alpha) = S(r,f)$. We define difference operators $\triangle_c f(z) = f(z+c) - f(z)$, $\triangle_c^n f(z) = \triangle_c^{n-1}(\triangle_c f(z))$, where c is a nonzero complex number and $n \ge 2$ is a positive integer. In particular, if $c = 1$, we use the usual difference notation $\triangle_c f(z) = \triangle f(z)$.

A lot of research works on entire and meromorphic functions whose differential polynomials share certain value or fixed point have been done by many mathematicians in the world (see [4], [13], [14], [16]). In recent value distribution in difference analogue has become a subject of great interest among the researchers. In 2006 R.G. Halburd and R.J. Korhonen [5] established a version of Nevanlinna theory based on difference operators. The difference logarithmic derivative lemma, given by R.G. Halburd and R.J. Korhonen [6] in 2006, Y.M. Chiang and S.J. Feng [3] in 2008 plays an important role in considering the difference analogues of Nevanlinna theory. With the development of difference analogue of Nevanlinna theory, the researchers concentrate their attention on the distribution of zeros of different types of difference polynomials and their corresponding uniqueness results. In 2007 I. Laine and C.C. Yang [9] proved the following result for difference polynomials.

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Theorem A. *Let f be a transcendental entire function with finite order and η be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z+\eta)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

In 2010 X.G. Qi, L.Z. Yang and K. Liu [12] proved the following uniqueness result which corresponds to Theorem A.

Theorem B. *Let f and g be two transcendental entire functions of finite order, and η be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share the value 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0)$, a_{n-1}, ... , a_0 are complex constants. We denote Γ_1, Γ_2 by $\Gamma_1 = m_1 + m_2$, $\Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. Throughout the paper we denote $d = \gcd(\lambda_0, \lambda_1, ..., \lambda_n)$, where $\lambda_i = n+1$ if $a_i = 0$, $\lambda_i = i+1$ if $a_i \neq 0$. In 2011 L. Xudan and W.C. Lin [15] considered the zeros of one certain type of difference polynomial and obtained the following result.

Theorem C. *Let f be a transcendental entire function of finite order and η be a fixed nonzero complex constant. Also suppose that $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0)$, a_{n-1}, ... , a_0 are complex constants, and m is the number of distinct zeros of $P(z)$. Then for $n > m$, $P(f(z))f(z+\eta) - \alpha(z) = 0$ has infinitely many solutions, where $\alpha(z)(\not\equiv 0)$ is a small function with respect to f.*

In the same paper the author also proved the following uniqueness result corresponding to Theorem C.

Theorem D. *Let f and g be two transcendental entire functions of finite order, η be a nonzero complex constant, and $n > 2\Gamma_2 + 1$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share the value 1 CM, then one of the following cases hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$;
(iii) $f = e^{\alpha}$, $g = e^{\beta}$, where α and β are two polynomials and $\alpha + \beta = c$, c is a constant satisfying $a_n^2 e^{(n+1)c} = 1$.

We recall the following example due to L. Xudan and W.C. Lin [15].

Example 1.1. *Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $\eta = 2\pi$. It is easily seen that $n > 2\Gamma_2 + 1$ and $P(f(z))f(z+\eta) = P(g(z))g(z+\eta)$. Therefore $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share the value 1 CM. It is also clear that though f and g satisfy $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$; $f \not\equiv tg$ for a constant t satisfying $t^m = 1$, where $m \in Z^+$.*

From the above example, we see that f and g do not share the value 0 CM. Regarding this one may ask the following question.

Question 1. *What can be said about the relationship between f and g, if f and g share the value 0 CM in Theorem D?*

Keeping the above question in mind, recently W.L. Li and X.M. Li [10] proved the following results.

Theorem E. *Let f and g be two transcendental entire functions of finite order such that f and g share the value 0 CM, let η be a nonzero complex constant, and let $n > 2\Gamma_2 + 1$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share the value 1 CM, then one of the following two cases hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = ce^{-\alpha}$, where α is a nonconstant polynomial and c is a constant satisfying $a_n^2 c^{n+1} = 1$.

Theorem F. *Let f and g be two transcendental entire functions of finite order such that f and g share the value 0 CM, η be a nonzero complex constant, and let $n > 3\Gamma_1 + 2\Gamma_2 + 4$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share the value 1 IM, then one of the following two cases hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = ce^{-\alpha}$, where α is a nonconstant polynomial and c is a constant satisfying $a_n^2 c^{n+1} = 1$.

Regarding Theorems E and F it is natural to ask the following question which is the motivation of the paper.

Question 2. *What happen if one consider the difference polynomials of the form $(P(f(z))f(z+\eta))^{(k)}$ where $k(\geq 0)$ is an integer ?*

In the paper, our main purpose is to find out the possible answer of the above question. We prove three results first one of which extends Theorem C, second one improves and generalizes Theorem E and the remaining improves and generalizes Theorem F. We are now ready to state our main results.

Theorem 1.1. *Let f be a transcendental entire function with finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that η is a nonzero complex constant, $n(\geq 1)$ and $k(\geq 0)$ are integers. Also suppose that $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n(\neq 0)$, a_{n-1}, ... , a_0 are complex constants. Then for $n > \Gamma_1 + km_2$, $(P(f(z))f(z+\eta))^{(k)} - \alpha(z) = 0$ has infinitely many solutions.*

Theorem 1.2. *Let f and g be two transcendental entire functions of finite order such that f and g share the value 0 CM. Suppose that η is a nonzero complex constant, $n(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n > 2\Gamma_2 + 2km_2 + 1$. If $(P(f(z))f(z+\eta))^{(k)}$ and $(P(g(z))g(z+\eta))^{(k)}$ share the value 1 CM, then one of the following two cases hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = ce^{-\alpha}$, where α is a nonconstant polynomial and c is a constant satisfying $a_n^2 c^{n+1} = 1$.

Theorem 1.3. *Let f and g be two transcendental entire functions with finite order such that f and g share the value 0 CM. Suppose that η is a nonzero complex constant, $n(\geq 1)$ and $k(\geq 0)$ are integers such that $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$. If $(P(f(z))f(z+\eta))^{(k)}$ and $(P(g(z))g(z+\eta))^{(k)}$ share the value 1 IM, then one of the following two cases hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = ce^{-\alpha}$, where α is a nonconstant polynomial and c is a constant satisfying $a_n^2 c^{n+1} = 1$.

2. Lemmas

In this section, we state some lemmas which will be needed in the sequel. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1. *[2] Let f be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\{r^{\rho-1+\varepsilon}\}.$$

Lemma 2.2. *[3] Let f be a meromorphic function of finite order ρ, $c(\neq 0)$ be a fixed complex constant. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O\{r^{\rho-1+\varepsilon}\} + O\{\log r\}.$$

Lemma 2.3. *[11] Let f be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then*

$$N(r, 0; f(z+c)) \leq N(r, 0; f) + S(r, f),$$

$$N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f),$$

$$\overline{N}(r, 0; f(z+c)) \leq \overline{N}(r, 0; f) + S(r, f),$$

$$\overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r, f),$$

outside of possible exceptional set with finite logarithmic measure.

Lemma 2.4. *Let f be a transcendental entire function of finite order and $F = P(f(z))f(z+c)$. Then*

$$T(r, F) = (n+1)T(r, f) + S(r, f).$$

From the lemma it is clear that $S(r, F) = S(r, f)$ and similarly $S(r, G) = S(r, g)$.

Proof. Noting that f is an entire function of finite order we deduce from Lemma 2.1 and the standard Valiron Mohon'ko theorem that

$$\begin{aligned}(n+1)T(r,f) &= T(r,f(z)P(f(z)))+S(r,f)\\ &= m(r,f(z)P(f(z)))+S(r,f)\\ &\leq m\left(r,\frac{f(z)P(f(z))}{f(z+c)P(f(z))}\right)+m(r,F)+S(r,f)\\ &\leq m\left(r,\frac{f(z)}{f(z+c)}\right)+m(r,F)+S(r,f)\\ &\leq T(r,F)+S(r,f). \end{aligned} \tag{2.1}$$

On the other hand, by Lemma 2.2 and the fact that f is a transcendental entire function of finite order, we obtain

$$\begin{aligned}T(r,F) &\leq T(r,P(f(z)))+T(r,f(z+c))+S(r,f)\\ &= nT(r,f)+T(r,f(z+c))+S(r,f)\\ &\leq (n+1)T(r,f)+S(r,f). \end{aligned} \tag{2.2}$$

Now the lemma follows from (2.1) and (2.2). □

Lemma 2.5. *[18] Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\leq T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f), \tag{2.3}$$

$$N_p\left(r,0;f^{(k)}\right)\leq k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{2.4}$$

Lemma 2.6. *[16] Let f and g be two nonconstant meromorphic functions sharing the value 1 CM. Then one of the following cases hold:*
(i) $T(r)\leq N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r)$,
(ii) $f=g$,
(iii) $fg=1$,
where $T(r)=\max\{T(r,f),T(r,g)\}$ *and* $S(r)=o\{T(r)\}$.

Lemma 2.7. *[1] Let F and G be two nonconstant meromorphic functions sharing the value 1 IM and $H\not\equiv 0$. Then*
$T(r,F)\leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G)$,
and the same inequality holds for $T(r,G)$.

Lemma 2.8. *Let f and g be two entire functions, $n(\geq 1)$, $k(\geq 0)$ be integers, and let*

$$F=(P(f(z))f(z+\eta))^{(k)},\ G=(P(g(z))g(z+\eta))^{(k)}.$$

If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n\leq 2\Gamma_1+2km_2+1$.

Proof. We put $F_1=P(f(z))f(z+\eta)$ and $G_1=P(g(z))g(z+\eta)$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}(r,c_1;F)+S(r,F)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F). \end{aligned} \tag{2.5}$$

Using (2.3), (2.4), (2.5), Lemma 2.3 and Lemma 2.4 we obtain

$$\begin{aligned}(n+1)T(r,f) &\leq T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;F_1)+S(r,f)\\ &\leq \overline{N}(r,0;G)+N_{k+1}(r,0;F_1)+S(r,f)\\ &\leq N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\\ &\leq (m_1+m_2+km_2+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \end{aligned} \tag{2.6}$$

Similarly

$$(n+1)T(r,g) \leq (m_1+m_2+km_2+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.7}$$

Combining (2.6) and (2.7) we obtain

$$(n-2m_1-2m_2-2km_2-1)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which gives $n \leq 2\Gamma_1+2km_2+1$. This proves the lemma. □

3. Proof of the Theorem

Proof of Theorem 1.1. Let $F_1 = P(f(z))f(z+\eta)$. Then F_1 is a transcendental entire function. If possible, we may assume that $F_1^{(k)}-\alpha(z)$ has only finitely many zeros. Then we have

$$N(r,\alpha;F_1^{(k)}) = O\{\log r\} = S(r,f). \tag{3.1}$$

Using (2.3), (3.1) and Nevanlinna's three small function theorem we obtain

$$\begin{aligned} T(r,F_1^{(k)}) &\leq \overline{N}(r,0;F_1^{(k)})+\overline{N}(r,\alpha;F_1^{(k)})+S(r,f) \\ &\leq T(r,F_1^{(k)})-T(r,F_1)+N_{k+1}(r,0;F_1)+S(r,f). \end{aligned} \tag{3.2}$$

Applying Lemma 2.4 we obtain from (3.2)

$$\begin{aligned} (n+1)T(r,f) &\leq N_{k+1}(r,0;F_1)+S(r,f) \\ &\leq (m_1+m_2+km_2+1)T(r,f)+S(r,f). \end{aligned}$$

This gives

$$(n-m_1-m_2-km_2)T(r,f) \leq S(r,f),$$

a contradiction with the assumption that $n > \Gamma_1+km_2$. This proves the theorem. □

Proof of Theorem 1.2. Let $F_1 = P(f(z))f(z+\eta)$, $G_1 = P(g(z))g(z+\eta)$, $F = F_1^{(k)}$ and $G = G_1^{(k)}$. Then F and G are transcendental entire functions that share the value 1 CM. Using (2.3) and Lemma 2.4 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(k)})+S(r,f) \\ &\leq T(r,(F_1)^{(k)})-(n+1)T(r,f)+N_{k+2}(r,0;F_1)+S(r,f) \\ &\leq T(r,F)-(n+1)T(r,f)+N_{k+2}(r,0;F_1)+S(r,f). \end{aligned}$$

From this we get

$$(n+1)T(r,f) \leq T(r,F)+N_{k+2}(r,0;F_1)-N_2(r,0;F)+S(r,f). \tag{3.3}$$

Also by (2.4) we obtain

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)})+S(r,f) \\ &\leq N_{k+2}(r,0;F_1)+S(r,f). \end{aligned} \tag{3.4}$$

Suppose, if possible, that (i) of Lemma 2.6 holds. Then using (3.4) we obtain from (3.3)

$$\begin{aligned} (n+1)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+N_{k+2}(r,0;F_1) \\ &\quad +S(r,f)+S(r,g) \\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+S(r,f)+S(r,g) \\ &\leq (m_1+2m_2+km_2+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \end{aligned} \tag{3.5}$$

Similarly

$$(n+1)T(r,g) \leq (m_1+2m_2+km_2+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \tag{3.6}$$

(3.5) and (3.6) together gives

$$(n-2m_1-4m_2-2km_2-1)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

contradicting with the fact that $n > 2\Gamma_2 + 2km_2 + 1$. Therefore by Lemma 2.6 we have either $FG = 1$ or $F = G$. Let $FG = 1$. Then

$$(P(f(z))f(z+\eta))^{(k)}(P(g(z))g(z+\eta))^{(k)} = 1. \tag{3.7}$$

Since f and g are entire functions, from (3.7) we deduce that $P(f(z)) \neq 0$ and $P(g(z)) \neq 0$. If possible, we assume that $P(z) = 0$ has two distinct roots, say, z_1 and z_2. Then

$$P(f(z)) = a_n(f-z_1)^{n_1}(f-z_2)^{n_2}$$

where n_1, n_2 are positive integers with $n_1 + n_2 = n$. Therefore $N(r, z_1; f) = O\{\log r\}$ and $N(r, z_2; f) = O\{\log r\}$. Now using Nevanlinna second fundamental theorem we immediately obtain a contradiction. Next we suppose that $P(z) = 0$ has only one root. Then $P(f(z)) = a_n(f-a)^n$ and $P(g(z)) = a_n(g-a)^n$, where a is a complex constant. Hence, from the assumption that f and g are two transcendental entire functions of finite order, we have $f(z) = e^{\alpha(z)} + a$ and $g(z) = e^{\beta(z)} + a$, $\alpha(z)$, $\beta(z)$ being nonconstant polynomials. From (3.7), we also see that $f(z+\eta) \neq 0$ and $g(z+\eta) \neq 0$ and therefore $a = 0$. Thus $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, $P(z) = a_n z^n$ and $[a_n e^{n\alpha(z)+\alpha(z+\eta)}]^{(k)}[a_n e^{n\beta(z)+\beta(z+\eta)}]^{(k)} = 1$. If $k = 0$ then proceeding similarly as in case 3 of the proof of Theorem 2 [10] we obtain $f(z) = e^{\alpha(z)}$, $g(z) = ce^{-\alpha(z)}$ where c is a constant satisfying $a_n^2 c^{n+1} = 1$. If $k \ge 1$ then we deduce

$$[a_n e^{n\alpha(z)+\alpha(z+\eta)}]^{(k)} = a_n e^{n\alpha(z)+\alpha(z+\eta)} P(\alpha', \alpha'_\eta, ..., \alpha^{(k)}, \alpha_\eta^{(k)}),$$

where $\alpha_\eta = \alpha(z+\eta)$. Obviously, $P(\alpha', \alpha'_\eta, ..., \alpha^{(k)}, \alpha_\eta^{(k)})$ has infinite zeros, so it is impossible. Next we assume that $F = G$. Then

$$(P(f(z))f(z+\eta))^{(k)} = (P(g(z))g(z+\eta))^{(k)}.$$

Integrating once we obtain

$$(P(f(z))f(z+\eta))^{(k-1)} = (P(g(z))g(z+\eta))^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 2.8 it follows that $n \le 2\Gamma_1 + 2(k-1)m_2 + 1$, a contradiction as $n > 2\Gamma_2 + 2km_2 + 1$ and $\Gamma_2 \ge \Gamma_1$. Hence $c_{k-1} = 0$. Repeating the process k-times, we deduce that

$$P(f(z))f(z+\eta) = P(g(z))g(z+\eta).$$

Then arguing similarly as in Case 2 in the proof of Theorem 2 [10] we obtain $f = tg$ for a constant t such that $t^d = 1$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let F, G, F_1 and G_1 be defined as in Theorem 1.2. Then F and G are transcendental entire functions that share the value 1 IM. We assume, if possible, that $H \not\equiv 0$. Using Lemma 2.7 and (3.4) we obtain from (3.3)

$$\begin{aligned}(n+1)T(r,f) &\le N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G)\\ &\quad +2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g)\\ &\le N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + 2N_{k+1}(r,0;F_1)\\ &\quad +N_{k+1}(r,0;G_1) + S(r,f) + S(r,g)\\ &\le (3m_1+4m_2+3km_2+3)T(r,f) + (2m_1+3m_2+2km_2+2)T(r,g)\\ &\quad +S(r,f) + S(r,g)\\ &\le (5m_1+7m_2+5km_2+5)T(r) + S(r). \end{aligned} \tag{3.8}$$

Similarly

$$(n+1)T(r,g) \le (5m_1+7m_2+5km_2+5)T(r) + S(r). \tag{3.9}$$

(3.8) and (3.9) together yields

$$(n-5m_1-7m_2-5km_2-4)T(r)\leq S(r),$$

a contradiction with the assumption that $n>3\Gamma_1+2\Gamma_2+5km_2+4$. We now assume that $H=0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{3.10}$$

where $A(\neq 0)$ and B are constants. From (3.10) it is obvious that F, G share the value 1 CM. Therefore $n>2\Gamma_2+2km_2+1$. We now discuss the following three cases separately.

Case 1. *We first assume that $B\neq 0$ and $A=B$. Then from (3.10) we obtain*

$$\frac{1}{F-1}=\frac{BG}{G-1}. \tag{3.11}$$

If $B=-1$, then from (3.11) we obtain $FG=1$. Then the result follows from the proof of Theorem 1.2.
If $B\neq -1$, from (3.11), we have $\frac{1}{F}=\frac{BG}{(1+B)G-1}$ and so $\overline{N}(r,\frac{1}{1+B};G)=\overline{N}(r,0;F)$. Using (2.3), (2.4) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{1+B};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,G)\\ &\leq N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)\\ &\quad -(n+1)T(r,g)+S(r,g).\end{aligned}$$

This gives

$$(n+1)T(r,g)\quad\leq\quad(m_1+m_2+km_2+1)\{T(r,f)+T(r,g)\}+S(r,g).$$

Thus we obtain

$$(n-2m_1-2m_2-2km_2-1)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction as $n>2\Gamma_2+2km_2+1$.

Case 2. *Next we assume that $B\neq 0$ and $A\neq B$. Then from (3.10) we get $F=\frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r,\frac{B-A+1}{B+1};G)=\overline{N}(r,0;F)$. Proceeding in a manner similar to Case 1 we arrive at a contradiction.*

Case 3. *Let $B=0$ and $A\neq 0$. Then from (3.10) we get $F=\frac{G+A-1}{A}$ and $G=AF-(A-1)$. If $A\neq 1$, it follows that $\overline{N}(r,\frac{A-1}{A};F)=\overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$. Then by Lemma 2.8 it follows that $n\leq 2\Gamma_1+2km_2+1$, a contradiction. Thus $A=1$ and hence $F=G$. Now the result follows from the proof of Theorem 1.2.*

This completes the proof of Theorem 1.3. □

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