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On The Darboux Vector Belonging To Involute Curve A Different View

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Abstract
In this paper, we investigated special Smarandache curves in terms of Sabban frame drawn on the surface of the sphere by the unit Darboux vector of involute curve. We created Sabban frame belonging to this curve. It was explained Smarandache curves position vector is composed by Sabban vectors belonging to this curve. Then, we calculated geodesic curvatures of this Smarandache curves. Found results were expressed depending on the base curve. We also gave example belonging to the results found.

Keywords: *involute curve; Darboux vector; Smarandache curves; Sabban frame; geodesic curvature.*

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1. Introduction and Preliminaries

The involute of the curve is well known by the mathematicians especially the differential geometry scientists. There are many essential consequences and properties of curves. Involute curves have been studied by some authors [3, 7]. Whose position vector is composed by Frenet frame vectors regular curve is called a Smarandache curve [10]. Special Smarandache curves have been studied by some authors [1, 2, 5, 8, 9]. K. Taşköprü, M. Tosun studied special Smarandache curves according to Sabban frame on S^2 [11]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves [4]. Let $\alpha : I \to E^3$ be a unit speed curve, we defined the quantities of the Frenet-Serret apparatus, respectively

$$T(s) \quad = \quad \alpha'(s), \;\; N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \;\; B(s) = T(s) \wedge N(s), \tag{1.1}$$

$$\kappa(s) = \|T'(s)\|, \;\; \tau(s) = \langle N'(s), B(s)\rangle. \tag{1.2}$$

we have an orthonormal frame $\{T, N, B\}$ along α. This frames is called the Frenet frame of α. This curve the Frenet formulae are, respectively, [7]

$$\begin{cases} T'(s) = \kappa(s)N(s) \\ N'(s) = -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) = -\tau(s)N(s). \end{cases} \tag{1.3}$$

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

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$$W = \tau T + \kappa B.$$

If we consider the normalization of the Darboux, we have

$$\sin\varphi = \frac{\tau}{\|W\|}, \quad \cos\varphi = \frac{\kappa}{\|W\|}$$

and

$$C = \sin\varphi T + \cos\varphi B$$

where $\angle(W, B) = \varphi$, [6]. Let $\alpha : I \to \mathbb{E}^3$ unit speed and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable two curves. If the tangent vector of the curve α is orthogonal to the tangent vector of the curve α^* which is called involute of the α. According to definition, if the tangent of the curve α is denoted by T and the tangent of the curve α^* is denoted by T^*, we can write [7]

$$\langle T, T^* \rangle = 0. \tag{1.4}$$

If the curve α^* is involute of α, then we may write that [7]

$$\alpha^*(s^*) = \alpha(s) + (c - s)T(s). \tag{1.5}$$

Let $\alpha : I \to E^3$ and $\alpha^* : I \to E^3$ be the C^2-class differentiable unit speed two curves and the amounts of $\{T(s), N(s), B(s)\}$ and $\{T^*(s^*), N^*(s^*), B^*(s^*)\}$ are entirely Frenet- Serret frame of the curves α and the involute α^*, respectively, then [3]

$$\begin{cases} T^* = N \\ N^* = -\cos\varphi T + \sin\varphi B \\ B^* = \sin\varphi T + \cos\varphi B. \end{cases} \tag{1.6}$$

where $\angle(W, B) = \varphi$. For the curvatures and the torsions we have

$$\kappa^* = \frac{\|W\|}{|c - s|\kappa}, \qquad \tau^* = \frac{(\kappa\tau' - \tau\kappa')}{\kappa|c - s|\,\|W\|^2}. \tag{1.7}$$

where $\dfrac{ds^*}{ds} = |c - s|\kappa$. From (1.9) equation, we have

$$\sin\varphi^* = \frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}, \quad \cos\varphi^* = \frac{\|W\|}{\sqrt{\varphi'^2 + \|W\|^2}}, \quad \varphi^{*\prime} = \Big(\frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}\Big)' \frac{\sqrt{\varphi'^2 + \|W\|^2}}{\|W\|}. \tag{1.8}$$

Let (α, α^*) be a curve pair in $\mathbb{E}^3$. For the vector C^* is the direction of the involute curve α^* we have

$$C^* = \frac{\sin\varphi\|W\|}{\sqrt{(\varphi')^2 + \|W\|^2}}T + \frac{\varphi'}{\sqrt{(\varphi')^2 + \|W\|^2}}N + \frac{\cos\varphi\|W\|}{\sqrt{(\varphi')^2 + \|W\|^2}}B. \tag{1.9}$$

where the vector C is the direction of the Darboux vector W of the base curve α, [3]. Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote by

$$\gamma(s) = \gamma(s), \;\; t(s) = \gamma'(s), \;\; d(s) = \gamma(s) \wedge t(s) \tag{1.10}$$

$\{\gamma(s), t(s), d(s)\}$ frame is called the Sabban frame of γ on S^2. Then we have the following spherical Frenet formulae of γ

$$\gamma'(s) = t(s), \;\; t'(s) = -\gamma(s) + \kappa_g(s)d(s), \;\; d'(s) = -\kappa_g(s)t(s) \tag{1.11}$$

where κ_g is called the geodesic curvature of the curve γ on S^2 which is, [11]

$$\kappa_g(s) = \langle t'(s), d(s) \rangle. \tag{1.12}$$

2. On The Darboux Vector Belonging To Involute Curve A Different View

In this section, we investigated special Smarandache curves created by Sabban frame $\{C^*, T_{C^*}, C^* \wedge T_{C^*}\}$, that belongs to drawn on the surface of the sphere by the unit Darboux vector of a α^* curve are defined. We found some results. These results will be expressed depending on the base curve. Let $\alpha_{C^*}(s_{C^*}) = C^*(s^*)$ be a unit speed regular spherical curves on S^2. We denote s_{C^*} as the arc-length parameter for drawn on the surface of the sphere by the unit Darboux vector of involute (C^*). Sabban frame for (C^*) is

$$\alpha_{C^*}(s_{C^*}) = C^*(s^*) \tag{2.1}$$

Differentiating (2.1), we found

$$T_{C^*}\frac{ds_{C^*}}{ds^*} = \varphi^{*\prime}\cos\varphi^* T^* - \varphi^{*\prime}\sin\varphi^* B^* \tag{2.2}$$

and we can write

$$\frac{ds_{C^*}}{ds^*} = \varphi^{*\prime} \tag{2.3}$$

Hence we have

$$T_{C^*} = \cos\varphi^* T^* - \sin\varphi^* B^*$$

and

$$C^* \wedge T_{C^*} = N^*.$$

From the equation (1.10), we have

$$\begin{aligned} C^* &= \sin\varphi^* T^* + \cos\varphi^* B^* \\ T_{C^*} &= \cos\varphi^* T^* - \sin\varphi^* B^* \\ C^* \wedge T_{C^*} &= N^*. \end{aligned} \tag{2.4}$$

Then from the equation (1.11) we have the following spherical Frenet formulae of (C^*) is:

$$\begin{aligned} C^{*\prime} &= T_{C^*} \\ (T_{C^*})' &= -C^* + \frac{\|W^*\|}{\varphi^{*\prime}} C^* \wedge T_{C^*} \\ (C^* \wedge T_{C^*})' &= -\frac{\|W^*\|}{\varphi^{*\prime}} T_{C^*}. \end{aligned} \tag{2.5}$$

From the equation (1.12), we have the following geodesic curvatures of (C^*) is

$$\kappa_g = \langle T'_{C^*}, C^* \wedge T_{C^*}\rangle \Longrightarrow \kappa_g = \frac{\|W^*\|}{\varphi^{*\prime}}. \tag{2.6}$$

β_1**-Smarandache curve** can be defined by

$$\beta_1(s_{C^*}) = \frac{1}{\sqrt{2}}(C^* + T_{C^*}) \tag{2.7}$$

or substituting the equation (2.4) into equation (2.7), we reach

$$\beta_1(s^*) = \frac{1}{\sqrt{2}}\Big((\sin\varphi^* + \cos\varphi^*)T^* + (\cos\varphi^* - \sin\varphi^*)B^*\Big). \tag{2.8}$$

Differentiating (2.7), we can write

$$T_{\beta_1}(s^*) = \frac{\varphi^{*\prime}(\cos\varphi^* - \sin\varphi^*)}{\sqrt{2\varphi^{*\prime 2} + \|W^*\|^2}}T^* + \frac{\|W^*\|}{\sqrt{2\varphi^{*\prime 2} + \|W^*\|^2}}N^* - \frac{\varphi^{*\prime}(\cos\varphi^* + \sin\varphi^*)}{\sqrt{2\varphi^{*\prime 2} + \|W^*\|^2}}B^*. \tag{2.9}$$

Considering the equations (2.8) and (2.9), it easily seen that

$$\beta_1 \wedge T_{\beta_1}(s^*) = \frac{\|W^*\|(\cos\varphi^* + \sin\varphi^*)}{\sqrt{2\|W^*\|^2 + 4{\varphi^*}'^2}}T^* - \frac{{\varphi^*}'}{\sqrt{2\|W^*\|^2 + 4{\varphi^*}'^2}}N^* + \frac{\|W^*\|(\cos\varphi^* + \sin\varphi^*)}{\sqrt{2\|W^*\|^2 + 4{\varphi^*}'^2}}B^*. \tag{2.10}$$

Differentiating (2.9), where

$$\begin{cases} \chi_1 = -2 - \left(\frac{\|W^*\|}{{\varphi^*}'}\right)^2 + \left(\frac{\|W^*\|}{{\varphi^*}'}\right)'\left(\frac{\|W^*\|}{{\varphi^*}'}\right) \\ \chi_2 = -2 - 3\left(\frac{\|W^*\|}{{\varphi^*}'}\right)^2 - \left(\frac{\|W^*\|}{{\varphi^*}'}\right)^4 - \left(\frac{\|W^*\|}{{\varphi^*}'}\right)'\left(\frac{\|W^*\|}{{\varphi^*}'}\right) \\ \chi_3 = 2\left(\frac{\|W^*\|}{{\varphi^*}'}\right) + \left(\frac{\|W^*\|}{{\varphi^*}'}\right)^3 + \left(\frac{\|W^*\|}{{\varphi^*}'}\right)' \end{cases} \tag{2.11}$$

including we can reach,

$$T'_{\beta_1}(s^*) = \frac{({\varphi^*}')^4\sqrt{2}(\chi_1\sin\varphi^* + \chi_2\cos\varphi^*)}{\Big(\|W^*\|^2 + ({\varphi^*}')^2\Big)^2}T^* + \frac{\chi_3({\varphi^*}')^4\sqrt{2}}{\Big(\|W^*\|^2 + ({\varphi^*}')^2\Big)^2}N^* + \frac{({\varphi^*}')^4\sqrt{2}(\chi_1\cos\varphi^* - \chi_2\sin\varphi^*)}{\Big(\|W^*\|^2 + ({\varphi^*}')^2\Big)^2}B^*. \tag{2.12}$$

From the equation (2.10) and (2.12), $\kappa_g^{\beta_1}$ geodesic curvature for involute curve $\beta_1(s^*)$ is

$$\begin{aligned} \kappa_g^{\beta_1} &= \langle T'_{\beta_1}, \beta_1 \wedge T_{\beta_1}\rangle \\ &= \frac{1}{\left(2 + \left(\frac{\|W^*\|}{{\varphi^*}'}\right)^2\right)^{\frac{5}{2}}}\Big(\frac{\|W^*\|}{{\varphi^*}'}\chi_1 - \frac{\|W^*\|}{{\varphi^*}'}\chi_2 + 2\chi_3\Big). \end{aligned} \tag{2.13}$$

From the equation (1.6) and (1.9), Sabban apparatus of the β_1-Smarandache curve for base curve are

$$\beta_1(s) = \frac{(\|W\| - \varphi')\sin\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}T - \frac{\varphi' + \|W\|}{\sqrt{2\varphi'^2 + 2\|W\|^2}}N + \frac{(\|W\| - \varphi')\cos\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}B,$$

$$\begin{aligned} T_{\beta_1}(s) &= \frac{(-\|W\| - \varphi')\eta\sin\varphi - \sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{\|W\|^2 + \varphi'^2}\sqrt{1 + 2\eta^2}}T + \frac{\eta(\varphi' - \|W\|)}{\sqrt{\|W\|^2 + \varphi'^2}\sqrt{1 + 2\eta^2}}N \\ &+ \frac{\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi - \eta(\|W\| + \varphi')\cos\varphi}{\sqrt{\|W\|^2 + \varphi'^2}\sqrt{1 + 2\eta^2}}B, \end{aligned}$$

$$\begin{aligned} (\beta_1 \wedge T_{\beta_1})(s) &= \frac{(\|W\| + \varphi')\sin\varphi - 2\eta\sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{2 + 4\eta^2}\sqrt{\|W\|^2 + \varphi'^2}}T - \frac{\varphi' - \|W\|}{\sqrt{2 + 4\eta^2}\sqrt{\|W\|^2 + \varphi'^2}}N \\ &+ \frac{(\|W\| + \varphi')\cos\varphi + 2\eta\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi}{\sqrt{2 + 4\eta^2}\sqrt{\|W\|^2 + \varphi'^2}}B, \end{aligned}$$

$$\begin{aligned} T'_{\beta_1}(s) &= \frac{(\overline{\chi}_1\|W\| - \overline{\chi}_2\varphi')\eta^4\sqrt{2}\sin\varphi - \overline{\chi}_3\eta^4\sqrt{2\|W\|^2 + 2\varphi'^2}\cos\varphi}{(1 + 2\eta^2)^2\sqrt{\|W\|^2 + \varphi'^2}}T + \frac{\eta^4\sqrt{2}(\overline{\chi}_1\varphi' - \overline{\chi}_2\|W\|)}{(1 + 2\eta^2)^2\sqrt{\|W\|^2 + \varphi'^2}}N \\ &+ \frac{(\overline{\chi}_1\|W\| - \overline{\chi}_2\varphi')\eta^4\sqrt{2}\cos\varphi + \overline{\chi}_3\eta^4\sqrt{2\|W\|^2 + 2\varphi'^2}\sin\varphi}{(1 + 2\eta^2)^2\sqrt{\|W\|^2 + \varphi'^2}}B \end{aligned}$$

and

$$\kappa_g^{\beta_1} = \frac{1}{(2+\frac{1}{\eta^2})^{\frac{5}{2}}}\big(\frac{1}{\eta}\overline{\chi}_1 - \frac{1}{\eta}\overline{\chi}_2 + 2\overline{\chi}_3\big), \tag{2.14}$$

where

$$\frac{1}{\eta} = \frac{(\varphi^*)'}{\|W^*\|} = \Big(\frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|}^2}\Big)'\cos\varphi(c-s) \tag{2.15}$$

and

$$\begin{cases} \overline{\chi}_1 = -2 - \frac{1}{\eta^2} + \frac{1}{\eta'}\frac{1}{\eta} \\ \overline{\chi}_2 = -2 - 3\frac{1}{\eta^2} - \frac{1}{\eta^4} - \frac{1}{\eta'}\frac{1}{\eta} \\ \overline{\chi}_3 = 2\frac{1}{\eta} + \frac{1}{\eta^3} + \frac{1}{\eta'}. \end{cases} \tag{2.16}$$

β_2-**Smarandache curve** can be defined by

$$\beta_2(s_{C^*}) = \frac{1}{\sqrt{2}}(C^* + C^* \wedge T_{C^*}) \tag{2.17}$$

or from the equation (1.6), (1.8) and (2.4), we can write

$$\beta_2(s) = \frac{\|W\|\sin\varphi - \sqrt{\varphi'^2 + \|W\|^2}\cos\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}T + \frac{\varphi'}{\sqrt{2\varphi'^2 + 2\|W\|^2}}N + \frac{\|W\|\cos\varphi + \sqrt{\varphi'^2 + \|W\|^2}\sin\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}B. \tag{2.18}$$

Differentiating (2.18), we can write

$$T_{\beta_2}(s) = \frac{-\varphi'\sin\varphi}{\sqrt{\|W\|^2 + \varphi'^2}}T + \frac{\|W\|}{\sqrt{\|W\|^2 + \varphi'^2}}N - \frac{\varphi'\cos\varphi}{\sqrt{\|W\|^2 + \varphi'^2}}B. \tag{2.19}$$

Considering the equations (2.18) and (2.19), with ease seen that

$$\big(\beta_2 \wedge T_{\beta_2}\big)(s) = \frac{-\|W\|\sin\varphi - \sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{2\|W\|^2 + 2\varphi'^2}}T - \frac{\varphi'}{\sqrt{2\|W\|^2 + 2\varphi'^2}}N + \frac{\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi - \|W\|\cos\varphi}{\sqrt{2\|W\|^2 + 2\varphi'^2}}B. \tag{2.20}$$

Differentiating (2.19), we can write

$$\begin{aligned} T'_{\beta_2}(s) \quad = \quad & \frac{-\eta\sqrt{2}\|W\|\sin\varphi - \sqrt{2\|W\|^2 + 2\varphi'^2}\cos\varphi}{(\eta-1)\sqrt{\|W\|^2 + \varphi'^2}}T + \frac{\eta\sqrt{2}\varphi'}{(\eta-1)\sqrt{\|W\|^2 + \varphi'^2}}N \\ & + \frac{\sqrt{2\|W\|^2 + 2\varphi'^2}\sin\varphi - \eta\sqrt{2}\|W\|\cos\varphi}{(\eta-1)\sqrt{\|W\|^2 + \varphi'^2}}B. \end{aligned} \tag{2.21}$$

$\kappa_g^{\beta_2}$ geodesic curvature for base curve $\beta_2(s_{\beta_2})$ is

$$\kappa_g^{\beta_2} = \frac{1+\eta}{\eta - 1}. \tag{2.22}$$

β_3-**Smarandache curve** can be defined by

$$\beta_3(s_{C^*}) = \frac{1}{\sqrt{2}}(T_{C^*} + C^* \wedge T_{C^*}) \tag{2.23}$$

or from the equation (2.4), (1.6) and (1.8), we can write

$$\beta_3(s) = \frac{-\varphi'\sin\varphi - \sqrt{\varphi'^2 + \|W\|^2}\cos\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}T + \frac{\|W\|}{\sqrt{2\varphi'^2 + 2\|W\|^2}}N + \frac{\sqrt{\varphi'^2 + \|W\|^2}\sin\varphi - \varphi'\cos\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}B. \tag{2.24}$$

Differentiating (2.24), we reach

$$\begin{aligned} T_{\beta_3}(s) &= \frac{(\varphi'-\eta\|W\|)\sin\varphi-\sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{2+\eta^2}\sqrt{\|W\|^2+\varphi'^2}}T-\frac{\eta\varphi'+\|W\|}{\sqrt{2+\eta^2}\sqrt{\|W\|^2+\varphi'^2}}N \\ &+\frac{(\varphi'-\eta\|W\|)\cos\varphi+\sqrt{\|W\|^2+\varphi'^2}\sin\varphi}{\sqrt{2+\eta^2}\sqrt{\|W\|^2+\varphi'^2}}B. \end{aligned} \tag{2.25}$$

Considering the equations (2.24) and (2.25), it is easily seen

$$\begin{aligned} \big(\beta_3\wedge T_{\beta_3}\big)(s) &= \frac{(2\|W\|+\eta\varphi')\sin\varphi-\eta\sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\varphi'^2}}T+\frac{2\varphi'-\eta\|W\|}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\varphi'^2}}N \\ &+\frac{(2\|W\|+\eta\varphi')\cos\varphi+\eta\sqrt{\|W\|^2+\varphi'^2}\sin\varphi}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\varphi'^2}}B. \end{aligned} \tag{2.26}$$

Differentiating (2.25), where

$$\overline{\sigma}_1=\frac{1}{\eta}+2\frac{1}{\eta^3}+2\frac{1}{\eta'}\frac{1}{\eta},\ \ \overline{\sigma}_2=-1-3\frac{1}{\eta^2}-2\frac{1}{\eta^4}-\frac{1}{\eta'},\ \ \overline{\sigma}_3=-\frac{1}{\eta^2}-2\frac{1}{\eta^4}+\frac{1}{\eta'} \tag{2.27}$$

including we have

$$\begin{aligned} T'_{\beta_3}(s) &= \frac{(\overline{\sigma}_2\|W\|+\overline{\sigma}_1\varphi')\eta^4\sqrt{2}\sin\varphi-\overline{\sigma}_3\eta^4\sqrt{2\|W\|^2+2\varphi'^2}\cos\varphi}{(2+\eta^2)^2\sqrt{\varphi'^2+\|W\|^2}}T+\frac{(\overline{\sigma}_2\varphi'-\overline{\sigma}_1\|W\|)\eta^4\sqrt{2}}{(2+\eta^2)^2\sqrt{\varphi'^2+\|W\|^2}}N \\ &+\frac{(\overline{\sigma}_2\|W\|+\overline{\sigma}_1\varphi')\eta^4\sqrt{2}\cos\varphi+\overline{\sigma}_3\eta^4\sqrt{2\|W\|^2+2\varphi'^2}\sin\varphi}{(2+\eta^2)^2\sqrt{\varphi'^2+\|W\|^2}}B. \end{aligned} \tag{2.28}$$

$\kappa_g^{\beta_3}$ geodesic curvature for base curve $\beta_3(s_{\beta_3})$ is

$$\kappa_g^{\beta_3}=\frac{1}{(2+\frac{1}{\eta^2})^{\frac{5}{2}}}\big(2\frac{1}{\eta}\overline{\sigma}_1-\overline{\sigma}_2+\overline{\sigma}_3\big). \tag{2.29}$$

β_4**-Smarandache curve** can be defined by

$$\beta_4(s_{C^*})=\frac{1}{\sqrt{3}}(C^*+T_{C^*}+C^*\wedge T_{C^*}) \tag{2.30}$$

or from the equation (1.6), (1.8) and (2.4), we can write

$$\begin{aligned} \beta_4(s) &= \frac{(\|W\|-\varphi')\sin\varphi-\sqrt{\varphi'^2+\|W\|^2}\cos\varphi}{\sqrt{3\varphi'^2+3\|W\|^2}}T+\frac{\varphi'+\|W\|}{\sqrt{3\varphi'^2+3\|W\|^2}}N \\ &+\frac{(\|W\|-\varphi')\cos\varphi+\sqrt{\varphi'^2+\|W\|^2}\sin\varphi}{\sqrt{3\varphi'^2+3\|W\|^2}}B. \end{aligned} \tag{2.31}$$

Differentiating (2.31), we reach

$$\begin{aligned} T_{\beta_4}(s) &= \frac{\big((1-\eta)\varphi'-\eta\|W\|\big)\sin\varphi-\sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2+\varphi'^2}}T+\frac{(\eta-1)\|W\|-\eta\varphi'}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2+\varphi'^2}}N \\ &+\frac{\big((\eta-1)\varphi'-\eta\|W\|\big)\cos\varphi+\sqrt{\|W\|^2+\varphi'^2}\sin\varphi}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2+\varphi'^2}}B. \end{aligned} \tag{2.32}$$

Considering the equations (2.31) and (2.32), it is easily seen

$$
\begin{aligned}
(\beta_4 \wedge T_{\beta_4})(s) &= \frac{((2-\eta)\|W\| + (1+\eta)\varphi')\sin\varphi - (2\eta-1)\sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\varphi'^2}}T + \frac{(2-\eta)\varphi' - (1+\eta)\|W\|}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\varphi'^2}}N \\
&+ \frac{((2-\eta)\|W\| + (1+\eta)\varphi')\cos\varphi + (2\eta-1)\sqrt{\|W\|^2+\varphi'^2}\sin\varphi}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\varphi'^2}}B.
\end{aligned} \tag{2.33}
$$

Differentiating (2.32), where

$$
\begin{cases}
\overline{\rho}_1 = -2 + 4\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}\left(2\frac{1}{\eta} - 1\right) \\
\overline{\rho}_2 = -2 + 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} - 2\frac{1}{\eta^4} - \frac{1}{\eta'}\left(1 + \frac{1}{\eta}\right) \\
\overline{\rho}_3 = 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 4\frac{1}{\eta^3} - 2\frac{1}{\eta^4} + \frac{1}{\eta'}\left(2 - \frac{1}{\eta}\right)
\end{cases} \tag{2.34}
$$

including we can write

$$
\begin{aligned}
T'_{\beta_4}(s) &= \frac{(\overline{\rho}_1\|W\| - \overline{\rho}_2\varphi')\eta^4\sqrt{3}\sin\varphi - \overline{\rho}_3\eta^4\sqrt{3\|W\|^2+3\varphi'^2}\cos\varphi}{4(1-\eta+\eta^2)^2\sqrt{\varphi'^2+\|W\|^2}}T + \frac{(\overline{\rho}_1\varphi' + \overline{\rho}_2\|W\|)\eta^4\sqrt{3}}{4(1-\eta+\eta^2)^2\sqrt{\varphi'^2+\|W\|^2}}N \\
&+ \frac{(\overline{\rho}_1\|W\| - \overline{\rho}_2\varphi')\eta^4\sqrt{3}\cos\varphi + \overline{\rho}_3\sqrt{3\|W\|^2+3\varphi'^2}\sin\varphi}{4(1-\eta+\eta^2)^2\sqrt{\varphi'^2+\|W\|^2}}B.
\end{aligned} \tag{2.35}
$$

$\kappa_g^{\beta_4}$ geodesic curvature for base curve $\beta_4(s_{\beta_4})$ is

$$
\kappa_g^{\beta_4} = \frac{(2\frac{1}{\eta} - 1)\overline{\rho}_1 + (-1 - \frac{1}{\eta})\overline{\rho}_2 + (2 - \frac{1}{\eta})\overline{\rho}_3}{4\sqrt{2}(1-\eta+\eta^2)^{\frac{5}{2}}}. \tag{2.36}
$$

Example. Let us consider the unit speed spherical curve:

$$
\alpha(s) = \{\frac{2}{5}\sin(2\,s) - \frac{1}{40}\sin(8\,s), -\frac{2}{5}\cos(2\,s) + \frac{1}{40}\cos(8\,s), \frac{4}{15}\sin(3\,s)\}
$$

in the context of definitions, we reach (C^*) curve (see Figure 1) and Smarandache curves according to Sabban frame on S^2. β_1, β_2, β_3 and β_4 (see Figure 2).

Figure 1. (C^*)-curve

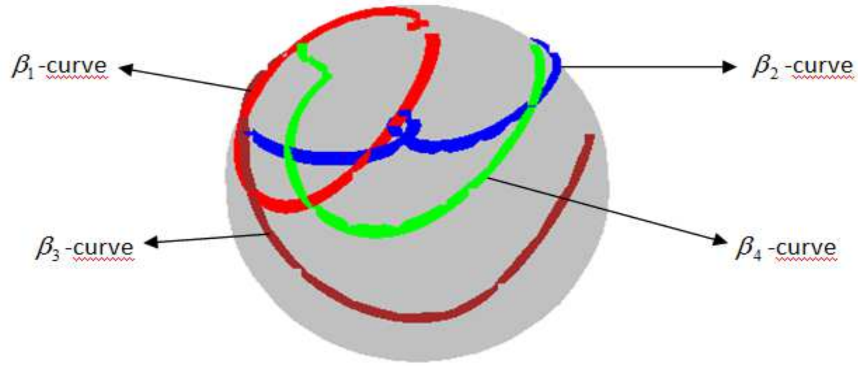

Figure 2. Smarandache curves

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