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Para-Quaternionic Structures on the 3-Jet Bundle

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Abstract
In this paper we construct an almost para-quaternionic structure on the 3-jet bundle of an almost para-hermitian manifold and we study its integrability. We give a necessary and sufficient conditions that are provided for these structures to become para-hyper-Kähler and we prove that the 3-jet bundle can not be a para-quaternionic Kähler manifold.

Keywords: *Para-quaternionic; para-hyperhermitian; 3-jet bundle; λ -lift; diagonal metric.*

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1. Introduction

A para-quaternionic structure on a manifold consists of an almost para hypercomplex structure which is a triple of endomorphisms of the tangent bundle $\mathbb{J} = \{J_1, J_2, J_3\}$, where J_1 is almost complex structure and $J_2,\ J_3$ are almost product structures satisfying anti-commutation relations and compatible with a semi-Riemannian metric necessarily of neutral signature. Moreover, if the structures $\mathbb{J}$ are parallel with respect to the Levi-Civita connection of the compatible metric, one arrives at the concept of para-hyper-Kähler structure and also named neutral hyper-Kähler or hypersymplectic structure ([3], [13], [14], [16], [18]...). The para-quaternionic and para-hyperhermitian structures are structures that appear in theorical physics, precisely, in string theory and integrable systems ([1], [9], [11]...).

On the other hand, the 3-jet bundle or the third order tangent bundle T^3M of a smooth n-dimensional manifold M is the $4n$-dimensional smooth manifold of equivalent classes of curves c on M that agree up to their 3-velocity or the manifold of 3-jet denoted j^3c. That is a generalization of the tangent bundle TM. This bundle has been studied with different names by many authors (see [5], [7]) for 2-jet bundle and generalized to the r-jet bundle in ([6]), where T^rM is the smooth manifold of equivalent classes of curves c on M that agree up to their r-velocity or a manifold of r-jets.

Dodson and Radivoiovici prove that T^2M becomes a vector bundle over M with structure group $GL(2n;R)$ if the manifold M is endowed with additional structure: a linear connection ∇ ([7]), this result was generalized to T^rM $(r \geq 2)$ in ([6]). Then, the 3-jet bundle of n-dimensional manifold M is a vector bundle when M is endowed with a linear connection ∇.

The linear connection ∇ on a manifold M defines a diffeomorphism S between the 3-jet bundle T^3M and the Whitney sum of three copies of the tangent bundles TM. S is a fibre diffeomorphism of locally trivial bundle but it is not an isomorphism of natural bundles. Next, using the vertical and horizontal lift (X^V, X^H) of vector fields $X \in \Gamma(TM)$ we define by the λ-lift the adapted frame $\{X^0, X^1, X^2, X^3\}$, so a sequence of distributions E_0, E_1, E_2 and E_3 on T^3M such that $T(T^3M) = \underset{i=\overline{0,3}}{\oplus} E_i$, when $\lambda = 1$. E_0, E_1 coincide with H and V respectively the horizontal and the vertical subspaces of TM. The λ-lift of tensor fields on manifold M to the 3-jet bundle T^3M, is a generalization of vertical and horizontal lift of geometric structures to the tangent bundle TM (see [6], [10]).

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The main purpose of this paper is to construct a para-quaternionic structure or para-hyperhermitian structure on the 3-jet bundle which is the generalization of this construction on tangent bundle (see [13], [20]), we also investigate its integrability, we obtain the necessary and sufficient conditions for these structures to become para-hyper-Kähler and finally we prove that the 3-jet bundle can not be a para-quaternionic Kähler manifold.

2. Preliminaries

An almost product (resp. complex) structure on a smooth manifold M is given by a tensor field P (resp. J) of type $(1,1)$ on M such that,

$$P \neq \pm Id \text{ and } P^2 = Id. \text{ (resp. } J^2 = -Id)$$

(M, P) (resp. (M, J)) is called an almost product (resp. complex) manifold. Moreover, if M is endowed with pseudo-Riemannian metric g satisfying

$$g(PX, PY) = -g(X, Y) \text{ (resp. } g(JX, JY) = g(X, Y))$$

for all vector fields X, Y on M, (M, g, P) (resp. (M, g, J)) is called an almost para-hermitian (resp. hermitian) structure.

When M support three tensor fields $\mathbb{J} = (J_\alpha)_{\alpha=1,2,3}$ where J_1 is an almost complex structure and J_2, J_3 are an almost product structures satisfying:

$$\begin{cases} J_\alpha^2 = -\varepsilon_\alpha Id \\ J_1 J_2 = -J_2 J_1 = J_3 \end{cases} \tag{2.1}$$

where $\alpha = 1, 2, 3,\ \varepsilon_1 = 1,\ \varepsilon_2 = \varepsilon_3 = -1$, then M is said to be an almost para-hypercomplex manifold and denoted $(M, \mathbb{J})$. Its dimension is multiple of 4.

A semi-Riemannian metric g on $(M, \mathbb{J})$ is said to be compatible or adapted to the almost para-hypercomplex structure $\mathbb{J}$ if it satisfies

$$g(J_1X, J_1Y) = -g(J_2X, J_2Y) = -g(J_3X, J_3Y) = g(X, Y) \tag{2.2}$$

for all vector fields X, Y on M. The pair $(g, \mathbb{J})$ is called an almost para-hyperhermitian structure on M and the triple $(M, g, \mathbb{J})$ is said to be an almost para-hyperhermitian manifold. Its adapted metric is of neutral signature $(2n, 2n)$. If $\mathbb{J}$ is parallel with respect to the Levi-Civita connection of g, then the manifold is called a para-hyper-Kähler.

Moreover, we say that $\mathbb{J}$ is integrable if its Nijenhuis tensor

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] + J_\alpha^2[X, Y];\ \alpha = 1, 2, 3$$

is zero for all vector fields X and Y on M, then $(M, \mathbb{J})$ is called a para-hypercomplex manifold. If g is a semi-Riemannian metric adapted to structure $\mathbb{J}$, then the pair $(g, \mathbb{J})$ is said to be a para-hyperhermitian structure on M and $(M, g, \mathbb{J})$ is called a para-hyperhermitian manifold.

For a n-dimensional manifold M, let assume that there is a rank 3-sub bundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ of sections of σ exists satisfying the formula (2.1). Then the bundle σ is called a para-quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ. Moreover, (M, σ) is said to be an almost para-quaternionic manifold.

A pseudo-Riemannian metric g is said to be adapted to the para-quaternionic structure σ if any local basis $\{J_1, J_2, J_3\}$ of σ satisfies the formula (2.2). (M, σ, g) is said to be an almost hermitian para-quaternionic manifold.

3. Bundle of the 3-jet

Let M be an n-dimensional smooth differentiable manifold. For each $x \in M$, we define an equivalence relation on the set $C_x = \{c : (-\varepsilon, \varepsilon) \to M \ / \ c \text{ is smooth and } c(0) = x, \varepsilon > 0\}$ by

$$c \approx_x h \Longleftrightarrow c^{(i)}(0) = h^{(i)}(0)\ for\ i = \overline{1, 3},$$

where $c^{(i)}$ denote the derivation of order i of c :

$$c^{(i)} : (-\varepsilon, \varepsilon) \to TM\ ;\ t \to [\frac{dc^{(i)}(t)}{dt^{(i)}}](0)$$

Definition 3.1. The 3-jet space or the third order tangent space of M at the point x denoted by $T^3_x M$ is the quotient $C_x/\approx_x$ and the 3-jet-bundle or the third order tangent bundle of M is the union of all the 3-jets spaces

$$T^3M = \underset{x\in M}{\cup} T^3_x M.$$

We denote by $j^3_x c$ the equivalence class of c with respect to $\approx_x$and by $j^3 c$ an element of T^3M.

Moreover, when M is endowed with a linear connection, T^3M becomes a vector bundle with structure group the general linear group $GL(2n;\mathbb{R})$ and $(3+1)n$-dimensional smooth manifold.

Now, let ∇ be a linear connection on M. Let $\pi_3 : T^3M \to M$ be the projection defined by $\pi_3(j^3c) = c(0)$, if $(U, x^1, ..., x^n)$ is a chart on M, then we consider the induced chart $(\pi_3^{-1}(U), x^{i,\lambda})_{i=\overline{1,n},\lambda=\overline{0,3}}$ on T^3M defined by

$$x^{i,\lambda}([c]_3) = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}(x^i \circ c)(0).$$

Using the connection ∇ we can define the diffeomorphism S by

$$\begin{aligned} S \quad &: \quad T^3M \to TM + TM + TM \\ S([c]_3) \quad &= \quad (\dot{c}(0), (\nabla_c\dot{c})(0), (\nabla_c\nabla_c\dot{c})(0)) \end{aligned}$$

∇_c denotes the covariant derivation along c and $\dot{c}$ is the velocity vector field of c, with

$$\nabla_c \dot{c}^i = (\frac{d^2c^i}{dt^2} + p_1^i)\frac{\partial}{\partial x^i} \text{ and } \nabla_c\nabla_c \dot{c}^i = (\frac{d^3c^i}{dt^3} + p_2^i)\frac{\partial}{\partial x^i}$$

where p_1^i (resp. p_2^i) is a polynomial of degree one (resp. two) on $\frac{d^kc^j}{dt^k}$ with $k \le 1$ (resp. $k \le 2$) and the coefficients of p_1^i (resp. p_2^i) depend on the connection coefficients Γ^i_{jk} (resp. $D_\alpha\Gamma^i_{jk}$, with $|\alpha| \le 1$).

4. Lift from M to T^3M

Let (M, g) be a pseudo-Riemannian manifold, ∇ it's Levi-Civita connection and R it's curvature tensor.

Some results of the lift from M to TM

Let be f a function on M. For any vector field X on M, we denote by f^V the vertical lift of f to TM defined by

$$f^V = f\circ\pi \text{ ; } \pi \text{ is projection from } TM \text{ to } M$$

Let be X a vector field on M. Then there is one and only one vector field X^V on TM called the vertical lift of X such that

$$X^V(f^V) = 0, \text{ for every } f$$

The connection ∇ define a horizontal distribution H on TM such that

$$T(TM) = V \oplus H \quad \text{where } V = \ker d\pi \tag{4.1}$$

Since for every point z of TM

$$d_z\pi_{/H_z} : H_z \to T_{\pi(z)}M$$

is an isomorphism, then, if X is a vector field on M, we can define

$$X^H(z) = (d_z\pi_{/H_z})^{-1}(X_{\pi(z)})$$

X^H is a vector field on TM called the horizontal lift of X to TM.

Consequently, $\{X^H, X^V\}$ is a $2n$-frame which is called the adapted frame to ∇ in TM.

Lift from M to T^3M

Let $X \in \Gamma(TM)$ be a vector field on M. For $\lambda = \overline{0,3}$, the $\lambda-$lift of X to T^3M is defined by

$$\begin{array}{l} X^0 = S_*^{-1}(X^H, X^H, X^H) \\ X^1 = S_*^{-1}(X^V, 0, 0) \\ X^2 = S_*^{-1}(0, X^V, 0) \\ X^3 = S_*^{-1}(0, 0, X^V) \end{array}$$

when $\lambda = 1$ the $\lambda-$lift of (X^0, X^1) coincide with (X^H, X^V) in TM. If $\lambda = 2$, the $\lambda-$lift was studied in ([5]) and for any $\lambda \geq 1$, it was studied in ([6]). $\{X^\lambda\}_{\lambda=0,..,3}$ is a $4n$-frame called the adapted frame in T^3M.

Proposition 4.1. *For all vector fields $X, Y \in \Gamma(TM)$ and $p \in T^3M$, we have the identities*

$$\begin{array}{rcl} [X^0, Y^0]_p &=& [X,Y]_p^0 - ((R(X,Y)u)^1 + (R(X,Y)w)^2 + (R(X,Y)z)^3) \\ [X^0, Y^i] &=& (\nabla_X Y)^i \\ [X^i, Y^0] &=& -(\nabla_Y X)^i \\ [X^i, Y^j] &=& 0 \end{array}$$

where $(u, w, z) = S(p)$ and $i, j = \overline{1,3}$.

Proof. For proof see [5] and [6]. □

Definition 4.1. The diagonal lift of g to T^3M denoted by ${}^Dg$ is defined by

$$\left\{ \begin{array}{l} i)\ {}^Dg(X^i, Y^i) = \ g(X,Y) \\ ii)\ {}^Dg(X^i, Y^j) = 0 \end{array} \right. \tag{4.2}$$

for $i, j = \overline{0,3}$ $(i \neq j)$ and X, Y vector fields in TM. ${}^Dg$ coincide with Sasaki metric on TM. (see [6])

The Levi-Civita connection ${}^D\nabla$ of ${}^Dg$ is given by Koszul formula as following

$$\begin{array}{rcl} ({}^D\nabla_{X^0} Y^0)_p &=& (\nabla_X Y)_p^0 - \frac{1}{2}((R(X,Y)u)^1 + (R(X,Y)w)^2 + (R(X,Y)z)^3) \\ ({}^D\nabla_{X^0} Y^i)_p &=& (\nabla_X Y)_p^i + \frac{1}{2}(R(X,Y)u)^0 \\ ({}^D\nabla_{X^i} Y^0)_p &=& \frac{1}{2}(R(X,Y)u)^0 \\ ({}^D\nabla_{X^i} Y^j)_p &=& 0 \ \text{ for } i, j \neq 0 \end{array} \tag{4.3}$$

for all vector fields $X,\ Y$ in TM, $p \in T^3M$ and $(u, w, z) = S(p)$.

Now, we suppose for sections 5 and 6 that (M, P, g) be an almost para-hermitian n-dimensional manifold.

5. Para-hyperhermitian structures

Definition 5.1. We define three tensor fields $\widetilde{\mathbb{J}} = (J_\alpha)_{\alpha=1,2,3}$ on T^3M by the equalities:

$$\left\{ \begin{array}{c} J_1X^0 = X^2 \\ J_1X^1 = X^3 \\ J_1X^2 = -X^0 \\ J_1X^3 = -X^1 \end{array} \right. , \quad \left\{ \begin{array}{c} J_2X^0 = PX^2 \\ J_2X^1 = PX^3 \\ J_2X^2 = PX^0 \\ J_2X^3 = PX^1 \end{array} \right. , \quad \left\{ \begin{array}{c} J_3X^0 = PX^0 \\ J_3X^1 = PX^1 \\ J_3X^2 = -PX^2 \\ J_3X^3 = -PX^3 \end{array} \right. ,$$

Then we have the theorem

Theorem 5.1. *The 3-jet bundle T^3M admits an almost para-hypercomplex structure $\widetilde{\mathbb{J}}$ which is a para-hyperhermitian with respect to ${}^Dg$.*

Proof. From the definition (5.1), we have for J_1

$$\begin{cases} J_1^2 X^0 = J_1 X^2 = -X^0 \\ J_1^2 X^1 = J_1 X^3 = -X^1 \\ J_1^2 X^2 = -J_1 X^0 = -X^2 \\ J_1^2 X^3 = -J_1 X^1 = -X^3 \end{cases} \Longrightarrow J_1^2 \widetilde{X} = -\widetilde{X}$$

the calculations for J_2^2 and J_3^2 are analogous to J_1^2.
For the anti-commutation rules, we have

$$\begin{cases} J_2 J_1 X^0 = J_2 X^2 = P X^0 = J_3 X^0 \\ J_2 J_1 X^1 = J_2 X^3 = P X^1 = J_3 X^1 \\ J_2 J_1 X^2 = -J_2 X^0 = -P X^2 = J_3 X^2 \\ J_2 J_1 X^3 = -J_2 X^1 = -P X^3 = J_3 X^3 \end{cases} \Longrightarrow J_2 J_1 \widetilde{X} = J_3 \widetilde{X}.$$

Its similar for $-J_1 J_2 = J_3$, thus

$$J_1^2 = -J_2^2 = -J_3^2 = -Id \text{ and } J_2 J_1 = -J_1 J_2 = J_3$$

Using the definition (5.1) for J_1, we have

$$\begin{cases} {}^D g(J_1 X^0, J_1 Y^0) = {}^D g(X^0, Y^0) \\ {}^D g(J_1 X^0, J_1 Y^1) = 0 = {}^D g(X^0, Y^1) \\ {}^D g(J_1 X^0, J_1 Y^2) = 0 = {}^D g(X^0, Y^2) \\ {}^D g(J_1 X^0, J_1 Y^3) = 0 = {}^D g(X^0, Y^3) \\ {}^D g(J_1 X^1, J_1 Y^1) = {}^D g(X^1, Y^1) \end{cases} \text{ and } \begin{cases} {}^D g(J_1 X^1, J_1 Y^2) = 0 = {}^D g(X^1, Y^2) \\ {}^D g(J_1 X^1, J_1 Y^3) = 0 = {}^D g(X^1, Y^3) \\ {}^D g(J_1 X^2, J_1 Y^2) = {}^D g(X^2, Y^2) \\ {}^D g(J_1 X^2, J_1 Y^3) = 0 = {}^D g(X^2, Y^3) \\ {}^D g(J_1 X^3, J_1 Y^3) = 0 = {}^D g(X^3, Y^3) \end{cases}$$

then

$${}^D g(J_1 \widetilde{X}, J_1 \widetilde{Y}) = {}^D g(\widetilde{X}, \widetilde{Y}).$$

And similarly for J_2 and J_3, we get the compatibility of $\widetilde{\mathbb{J}}$ with g^D defined in formula (4.2)

$${}^D g(J_1 \widetilde{X}, J_1 \widetilde{Y}) = -{}^D g(J_2 \widetilde{X}, J_2 \widetilde{Y}) = -{}^D g(J_3 \widetilde{X}, J_3 \widetilde{Y}) = {}^D g(\widetilde{X}, \widetilde{Y}).$$

Thus, $\widetilde{\mathbb{J}}$ is a para-hyperhermitian structure with respect to ${}^D g$. □

6. Study of integrability

First of all, we mention a general proposition:

Proposition 6.1. *Let (M, g, P) be an almost product manifold, then we have*

$$\begin{aligned} &1/\ \ [PX, PY] = (\nabla_{PX} P)(Y) - (\nabla_{PY} P)(X) + P(\nabla_{PX} Y - \nabla_{PY} X), \\ &2/\ \ P[PX, Y] = P\nabla_{PX} Y - P\nabla_Y PX, \\ &3/\ \ P[X, PY] = P\nabla_X PY - P\nabla_{PY} X, \end{aligned}$$

for all vector fields X, Y on M.

The integrability of structure $\widetilde{\mathbb{J}}$ is given by 16 equations for each $j, i = \overline{0,3}$, in following proposition.

Proposition 6.2. *The Nijenhuis tensor of structure J_1 is given by*

$$\begin{aligned} N_1(X^0, Y^0) &= (R(X,Y)u)^1 + (R(X,Y)v)^2 + (R(X,Y)w)^3 \\ N_1(X^2, Y^2) &= (R(X,Y)u)^1 + (R(X,Y)v)^2 + (R(X,Y)w)^3 \\ N_1(X^2, Y^0) &= N_1(X^2, Y^0) \\ &= J_1((R(X,Y)u)^1 + (R(X,Y)v)^2 + (R(X,Y)w)^3) \\ &= (R(X,Y)u)^3 - (R(X,Y)v)^0 - (R(X,Y)w)^1 \\ N_1(X^i, Y^j) &= 0 \textit{ for all } i, j = \overline{0,3} \textit{ and } (i,j) \neq \{(0,0), (0,2), (2,0), (2,2)\} \end{aligned}$$

Similarly, we deduce for J_2

$$\begin{array}{rcl} N_2(X^0,Y^0) &=& -(P(\nabla_X P)Y)^0 + (R(X,Y)u)^1 + (R(X,Y)v)^2 + (R(X,Y)w)^3 \\ N_2(X^0,Y^2) &=& -(P(\nabla_X P)Y + (\nabla_{PY}P)Y)^2 \\ && +(R(X,PY)u)^3 + (R(X,PY)v)^0 + (R(X,PY)w)^1 \\ N_2(X^2,Y^0) &=& (P(\nabla_Y P)X + (\nabla_{PX}P)X)^2 \\ && +(R(PX,Y)u)^3 + (R(PX,Y)v)^0 + (R(PX,Y)w)^1 \\ N_2(X^0,Y^i) &=& (P(\nabla_X P)Y)^i \quad \textit{for } i = 1{,}3 \\ N_2(X^i,Y^0) &=& -(P(\nabla_Y P)X)^i \ \textit{ for } i = 1{,}3 \\ N_2(X^i,Y^j) &=& 0 \textit{ for } i,j = 1{,}3 \\ N_2(X^1,Y^2) &=& -((\nabla_{PY}P)X)^3 \\ N_2(X^2,Y^1) &=& ((\nabla_{PX}P)Y)^3 \\ N_2(X^2,Y^2) &=& ((\nabla_{PX}P)Y - (\nabla_{PY}P)X)^0 \\ && -(R(PX,PY)u)^1 + (R(PX,PY)v)^2 + (R(PX,PY)w)^3 \\ N_2(X^2,Y^3) &=& ((\nabla_{PX}P)Y)^1 \\ N_2(X^3,Y^2) &=& -((\nabla_{PY}P)X)^1 \end{array}$$

and finally for J_3

$$\begin{array}{rcl} N_3(X^0,Y^0) &=& ((\nabla_{PX}P)Y - (\nabla_{PY}P)X - P(\nabla_X P)Y + P(\nabla_Y PX))^0 + \\ && +(P(R(PX,Y)u) + P(R(X,PY)u) - R(PX,PY)u - R(X,Y)u)^1 \\ && +(P(R(PX,Y)v) + P(R(X,PY)v) - R(PX,PY)v - R(X,Y)v)^2 \\ && +(P(R(PX,Y)w) + P(R(X,PY)w) - R(PX,PY)w - R(X,Y)w)^3 \end{array}$$

$$\begin{array}{rcl} N_3(X^0,Y^1) &=& (\nabla_{PX}PY - P(\nabla_X P)Y - P\nabla_{PX}Y)^1 \\ &=& ((\nabla_{PX}P)Y - P(\nabla_X P)Y)^1 \\ N_3(X^1,Y^0) &=& -(\nabla_{PY}PX - P(\nabla_Y P)X - P\nabla_{PY}X)^1 \\ &=& (P(\nabla_Y P)X - (\nabla_{PY}P)X)^1 \\ N_3(X^0,Y^i) &=& (P\nabla_{PX}Y - P(\nabla_X P)Y - \nabla_{PX}PY)^i \\ &=& -(P(\nabla_X P)Y + (\nabla_{PX}P)Y)^i \textit{ for } i = 2,3 \\ N_3(X^i,Y^0) &=& -(P\nabla_{PY}X - P(\nabla_Y P)X - \nabla_{PY}PX)^i \\ &=& (P(\nabla_Y P)X + (\nabla_{PY}P)X)^i \textit{ for } i = 2,3 \\ N_3(X^i,Y^j) &=& 0 \textit{ for } i,j = \overline{1,3} \end{array}$$

for all X, Y *vector fields on* M *and* $S(p) = (u, w, z)$.

Proof. We recall that the Nijenhuis tensor of the structures $\widetilde{\mathbb{J}} = (J_\alpha)_{\alpha=1,2,3}$ is

$$\begin{array}{rcl} N_\alpha(X^i,Y^j) &=& [J_\alpha X^i, J_\alpha Y^j] - J_\alpha[X^i, J_\alpha Y^j] - J_\alpha[J_\alpha X^i, Y^j] + J_\alpha^2[X^i,Y^j] \\ \text{for } \alpha &=& 1,2,3 \text{ and } j,i = \overline{0,3}. \end{array}$$

Using the definition (5.1) and the formula (4.3), we get the Nijenhuis tensor of J_1 (i.e N_1). For N_2 and N_3, we use also the proposition (6.1). □

Then, we have the following theorem

Theorem 6.1. *The structure* $\widetilde{\mathbb{J}} = (J_\alpha)_{\alpha=\overline{1,3}}$ *is an almost para-hypercomplex structure on* T^3M *which becomes almost para-hyperhermitian with respect to the diagonal lift metric* ${}^Dg$*. Moreover,* $(T^3M, {}^Dg, \widetilde{\mathbb{J}})$ *is para-hyperhermitian (i.e* $\widetilde{\mathbb{J}}$ *is integrable) if and only if* (M, P, g) *is a flat para-Kähler manifold.*

7. Para-quaternionic structures

Let (M, σ, g) be an almost hermitian para-quaternionic $4n$-dimensional manifold. We can locally choose a para-hypercomplex structure $\mathbb{J} = \{J_1, J_2, J_3\}$ which is a basis of σ. In fact, by the definition an almost hermitian para-quaternionic manifold is locally almost hermitian para-hypercomplex.

We can locally define a nondegenerate 2-forms

$$\Omega_{J_\alpha}(X, Y) = g(X, J_\alpha Y),\ \alpha = 1, 2, 3 \tag{7.1}$$

However, the 4-form

$$\Omega = \Omega_{J_1} \wedge \Omega_{J_1} - \Omega_{J_2} \wedge \Omega_{J_2} - \Omega_{J_3} \wedge \Omega_{J_3}$$

is defined globally on M.

Definition 7.1. An almost hermitian para-hypercomplex $4n$-dimensional manifold $(M, g, \mathbb{J})$ $(n \geq 1)$ is para-hyperKähler if $\nabla \mathbb{J} = 0$ (i.e $\nabla J_\alpha = 0$, $\alpha = \overline{1,3}$), where ∇ is the Levi-Civita connection with respect to the metric g. An almost hermitian para-quaternionic $4n$-dimensional manifold (M, σ, g) $(n \geq 2)$ is called para-quaternionic Kähler manifold if $\nabla \Omega = 0$ and M is not para-hyperKähler. (see [17],[13],[19])

Let (M, P, g) be an almost para-hermitian n-dimensional manifold. We shall call an almost para-quaternionic structure on T^3M any sub bundle $\tilde{\sigma}$ of the vector bundle $End(T^3M)$, locally spanned by a para-hyperhermitian structure $\widetilde{\mathbb{J}}$. The pair $(T^3M, \tilde{\sigma})$ will be called an almost para-quaternionic manifold and $(T^3M, \tilde{\sigma}, {}^D g)$ is called an almost hermitian para-quaternionic manifold.

We define three nondegenerate 2-forms in $(T^3M, \tilde{\sigma}, {}^D g)$ by

$$\tilde{\Omega}_{J_\alpha}(\widetilde{X}, \widetilde{Y}) = {}^D g(\widetilde{X}, J_\alpha \widetilde{Y}),\ \alpha = \overline{1,3} \tag{7.2}$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ in T^3M. The 4-form is given by

$$\tilde{\Omega} = \tilde{\Omega}_{J_1} \wedge \tilde{\Omega}_{J_1} - \tilde{\Omega}_{J_2} \wedge \tilde{\Omega}_{J_2} - \tilde{\Omega}_{J_3} \wedge \tilde{\Omega}_{J_3}$$

Proposition 7.1. *The Levi-Civita connection of ${}^D g$ satisfies ${}^D\nabla \tilde{\Omega} = 0$ if and only if (M, P, g) is a flat para-Kähler manifold.*

In order to prove the proposition, first, we need the following lemma.

Lemma 7.1. ***i.*** *From the formulas (4.2), (7.1) and the definition (5.1), the three 2-forms $\tilde{\Omega}_{J_\alpha}$ are given by*

$$\begin{cases} \tilde{\Omega}_{J_1}(X^i, Y^j) = g(X, Y) \text{ for } (i,j) = (2,0) \text{ and } (3,1) \\ \tilde{\Omega}_{J_1}(X^i, Y^j) = -g(X, Y) \text{ for } (i,j) = (0,2) \text{ and } (1,3) \\ \tilde{\Omega}_{J_1}(X^i, Y^j) = 0 \text{ for the remaining cases } (i,j) \end{cases}$$

$$\begin{cases} \tilde{\Omega}_{J_2}(X^i, Y^j) = g(X, PY) \text{ for } (i,j) = (2,0), (3,1), (0,2) \text{ and } (1,3) \\ \tilde{\Omega}_{J_2}(X^i, Y^j) = 0 \text{ for the remaining cases } (i,j) \end{cases}$$

$$\begin{cases} \tilde{\Omega}_{J_3}(X^i, Y^j) = g(X, PY) \text{ for } (i,j) = (0,0), (1,1) \\ \tilde{\Omega}_{J_3}(X^i, Y^j) = -g(X, PY) \text{ for } (i,j) = (2,2), (3,3) \\ \tilde{\Omega}_{J_3}(X^i, Y^j) = 0 \text{ for the remaining cases } (i,j) \end{cases}$$

with $i, j = \overline{0,3}$ and for any vector fields X, Y in TM.

ii.

$$\begin{aligned} {}^D\nabla_{U^i}(\tilde{\Omega}_{J_\alpha} \wedge \tilde{\Omega}_{J_\alpha})(X^i, Y^j, Z^k, W^l) &= 2 \sum_{Y^j, Z^k, W^l} ({}^D g(X^i, ({}^D\nabla_{U^i} J_\alpha) Y^j) \tilde{\Omega}_{J_\alpha}(Z^k, W^l) \\ &\quad + \tilde{\Omega}_{J_\alpha}(X^i, Y^j)\, {}^D g(Z^k, ({}^D\nabla_{U^i} J_\alpha) W^l)) \end{aligned}$$

where the sum is taken over cyclic permutations of Y^j, Z^k, W^l and $i, j, k, l = \overline{0,3}$.

iii.

$$({}^D\nabla_{X^i} J_\alpha)Y^j = {}^D\nabla_{X^i}(J_\alpha Y^j) - J_\alpha({}^D\nabla_{X^i} Y^j) \tag{7.3}$$

Using the formulas (4.3) and (7.3), we have

$$\begin{aligned}
({}^D\nabla_{X^0} J_\alpha)Y^0 &= {}^D\nabla_{X^0}(J_\alpha Y^0) - J_\alpha(\nabla_X Y)^0 \\
&\quad + \tfrac{1}{2}J_\alpha((R(X,Y)u)^1 + (R(X,Y)w)^2 + (R(X,Y)z)^3) \\
({}^D\nabla_{X^0} J_\alpha)Y^i &= {}^D\nabla_{X^0}(J_\alpha Y^j) - J_\alpha(\nabla_X Y)^i + \tfrac{1}{2}J_\alpha(R(u,Y)X)^0 \\
({}^D\nabla_{X^i} J_\alpha)Y^0 &= {}^D\nabla_{X^i}(J_\alpha Y^0) + \tfrac{1}{2}J_\alpha(R(X,Y)u)^0 \\
({}^D\nabla_{X^i} J_\alpha)Y^j &= {}^D\nabla_{X^i}(J_\alpha Y^j) \textit{ for } i,j = \overline{1,3}
\end{aligned} \tag{7.4}$$

Proof of the proposition 10. ${}^D\nabla_{X^i}\tilde{\Omega}$ is given by $4^4 = 256$ identities when the indices i,j,k,l varies from 0 to 3. For this, we have used a computer program with matlab software. ${}^D\nabla_{X^i}\tilde{\Omega}$ is calculated in three parts as follows

$$\begin{aligned}
{}^D\nabla_{X^i}\tilde{\Omega} &= {}^D\nabla_{X^i}(\tilde{\Omega}_{J_1}\wedge\tilde{\Omega}_{J_1}) - {}^D\nabla_{X^i}(\tilde{\Omega}_{J_2}\wedge\tilde{\Omega}_{J_2}) - {}^D\nabla_{X^i}(\tilde{\Omega}_{J_3}\wedge\tilde{\Omega}_{J_3}) \\
&= I_1 - I_2 - I_3 \quad i = \overline{0,3}
\end{aligned} \tag{7.5}$$

Taking account of lemma (7.1)-(i,ii) as databases of computer program, each part $I_\alpha(\alpha = 1,2,3)$ is calculated in (256) identities when the indices i,j,k,l vary from 0 to 3. We remark that all (256) identities are a sum of terms of types

$$g(X,Y)\ {}^Dg(Z^k, ({}^D\nabla_{X^i}J_\alpha)W^l) \text{ and } g(X,PY)\ {}^Dg(Z^k, ({}^D\nabla_{X^i}J_\alpha)W^l) \tag{7.6}$$

where X,Y,Z,W commutes over cyclic permutations except X in ${}^D\nabla_{X^i}$.
However, from the lemma (7.1) (iii), we have for the structure J_1

$$\begin{aligned}
({}^D\nabla_{X^0} J_1)Y^0 &= \tfrac{1}{2}((R(X,Y)u)^3 - (R(X,Y)z)^1) \\
({}^D\nabla_{X^0} J_1)Y^1 &= \tfrac{1}{2}((R(u,Y)X)^0 + (R(u,Y)X)^2)) \\
({}^D\nabla_{X^0} J_1)Y^2 &= \tfrac{1}{2}((R(X,Y)u)^1 + 2(R(X,Y)w)^2 + (R(X,Y)z)^3) \\
({}^D\nabla_{X^0} J_1)Y^3 &= \tfrac{1}{2}((R(u,Y)X)^2 - (R(u,Y)X)^0) \\
({}^D\nabla_{X^i} J_1)Y^0 &= \tfrac{1}{2}(R(u,X)Y)^2 \text{ for } i = \overline{1,3} \\
({}^D\nabla_{X^2} J_1)Y^2 &= -\tfrac{1}{2}(R(u,X)Y)^0 \\
({}^D\nabla_{X^i} J_1)Y^j &= 0 \text{ for } i,j = 1,3
\end{aligned} \tag{7.7}$$

and for J_2

$$\begin{aligned}
({}^D\nabla_{X^0} J_2)Y^0 &= ((\nabla_X P)Y)^2 + \tfrac{1}{2}((R(u,PY)X)^0 \\
&\quad + (PR(X,Y)w)^0 + (PR(X,Y)z)^1 + (PR(X,Y)u)^3) \\
({}^D\nabla_{X^0} J_2)Y^1 &= ((\nabla_X P)Y)^3 + \tfrac{1}{2}((R(u,PY)X)^0 + (PR(u,Y)X)^2) \\
({}^D\nabla_{X^0} J_2)Y^2 &= (\nabla_X P)Y)^0 - \tfrac{1}{2}((R(X,PY)u)^1 + ((R(PX,Y)w)^2 \\
&\quad - (PR(u,Y)X))^2 + (R(PX,Y)z)^3) \\
({}^D\nabla_{X^0} J_2)Y^3 &= (\nabla_X P)Y)^1 + \tfrac{1}{2}((R(u,PY)X)^0 + (PR(u,Y)X)^2) \\
({}^D\nabla_{X^i} J_2)Y^0 &= \tfrac{1}{2}(PR(X,Y)u)^2 \text{ for } i = \overline{1,3} \\
({}^D\nabla_{X^i} J_2)Y^2 &= \tfrac{1}{2}(R(u,X)PY)^0 \text{ for } i = \overline{1,3} \\
({}^D\nabla_{X^i} J_2)Y^j &= 0 \text{ for } i = 1,2,3 \text{ and } j = 1,3
\end{aligned} \tag{7.8}$$

finally for J_3

$$\begin{array}{rcl} ({}^D\nabla_{X^0}J_3)Y^0 &=& ((\nabla_X P)Y)^0 - \frac{1}{2}((R(X,PY)u)^1 + (R(X,PY)w)^2 + \\ && (R(X,PY)z)^3) + (P(R(X,Y)u)^1 - P(R(X,Y)w)^2 \\ && -P(R(X,Y)z)^3) \\ ({}^D\nabla_{X^0}J_3)Y^1 &=& ((\nabla_X P)Y)^1 + \frac{1}{2}(PR(u,Y)X)^0 \\ ({}^D\nabla_{X^0}J_3)Y^2 &=& -((\nabla_X P)Y)^2 + \frac{1}{2}(PR(u,Y)X)^0 \\ ({}^D\nabla_{X^0}J_3)Y^3 &=& -((\nabla_X P)Y)^3 + \frac{1}{2}(PR(u,Y)X)^0 \\ ({}^D\nabla_{X^i}J_3)Y^0 &=& \frac{1}{2}(R(u,X)PY + P(R(X,Y)u))^0 \text{ for } i = \overline{1,3} \\ ({}^D\nabla_{X^i}J_\alpha)Y^j &=& 0 \text{ for } i,j = \overline{1,3} \end{array} \tag{7.9}$$

Taking into account the formulas (7.7), (7.8) and (7.9), the terms (7.6) vanishes if and only if P is parallel and without curvature ($i.e$ $\nabla P = 0$ and $R \equiv 0$). Then, ${}^D\nabla_{X^i}\tilde{\Omega}$ vanishes if and only if (M,P,g) is a flat para-Kähler manifold. □

Remark 7.1. If P is not parallel or $R \neq 0$ then ${}^D\nabla_{X^i}\tilde{\Omega} \neq 0$.

Finally, we have the following theorems

Theorem 7.1. *Let $(T^3M,{}^D g)$ be the 3-jet bundle with para-hyperhermitian structure $\widetilde{\mathbb{J}}$ with respect to the diagonal lift metric g^D. $(T^3M,{}^D g)$ is para-hyperKähler manifold if and only if (M,P,g) is a flat para-Kähler manifold (i.e $\nabla P = 0$ and $R \equiv 0$).*

Proof. The proof is given from the formulas (7.7), (7.8) and (7.9). □

Theorem 7.2. *The almost hermitian para-quaternionic manifold $(T^3M,\tilde{\sigma},{}^D g)$ is never para-quaternionic Kähler manifold.*

Proof. From the proposition (7.1), we have ${}^D\nabla\tilde{\Omega} = 0$ if (M,P,g) is a flat para-Kähler manifold i.e $\nabla P = 0$ and $R \equiv 0$ or in this case, $(T^3M,\tilde{\sigma},{}^D g)$ is para-hyper Kähler manifold (i.e ${}^D\nabla\widetilde{\mathbb{J}} = 0$) and taking account about the definition (7.1), $(T^3M,\tilde{\sigma},{}^D g)$ is never para-quaternionic Kähler manifold. □

A possible extension of this paper is to construct a para-hyperhermitian (quaternionic) structures on r-jet bundle with $r = -1mod[4]$ as a naturally generalization of the tangent bundle of an almost para-hermitian manifold.

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References

[1] Barret, J., Gibbons, G.W., Perry, M.J., Pope, C.N. and Ruback P., Kleinian geometry and the $N = 2$ superstring, *Int . J. Mod. Phys. A* 9(1994), 1457-1494.

[2] Bielawski, R., Complexification and hypercomplexification of manifolds with a linear connection, *Internat. J. Math.* 14(2003), no.8, 813-824.

[3] Dancer, A. and Swann, A., Hypersymplectic manifolds, in "Recent developments in pseudo-Riemannian geometry". *ESI Lectures in Mathematics and Physics* (2008), 97-148.

[4] Davidov, J., Grantcharov, G., Mushkarov, O. and Yotov, M., Para-hyperhermitian surfaces. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 52(2009), no.3, 281-289.

[5] Dida, H.M., Hathout, F. and Djaa, M., On the geometry of the second order tangent bundle with the diagonal lift metric, *Int. Journal of Math. Anal.* 3(2009), 443-456.

[6] Djaa, M. and Gancarzewicz, J., The geometry of tangent bundles of order r, *Boletin Academia Galega de Ciencias spain* 4(1985), 147-165.

[7] Dodson, C.T.J. and Radivoiovici, M.S., Tangent and Frames Bundles of order two, *Analele stiintifice ale Univesitatii "AL. I Gusa"* 28(1982), 63-71.

[8] Dombrowski, P., On the geometry of the tangent bundle. *J.Reine Angew. Math.* 210(1962), 73-88.

[9] Feix, B., Hyperkahler metrics on cotangent bundles, *J. Reine Angew. Math* 532(2001), 33-46.

[10] Hathout, F. , Dida, H.M., Diagonal lift in the tangent bundle of order two and its applications, *Turk. J. Math*. 29(2005), 1-12.

[11] Hull, C.M., Actions for (2,1) sigma models and strings, *Nucl. Phys. B* 509(1998), 252-272.

[12] Ivanov, S., Sanov, V.T. and Zamkovoy, S., Hyper-para-Hermitian manifolds with torsion, *J. Geom. Phys.* 56(2006), no.4, 670-690.

[13] Ivanov, S. and Zamkovoy, S., Para-hermitianand para-quaternionic manifolds. *Differ. Geom. Appl*. 23(2005), 205-234.

[14] Kaledin, D., A canonical hyperkähler metric on the total space of a cotangent bundle. *Proceedings of the Second Meeting "Quaternionic structures in mathematics and physics" (Rome, 1999), World Sci. Publishing, River Edge, NJ* (2001), 195-230.

[15] Kamada, H., Neutral hyper-Kähler structures on primary Kodaira surfaces, *Tsukuba J. Math.* 23(1999) 321-332.

[16] Lanus, S. and Vîlcu, G.E., Some constructions of almost para-hyperhermitian structures on manifolds and tangent bundles, *Int. J. Geom. Methods Mod. Phys.*, 5(2008), no.6, 893-903.

[17] Lonescu, A.M. and Vîlcu, G.E., A Note on para-quaternionic Manifolds, *Missouri J. Math. Sci.*, 19(2007), 213-218.

[18] Nakashima, Y. and Watanabe, Y., Some constructions of almost Hermitian and quaternion metric structures, *Math. J. Toyama Univ.*, 13(1900), 119-138.

[19] Tahara, M., Marchiafava, S. and Watanabe, Y., Quaternionic Kähler structures on the tangent bundle of a complex space form, *Rend. Ist. Mat. Univ. Trieste*, 31(1999), 163-175.

[20] Vîlcu, G.E., Para-hyperhermitian structures on tangent bundles, *Proceedings of the Estonian Academy of Sciences* 60(2011), no.3, 165-173.

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