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A Generalization of Asymptotically $\mathcal{I}$-Lacunary Statistical Equivalence of Sequences of Sets

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Abstract
This paper presents, for sequences of sets, generalize the concepts of Wijsman asymptotically strongly $\mathcal{I}$-lacunary equivalence and Wijsman asymptotically strongly $\mathcal{I}$-Cesàro equivalence by using $p = (p_k)$ which is the sequence of positive real numbers where $\mathcal{I}$ is an ideal of the subset of the set $\mathbb{N}$ of natural numbers.

Keywords: *Asymptotically equivalence; statistical convergence; $\mathcal{I}$-convergence; lacunary sequence; Cesàro summability; sequences of sets; Wijsman convergence.*

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1. Introduction and Background

The concept of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [16] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set $\mathbb{N}$ of natural numbers. Das et al. [6] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal.

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. One of these extensions considered in this paper is the concept of Wijsman convergence (see, [2, 4, 12, 13, 18, 27, 31]). Nuray and Rhoades [18] extended the notions of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [27] defined the concept of Wijsman lacunary statistical convergence for sequence of sets and considered its relationship with Wijsman statistical convergence which was defined by Nuray and Rhoades [18]. Kişi and Nuray [13] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Also, the concepts of Wijsman $\mathcal{I}$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}$-lacunary convergence were given by Kişi et al. [15]. Recently, the concepts of Wijsman $\mathcal{I}$-Cesàro summability for sequences of sets were given by Ulusu and Kişi [26].

Marouf [17] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Patterson [19] extended these concepts by presenting an asymptotically statistical equivalent analogue of these definitions. Patterson and Savaş [20] extended the definitions presented in [19] to lacunary sequences. In addition to these definitions, natural inclusion theorems were presented. Savaş [23] presented the concept of $\mathcal{I}$-asymptotically lacunary statistically equivalence which is a natural combination of the definitions for asymptotically equivalence and $\mathcal{I}$-lacunary statistical convergence. Savaş and Gümüş [24] extended the concept of $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences by using the sequence $p = (p_k)$ which is the sequence of positive real numbers.

The concept of asymptotically equivalence of sequences which is defined by Marouf [17] has been extended by Ulusu and Nuray [28] to concepts of Wijsman asymptotically equivalence for sequences of sets. In addition to these definitions, natural inclusion theorems are presented. Kişi et al. [15] presented, for sequences of sets, the concept of

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Wijsman asymptotically $\mathcal{I}$-lacunary statistically equivalence which is a natural combination of the definitions for asymptotically equivalence and Wijsman $\mathcal{I}$-lacunary statistically convergence. Recently, the concepts of Wijsman asymptotically $\mathcal{I}$-Cesàro equivalence for sequences of sets were given by Ulusu [25].

Now, we recall the basic definitions and concepts (See [1–11, 13–18, 21, 22, 25–30, 32]).

Definition 1.1. A sequence (x_k) is said to be strongly Cesàro summable to the number L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|x_k-L|=0.$$

Definition 1.2. A sequence (x_k) is said to be statistically convergent to the number L if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\Big|\big\{k\le n:|x_k-L|\ge\varepsilon\big\}\Big|=0.$$

It is denoted by $st-\lim x_k=L$.

By a lacunary sequence we mean an increasing integer sequence $\theta=\{k_r\}$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 1.3. Let θ be a lacunary sequence. A sequence (x_k) is said to be lacunary statistically convergent to the number L if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\big\{k\in I_r:|x_k-L|\ge\varepsilon\big\}\Big|=0.$$

It is denoted by $S_\theta-\lim x_k=L$ or $x_k\to L(S_\theta)$.

Definition 1.4. Let θ be a lacunary sequence. A sequence (x_k) is said to be strongly lacunary summable to the number L if

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|x_k-L|=0.$$

A family of sets $\mathcal{I}\subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset\in\mathcal{I}$, (ii) For each $A,B\in\mathcal{I}$ we have $A\cup B\in\mathcal{I}$, (iii) For each $A\in\mathcal{I}$ and each $B\subseteq A$ we have $B\in\mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N}\notin\mathcal{I}$ and non-trivial ideal is called admissible if $\{n\}\in\mathcal{I}$ for each $n\in\mathbb{N}$.

A family of sets $\mathcal{F}\subseteq 2^{\mathbb{N}}$ is called a filter if and only if
(i) $\emptyset\notin\mathcal{F}$, (ii) For each $A,B\in\mathcal{F}$ we have $A\cap B\in\mathcal{F}$, (iii) For each $A\in\mathcal{F}$ and each $B\supseteq A$ we have $B\in\mathcal{F}$.

$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if $\mathcal{F}(\mathcal{I})=\big\{M\subset\mathbb{N}:(\exists A\in\mathcal{I})(M=\mathbb{N}\backslash A)\big\}$ is a filter in $\mathbb{N}$.

Definition 1.5. Let $\mathcal{I}\subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_k) is said to be $\mathcal{I}$-convergent to the number L if for every $\varepsilon>0$, the set

$$A(\varepsilon)=\big\{n\in\mathbb{N}:|x_k-L|\ge\varepsilon\big\}$$

belongs to $\mathcal{I}$.

Definition 1.6. Let θ be a lacunary sequence. A sequence (x_k) is said to be $\mathcal{I}$-lacunary statistically convergent to the number L if for every $\varepsilon,\delta>0$, the set

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\Big|\big\{k\in I_r:|\,x_k-L|\ge\varepsilon\big\}\Big|\ge\delta\right\}$$

belongs to $\mathcal{I}$. It is denoted by $x_k\to L(S_\theta(\mathcal{I}))$.

Definition 1.7. Let θ be a lacunary sequence. A sequence (x_k) is said to be strongly $\mathcal{I}$-lacunary convergent to the number L if for every $\varepsilon>0$, the set

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}|x_k-L|\ge\varepsilon\right\}$$

belongs to $\mathcal{I}$. It is denoted by $x_k\to L(N_\theta(\mathcal{I}))$.

Let (X,ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by
$$d(x,A) = \inf_{a\in A} \rho(x,a).$$

Definition 1.8. Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that a sequence $\{A_k\}$ is Wijsman convergent to A if for each $x \in X$,
$$\lim_{k\to\infty} d(x,A_k) = d(x,A).$$
It is denoted by $W - \lim A_k = A$.

Definition 1.9. Let (X,ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets $A, A_k \subset X$, we say that a sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$, the set
$$A(x,\varepsilon) = \left\{k \in \mathbb{N} : |d(x,A_k) - d(x,A)| \geq \varepsilon\right\}$$
belongs to $\mathcal{I}$. It is denoted by $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$.

Definition 1.10. Let (X,ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets A, $A_k \subseteq X$, we say that a sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$, the set
$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} |d(x,A_k) - d(x,A)| \geq \varepsilon\right\}$$
belongs to $\mathcal{I}$. It is denoted by $A_k \to A\left(C_1\left[\mathcal{I}_W\right]\right)$.

Definition 1.11. Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and θ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subset X$, we say that a sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-lacunary statistical convergent to A if for every $\varepsilon, \delta > 0$ and for each $x \in X$, the set
$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$
belongs to $\mathcal{I}$. It is denoted by $A_k \to A\left(S_\theta\left(I_W\right)\right)$.

Definition 1.12. Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and θ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subset X$, we say that a sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}$-lacunary convergent to A if for every $\varepsilon > 0$ and for each $x \in X$, the set
$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} |d(x,A_k) - d(x,A)| \geq \varepsilon\right\}$$
belongs to $\mathcal{I}$. It is denoted by $A_k \to A\left(N_\theta\left[\mathcal{I}_W\right]\right)$.

Definition 1.13. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if
$$\lim_k \frac{x_k}{y_k} = 1.$$
It is denoted by $x \sim y$.

Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$, the term $d(x;A_k,B_k)$ is defined as follows:
$$d(x;A_k,B_k) = \begin{cases} \dfrac{d(x,A_k)}{d(x,B_k)} & , \quad x \notin A_k \cup B_k \\ \\ L & , \quad x \in A_k \cup B_k. \end{cases}$$

Definition 1.14. Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically equivalent of multiple L if for each $x \in X$,

$$\lim_{k\to\infty} d(x;A_k,B_k) = L.$$

It is denoted by $A_k \overset{L}{\sim} B_k$ and simply Wijsman asymptotically equivalent if $L=1$.

Definition 1.15. Let (X,ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{I}$-equivalent of multiple L if for every $\varepsilon > 0$ and each $x \in X$,

$$\big\{k \in \mathbb{N} : |d(x;A_k,B_k) - L| \geq \varepsilon\big\} \in \mathcal{I}.$$

It is denoted by $A_k \overset{\mathcal{I}_W^L}{\sim} B_k$ and simply Wijsman asymptotically $\mathcal{I}$-equivalent if $L=1$.

Definition 1.16. Let (X,ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically strongly $\mathcal{I}$-Cesàro equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and for each $x \in X$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} |d(x;A_k,B_k) - L| \geq \varepsilon\right\}$$

belongs to $\mathcal{I}$. It is denoted by $A_k \overset{C_1^L[\mathcal{I}_W]}{\sim} B_k$ and simply asymptotically strongly $\mathcal{I}$-Cesàro equivalent (Wijsman sense) if $L=1$.

Definition 1.17. Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of multiple L provided that for every $\varepsilon, \delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\Big|\big\{k \in I_r : |d(x;A_k,B_k) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\} \in \mathcal{I}.$$

It is denoted by $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$ and simply Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent if $L=1$.

Definition 1.18. Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically strongly $\mathcal{I}$-lacunary equivalent of multiple L provided that for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} |d(x;A_k,B_k) - L| \geq \varepsilon\right\} \in \mathcal{I}.$$

It is denoted by $A_k \overset{N_\theta^L[\mathcal{I}_W]}{\sim} B_k$ and simply Wijsman asymptotically strongly $\mathcal{I}$-lacunary equivalent if $L=1$.

2. Main Results

In this section we introduce the concepts of asymptotically strongly $\mathcal{I}$-lacunary p-equivalence (Wijsman sense) and asymptotically strongly $\mathcal{I}$-Cesàro p-equivalence (Wijsman sense) for sequences of set. In addition to these definitions, natural inclusion theorems presented.

Definition 2.1. Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically strongly $\mathcal{I}$-lacunary p-equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} |d(x;A_k,B_k) - L|^{p_k} \geq \varepsilon\right\} \in \mathcal{I},$$

where $p=(p_k)$ be a sequence of positive real numbers. In this case, we write $A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k$ and simply asymptotically strongly $\mathcal{I}$-lacunary p-equivalent (Wijsman sense) if $L=1$.

If we take $p_k = p$ for all $k \in \mathbb{N}$, we write $A_k \overset{N_\theta^{L_p}[\mathcal{I}_W]}{\sim} B_k$ instead of $A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k$.

Theorem 2.1. *Let (X, ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and θ be a lacunary sequence. Then,*

(i) $A_k \overset{N_\theta^{L_p}[\mathcal{I}_W]}{\sim} B_k \Rightarrow A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$.

(ii) If $d(x, A_k) = O\big(d(x, B_k)\big)$, then $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k \Rightarrow A_k \overset{N_\theta^{L_p}[\mathcal{I}_W]}{\sim} B_k$.

Proof. (i). Let $A_k \overset{N_\theta^{L_p}[\mathcal{I}_W]}{\sim} B_k$. For every $\varepsilon > 0$ and for each $x \in X$, we can write

$$\begin{aligned} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^p &\geq \sum_{\substack{k \in I_r \\ |d(x;A_k,B_k)-L| \geq \varepsilon}} |d(x; A_k, B_k) - L|^p \\ &\geq \varepsilon^p \cdot \big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\big| \end{aligned}$$

and so we get

$$\frac{1}{\varepsilon^p \cdot h_r} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^p \geq \frac{1}{h_r} \Big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\Big|.$$

Then, for any $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \Big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^p \geq \varepsilon^p \cdot \delta\right\} \in \mathcal{I}.$$

Therefore, $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$.

(ii) Suppose that $d(x, A_k) = O\big(d(x, B_k)\big)$ and $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$. Since $d(x, A_k) = O\big(d(x, B_k)\big)$, there exists an $M > 0$ such that

$$|d(x; A_k, B_k) - L| \leq M$$

for all k and for each $x \in X$. For every $\varepsilon > 0$ and for each $x \in X$, we have

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^p &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |d(x;A_k,B_k)-L| \geq \varepsilon}} |d(x; A_k, B_k) - L|^p \\ &\quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |d(x;A_k,B_k)-L| < \varepsilon}} |d(x; A_k, B_k) - L|^p \\ &\leq \frac{1}{h_r} M^p \cdot \big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\big| \\ &\quad + \frac{1}{h_r} \varepsilon^p \cdot \big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\big| \\ &\leq \frac{M^p}{h_r} \cdot \big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\big| + \varepsilon^p. \end{aligned}$$

Then, for any $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^p \geq \varepsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r} \Big|\big\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\big\}\Big| \geq \frac{\varepsilon^p}{M^p}\right\} \in \mathcal{I}.$$

Therefore, $A_k \overset{N_\theta^{L_p}[\mathcal{I}_W]}{\sim} B_k$. $\square$

Theorem 2.2. *Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, θ be a lacunary sequence and $p=(p_k)$ be a sequence of positive real numbers. If $0<t=\inf_k p_k \leq \sup_k p_k = T < \infty$, then*

$$A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k \Rightarrow A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k.$$

Proof. Suppose that $\inf_k p_k = t$, $\sup_k p_k = T$ and $A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k$. For every $\varepsilon>0$ and for each $x\in X$, we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k} &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}|d(x;A_k,B_k)-L|^{p_k}\\
&\quad +\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|<\varepsilon}}|d(x;A_k,B_k)-L|^{p_k}\\
&\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}|d(x;A_k,B_k)-L|^{p_k}\\
&\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}(\varepsilon)^{p_k}\\
&\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}\min\left\{(\varepsilon)^t,(\varepsilon)^T\right\}\\
&\geq \frac{1}{h_r}\cdot\min\left\{(\varepsilon)^t,(\varepsilon)^T\right\}\cdot\left|\left\{k\in I_r : |d(x;A_k,B_k)-L|\geq\varepsilon\right\}\right|.
\end{aligned}$$

Then, for any $\delta>0$

$$\begin{aligned}
&\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in I_r : |d(x;A_k,B_k)-L|\geq\varepsilon\right\}\right|\geq\delta\right\}\\
&\qquad\subseteq\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k}\geq\delta\cdot\min\left\{(\varepsilon)^t,(\varepsilon)^T\right\}\right\}\in\mathcal{I}.
\end{aligned}$$

Hence, $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$. □

Theorem 2.3. *Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, θ be a lacunary sequence and $p=(p_k)$ be a sequence of positive real numbers. If $d(x,A_k)=O\big(d(x,B_k)\big)$ and $0<t=\inf_k p_k \leq \sup_k p_k = T<\infty$, then*

$$A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k \Rightarrow A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k.$$

Proof. Suppose that $d(x,A_k)=O\big(d(x,B_k)\big)$, $\inf_k p_k=t$, $\sup_k p_k=T$ and $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$. Since $d(x,A_k)=O\big(d(x,B_k)\big)$, there exists an $M>0$ such that

$$|d(x;A_k,B_k)-L|\leq M$$

for all k and for each $x\in X$. For every $\varepsilon>0$ and for each $x\in X$, we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k} &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}|d(x;A_k,B_k)-L|^{p_k}\\
&\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_k,B_k)-L|<\varepsilon}}|d(x;A_k,B_k)-L|^{p_k}\\
&\leq \frac{1}{h_r}\left|\left\{k\in I_r:|d(x;A_k,B_k)-L|\geq\frac{\varepsilon}{2}\right\}\right|\cdot\max\left\{M^t,M^T\right\}\\
&\quad+\frac{1}{h_r}\left|\left\{k\in I_r:|d(x;A_k,B_k)-L|<\frac{\varepsilon}{2}\right\}\right|\cdot\frac{\max\left\{(\varepsilon)^{p_k}\right\}}{2}\\
&\leq \max\left\{M^t,M^T\right\}\cdot\frac{1}{h_r}\left|\left\{k\in I_r:|d(x;A_k,B_k)-L|\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\max\left\{\varepsilon^t,\varepsilon^T\right\}}{2}
\end{aligned}$$

and so we get

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k}\geq\varepsilon\right\}$$

$$\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:|d(x;A_k,B_k)-L|\geq\frac{\varepsilon}{2}\right\}\right|\geq\frac{2\varepsilon-\max\left\{\varepsilon^t,\varepsilon^T\right\}}{2\max\left\{M^t,M^T\right\}}\right\}\in\mathcal{I}.$$

Therefore, $A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k$. □

Definition 2.2. Let (X,ρ) be a metric space, $\mathcal{I}\subseteq 2^{\mathbb{N}}$ be an admissible ideal and $p=(p_k)$ be a sequence of positive real numbers. For any non-empty closed subsets A_k, $B_k\subseteq X$, we say that two sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically strongly $\mathcal{I}$-Cesàro p-equivalent (Wijsman sense) of multiple L if for every $\varepsilon>0$ and for each $x\in X$, the set

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}|d(x;A_k,B_k)-L|^{p_k}\geq\varepsilon\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \overset{C_1^{L(p)}[\mathcal{I}_W]}{\sim} B_k$ and simply asymptotically strongly $\mathcal{I}$-Cesàro p-equivalent (Wijsman sense) if $L=1$.

Theorem 2.4. *Let (X,ρ) be a metric space, $\mathcal{I}\subseteq 2^{\mathbb{N}}$ be an admissible ideal and $p=(p_k)$ be a sequence of positive real numbers. If $\theta=\{k_r\}$ be a lacunary sequence with $\liminf_r q_r>1$, then*

$$A_k \overset{C_1^{L(p)}[\mathcal{I}_W]}{\sim} B_k \Rightarrow A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k.$$

Proof. Let $\theta=\{k_r\}$ be a lacunary sequence with $\liminf_r q_r>1$. Then, there exists $\lambda>0$ such that $q_r\geq 1+\lambda$ for all $r\geq 1$. Since $h_r=k_r-k_{r-1}$, we have

$$\frac{k_r}{h_r}\leq\frac{1+\lambda}{\lambda}\quad\text{and}\quad\frac{k_{r-1}}{h_r}\leq\frac{1}{\lambda}.$$

Given $\varepsilon>0$ and we define the set

$$S_r=\left\{k_r\in\mathbb{N}:\frac{1}{k_r}\sum_{k=1}^{k_r}|d(x;A_k,B_k)-L|^{p_k}<\varepsilon\right\}$$

for each $x \in X$. It is obvious that $S_r \in \mathcal{F}(\mathcal{I})$ which is the filter of the ideal $\mathcal{I}$. Then, for each $k_r \in S_r$ and for each $x \in X$, we get

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k} &= \frac{1}{h_r}\sum_{k=1}^{k_r}|d(x;A_k,B_k)-L|^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}}|d(x;A_k,B_k)-L|^{p_k} \\
&= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{k=1}^{k_r}|d(x;A_k,B_k)-L|^{p_k}\right) \\
&\quad -\frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}|d(x;A_k,B_k)-L|^{p_k}\right) \\
. \qquad &\leq \left(\frac{1+\lambda}{\lambda}\right)\varepsilon - \frac{1}{\lambda}\varepsilon'.
\end{aligned}$$

Now, choose

$$\mu = \left(\frac{1+\lambda}{\lambda}\right)\varepsilon + \frac{1}{\lambda}\varepsilon',$$

we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k} < \mu\right\} \in \mathcal{F}(\mathcal{I}).$$

This completes the proof. □

For the next result we assume that the lacunary sequence $\theta = \{k_r\}$ satisfies the condition that for any set $C \in \mathcal{F}(\mathcal{I})$, $\bigcup\{n : k_{r-1} < n < k_r,\ r \in C\} \in \mathcal{F}(\mathcal{I})$.

Theorem 2.5. *Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and $p = (p_k)$ be a sequence of positive real numbers. If $\theta = \{k_r\}$ be a lacunary sequence with $\limsup_r q_r < \infty$, then*

$$A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k \Rightarrow A_k \overset{C_1^{L(p)}[\mathcal{I}_W]}{\sim} B_k.$$

Proof. Let $\theta = \{k_r\}$ be a lacunary sequence with $\limsup_r q_r < \infty$. Then, there exists an $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $A_k \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} B_k$ and we define the sets B_r and D_n such that

$$B_r = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k} < \varepsilon_1\right\}$$

and

$$D_n = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}|d(x;A_k,B_k)-L|^{p_k} < \varepsilon_2\right\},$$

for each $x \in X$. It is obvious that $B_r \in \mathcal{F}(\mathcal{I})$.

Now, we take

$$a_j = \frac{1}{h_j}\sum_{k\in I_j}|d(x;A_k,B_k)-L|^{p_k} \leq \varepsilon_1$$

for all $j \in B_r$ and for each $x \in X$.

Choose n is any integer with $k_{r-1} < n < k_r$ where $r \in B_r$. Hence, we get

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}|d(x;A_k,B_k)-L|^{p_k} &\le \frac{1}{k_{r-1}}\sum_{k=1}^{k_r}|d(x;A_k,B_k)-L|^{p_k}\\
&= \frac{1}{k_{r-1}}\Bigg(\sum_{k\in I_1}|d(x;A_k,B_k)-L|^{p_k}+\sum_{k\in I_2}|d(x;A_k,B_k)-L|^{p_k}\\
&\quad +\cdots+\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k}\Bigg)\\
&= \frac{k_1}{k_{r-1}}\left(\frac{1}{h_1}\sum_{k\in I_1}|d(x;A_k,B_k)-L|^{p_k}\right)\\
&\quad +\frac{k_2-k_1}{k_{r-1}}\left(\frac{1}{h_2}\sum_{k\in I_2}|d(x;A_k,B_k)-L|^{p_k}\right)\\
&\quad +\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}\left(\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_k,B_k)-L|^{p_k}\right)\\
&= \frac{k_1}{k_{r-1}}a_1+\frac{k_2-k_1}{k_{r-1}}a_2+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}a_r\\
&\le \left(\sup_{j\in B_r} a_j\right)\frac{k_r}{k_{r-1}}\\
&< \varepsilon_1\cdot M.
\end{aligned}
$$

Choose $\varepsilon_2 = \dfrac{\varepsilon_1}{M}$ and in view of the fact that

$$\bigcup\{n : k_{r-1} < n < k_r,\ r \in B_r\} \subset D_n,$$

where $B_r \in \mathcal{F}(\mathcal{I})$, it follows from our assumption on $\theta = \{k_r\}$ that the set D_n also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

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