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Some Properties Concerning the Cheeger-Gromoll Type Deformation of Metric g on a Riemannian Manifold (M^m, g)

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Abstract. In this paper, we introduce a new class of metrics on a Riemannian manifold, which is obtained by deforming the metric of this Riemannian manifold into a Cheeger-Gromoll-type metric. We first investigate the Levi-Civita connection for this metric. Then we characterize the Riemannian curvature, the sectional curvature, and the scalar curvature. Finally, we explore a class of harmonic and biharmonic maps.

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1. Introduction

Let (M^m, g) be an m-dimensional Riemannian manifold and $\Im_s(M)$ the module of C^∞ tensor fields of type (r, s) over the ring of real-valued C^∞ functions on M. We denote by ∇ the Levi-Civita connection of g which is defined by the Koszul formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]) \tag{1.1}$$

for all vector fields X, Y and Z on M.

The Riemannian curvature tensor R and the Ricci curvature Ric of (M^m, g) are defined respectively by

$$\mathrm{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$\mathrm{Ric}(X, Y) = \mathrm{Tr}_g \mathrm{R}(*, X)Y = \sum_{i=1}^{m} g(\mathrm{R}(e_i, X)Y, e_i),$$

for all vector fields X, Y and Z on M, where $\{e_i\}_{i=1,\cdots,m}$ be a local orthonormal frame on M.

The map

$$\begin{array}{rccc} \sharp: & \Im^0_1(M) & \to & \Im^1_0(M) \\ & \omega & \mapsto & \sharp(\omega) \end{array}$$

defined by

$$g(\sharp(\omega), Y) = \omega(Y)$$

for all $Y \in \Im^1_0(M)$ is $C^\infty(M)$-linear isomorphism.

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Locally for all $\omega = \sum_{i=1}^{m} \omega_i dx^i \in \Im_1^0(M)$, we have $\sharp(\omega) = \sum_{i,j=1}^{m} g^{ij}\omega_i\partial_j$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}). The gradient of a smooth function f on M is defined by

$$\mathrm{grad} f = \sharp df.$$

If $\{e_i\}_{i=1,\ldots,n}$ is a local orthonormal frame on (M^n, g), then the $\mathrm{grad} f$ is expressed by

$$\mathrm{grad} f = \sum_{i=1}^{n} e_i(f)e_i.$$

The divergence of a vector field X on M, is defined by

$$\mathrm{div}X = \mathrm{Tr}_g(\nabla X) = \sum_{i=1}^{m} g(\nabla_{e_i}X, e_i).$$

The Laplacian of f is expressed by

$$\Delta f = \mathrm{div}(\mathrm{grad} f) = \sum_{i=1}^{m} g(\nabla_{e_i}\mathrm{grad} f, e_i) = \sum_{i=1}^{m}(e_ie_i(f) - \nabla_{e_i}e_i(f)).$$

The rough Laplacian of X is defined by

$$\bar{\Delta}X = -\mathrm{Tr}_g(\nabla^2 X) = -\sum_{i=1}^{n}(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})X.$$

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y).$$

Here, ∇ is the Levi-Civita connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$. The tension field of ϕ is defined by

$$\tau(\phi) = \mathrm{Tr}_g\nabla d\phi = \sum_{i=1}^{m}(\nabla^{\phi}_{e_i}d\phi(e_i) - d\phi(\nabla_{e_i}e_i)).$$

The map ϕ is called harmonic if and only if $\tau(\phi) = 0$. For more details see [12, 13, 15]. In recent years, this topic has been widely developed by many authors [1, 4, 7–10, 14, 16, 17].

The bitension field of ϕ is also defined by

$$\tau_2(\phi) = \Delta^{\phi}\tau(\phi) - \mathrm{Tr}_g R^N(\tau(\phi), d\phi(*))d\phi(*),$$

where

$$\Delta^{\phi}\tau(\phi) = -\mathrm{Tr}_g(\nabla^{\phi})^2\tau(\phi) = \sum_{i=1}^{m}(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i}e_i})\tau(\phi),$$

and

$$\mathrm{Tr}_g R^N(\tau(\phi), d\phi(*))d\phi(*) = \sum_{i=1}^{m} R^N(\tau(\phi), d\phi(e_i))d\phi(e_i).$$

$\Delta^{\phi}\tau(\phi)$ denotes the rough Laplacian of $\tau(\phi)$ on the pull-back bundle $\phi^{-1}TN$ and R^N denotes the curvature tensor of the target manifold N.

The map ϕ is biharmonic if and only if $\tau_2(\phi) = 0$. Biharmonic maps are a generalization of harmonic maps, meaning that every harmonic map is biharmonic. Therefore, it's interesting to construct proper biharmonic maps, also known as non-harmonic biharmonic maps see [2, 3, 5, 6, 9, 11].

In previous work [10], we introduced a new class of metrics on the Riemannian manifold (M^m, g), which are defined as follows:

$$G(X, Y) = fg(X, Y) + g(\xi, X)g(\xi, Y)$$

for all vector fields X, Y and ξ on M such that ξ is a unit parallel vector field on M and $\xi(f) = 0$, where f be a smooth function on M. The several properties of the Riemannian manifold associated with this metric were explored. A class of proper biharmonic maps is also characterized.

Moreover, in a different work [18], we presented a novel class of metrics on the anti-para Kähler manifold (M^m, φ, g) characterized by:

$$G(X, Y) = g(X, Y) + \delta^2 g(X, \varphi\xi)g(Y, \varphi\xi)$$

for all vector fields X, Y on M, where δ is positive constant and ξ is unit vector field on M such that $g(\nabla_X(\varphi\xi), Y) = g(\nabla_Y(\varphi\xi), X)$. In addition to characterizing various Riemannian manifold properties related to this class of metrics, we looked at a class of harmonic maps.

Motivated by the aforementioned study, we present a new category of metrics on the Riemannian manifold. (M^m, g), denoted as $\widetilde{g}$, defined by:

$$\widetilde{g}(X, Y) = g(X, Y) + \alpha g(X, \xi)g(Y, \xi),$$

for all vector fields X, Y and ξ on M, where α is a positive constant and ξ is a vector field such that $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X)$ and ∇ denote the Levi-Civita connection of (M^m, g).

In this paper, we present the Cheeger-Gromoll type deformation of the metric g on a Riemannian manifold (M^m, g). After describing the introduction, in Section 2, we provide the basic properties of the Cheeger-Gromoll type deformation of metric g, including the Levi-Civita connection of this metric (Theorem 2.2). In Section 3, we investigate all types of curvature. Initially, we examine the curvature tensor (Theorem 3.1) and the sectional curvature (Theorem 3.3). Furthermore, we characterize the Ricci tensor (Theorem 3.5), the Ricci curvature (Theorem 3.7), and the scalar curvature (Theorem 3.9). In Section 4, we explore the harmonicity of a mapping between Riemannian manifolds, one of which is endowed with a Cheeger-Gromoll type deformation of the metric of this manifold (Theorem 4.1 and Theorem 4.3). In the last section, we study the biharmonicity of the identity map between Riemannian manifolds, one of which is endowed with a Cheeger-Gromoll type deformation of the metric of this manifold (Theorem 5.1 and Theorem 5.3), where we establish the necessary and sufficient conditions under which the identity map is a proper biharmonic map (Theorem 5.2 and Theorem 5.4).

2. Cheeger-Gromoll Type Deformation of Metric g

Definition 2.1. Given a Riemannian manifold (M^m, g). We define the Cheeger-Gromoll type deformation of metric g on M noted $\widetilde{g}$ by

$$\widetilde{g}(X, Y) = g(X, Y) + \alpha g(X, \xi)g(Y, \xi), \tag{2.1}$$

for all vector fields X and Y on M, where α is positive constant and ξ is a vector field of gradient-type on M, i.e. $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X)$ and ∇ denote the Levi-Civita connection of (M^m, g).

Note that, if $\alpha = 0$ or $\xi = 0$, then $\widetilde{g} = g$. In the following, we consider $\alpha \neq 0$, $\xi \neq 0$ and $\lambda = 1 + \alpha|\xi|^2$, where $|.|$ denote the norm with respect to (M^m, g).

If ω is a covector field on M such that $\xi = \sharp\omega$, then we can write formula (2.1) in the form:

$$\widetilde{g}(X, Y) = g(X, Y) + \alpha\omega(X)\omega(Y)$$

for all vector fields X and Y on M. In this case, the metric $\widetilde{g}$ is written in the form

$$\widetilde{g} = g + \alpha\omega \otimes \omega.$$

We will compute the Levi-Civita connection $\widetilde{\nabla}$ for $(M^m, \widetilde{g})$, so we will need the following result.

$$X\widetilde{g}(Y, Z) = \widetilde{g}(\nabla_X Y, Z) + \widetilde{g}(Y, \nabla_X Z) + \alpha g(Y, \xi)g(Z, \nabla_X\xi) + \alpha g(Z, \xi)g(Y, \nabla_X\xi), \tag{2.2}$$

for all vector fields X, Y and Z on M.

Theorem 2.2. *Given a Riemannian manifold (M^m, g), the Levi-Civita connection $\widetilde{\nabla}$ of $(M^m, \widetilde{g})$, is expressed by:*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{\alpha}{\lambda} g(\nabla_X\xi, Y)\xi, \tag{2.3}$$

for all vector fields X and Y on M, where $\lambda = 1 + \alpha|\xi|^2$.

Proof. From Koszul formula (1.1), (2.1) and (2.2), we have

$$\begin{aligned}2\widetilde{g}(\widetilde{\nabla}_X Y, Z) &= X\widetilde{g}(Y,Z) + Y\widetilde{g}(Z,X) - Z\widetilde{g}(X,Y) + \widetilde{g}(Z,[X,Y]) + \widetilde{g}(Y,[Z,X]) - \widetilde{g}(X,[Z,Y])\\ &= \widetilde{g}(\nabla_X Y, Z) + \widetilde{g}(Y, \nabla_X Z) + \alpha g(Y,\xi)g(Z,\nabla_X\xi) + \alpha g(Z,\xi)g(Y,\nabla_X\xi) + \widetilde{g}(\nabla_Y Z, X) + \widetilde{g}(Z,\nabla_Y X)\\ &+ \alpha g(Z,\xi)g(X,\nabla_Y\xi) + \alpha g(X,\xi)g(Z,\nabla_Y\xi) - \widetilde{g}(\nabla_Z X, Y) - \widetilde{g}(X,\nabla_Z Y) + \widetilde{g}(Z,\nabla_X Y) - \alpha g(X,\xi)g(Y,\nabla_Z\xi)\\ &- \alpha g(Y,\xi)g(X,\nabla_Z\xi) - \widetilde{g}(Z,\nabla_Y X) + \widetilde{g}(Y,\nabla_Z X) - \widetilde{g}(Y,\nabla_X Z) - \widetilde{g}(X,\nabla_Y Z) + \widetilde{g}(X,\nabla_Z Y)\\ &= 2\widetilde{g}(\nabla_X Y, Z) + \frac{2\alpha}{\lambda} g(\nabla_X\xi, Y)\widetilde{g}(\xi, Z).\end{aligned}$$

This completes the proof. □

Note that by (2.3), we find:

$$\widetilde{\nabla}_X\xi = \nabla_X\xi + \frac{\alpha}{\lambda} g(\nabla_X\xi,\xi)\xi, \tag{2.4}$$

for all vector field X on M.

3. Curvatures of Cheeger-Gromoll Type Deformation of Metric g

We will compute the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ of $(M^m, \widetilde{g})$ as outlined below.

Theorem 3.1. *Given a Riemannian manifold (M^m, g), the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ of $(M^m, \widetilde{g})$, is expressed by:*

$$\begin{aligned}\widetilde{\mathrm{R}}(X,Y)Z &= \mathrm{R}(X,Y)Z + \frac{\alpha}{\lambda} g(\nabla_Y\xi, Z)\nabla_X\xi - \frac{\alpha}{\lambda} g(\nabla_X\xi, Z)\nabla_Y\xi + \frac{\alpha^2}{\lambda^2}\left(g(\nabla_X\xi,Z)g(\nabla_Y\xi,\xi) - g(\nabla_Y\xi,Z)g(\nabla_X\xi,\xi)\right)\xi\\ &- \frac{\alpha}{\lambda} g(\mathrm{R}(X,Y)Z,\xi)\xi\end{aligned} \tag{3.1}$$

for all vector fields X, Y and Z on M, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^m, g).

Proof. For all vector fields X, Y and Z on M, we have

$$\widetilde{\mathrm{R}}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z. \tag{3.2}$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_Y Z &= \widetilde{\nabla}_X\Big(\nabla_Y Z + \frac{\alpha}{\lambda} g(\nabla_Z\xi, Y)\xi\Big)\\ &= \nabla_X\nabla_Y Z + \frac{\alpha}{\lambda} g(\nabla_Y\xi, Z)\nabla_X\xi + \frac{\alpha}{\lambda} g(\nabla_Y Z, \nabla_X\xi)\xi + \frac{\alpha}{\lambda} g(\nabla_X Z, \nabla_Y\xi)\xi + \frac{\alpha}{\lambda} g(\nabla_X\nabla_Y\xi, Z)\xi\\ &- \frac{\alpha^2}{\lambda^2} g(\nabla_Y\xi, Z)g(\nabla_X\xi,\xi)\xi.\end{aligned} \tag{3.3}$$

By replacing X by Y in (3.3), we find

$$\begin{aligned}\widetilde{\nabla}_Y\widetilde{\nabla}_X Z &= \nabla_Y\nabla_X Z + \frac{\alpha}{\lambda} g(\nabla_X\xi, Z)\nabla_Y\xi + \frac{\alpha}{\lambda} g(\nabla_X Z, \nabla_Y\xi)\xi + \frac{\alpha}{\lambda} g(\nabla_Y Z, \nabla_X\xi)\xi + \frac{\alpha}{\lambda} g(\nabla_Y\nabla_X\xi, Z)\xi\\ &- \frac{\alpha^2}{\lambda^2} g(\nabla_X\xi, Z)g(\nabla_Y\xi,\xi)\xi.\end{aligned} \tag{3.4}$$

We also find

$$\widetilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \frac{\alpha}{\lambda} g(\nabla_{[X,Y]}\xi, Z)\xi. \tag{3.5}$$

Substituting (3.3), (3.4) and (3.5) into (3.2) we find (3.1). □

Corollary 3.2. *Given a Riemannian manifold (M^m, g). If ξ is unit vector field, then the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ of $(M^m, \widetilde{g})$ is expressed by:*

$$\widetilde{\mathrm{R}}(X,Y)Z = \mathrm{R}(X,Y)Z + \frac{\alpha}{1+\alpha} g(\nabla_Y\xi, Z)\nabla_X\xi - \frac{\alpha}{1+\alpha} g(\nabla_X\xi, Z)\nabla_Y\xi - \frac{\alpha}{1+\alpha} g(\mathrm{R}(X,Y)Z,\xi)\xi,$$

for all vector fields X, Y and Z on M.

Theorem 3.3. *Given a Riemannian manifold* (M^m, g)*. If* K *(resp.,* $\widetilde{\mathrm{K}}$*) denote the sectional curvature of* (M^m, g) *(resp.,* $(M^m, \widetilde{g})$*), then we have*

$$\widetilde{\mathrm{K}}(X,Y) = \frac{1}{1+\alpha g(X,\xi)^2+\alpha g(Y,\xi)^2}\Big(\mathrm{K}(X,Y) - \frac{\alpha}{\lambda}g(\nabla_X\xi, Y)^2 + \frac{\alpha}{\lambda}g(\nabla_Y\xi, Y)g(\nabla_X\xi, X)\Big), \tag{3.6}$$

for all X and Y two vector fields orthonormal with respect to g.

Proof. We have,

$$\widetilde{\mathrm{K}}(X,Y) = \frac{\widetilde{g}(\widetilde{\mathrm{R}}(X,Y)Y,X)}{\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2}. \tag{3.7}$$

On the one hand, we have

$$\widetilde{g}(\widetilde{\mathrm{R}}(X,Y)Y,X) = g(\widetilde{\mathrm{R}}(X,Y)Y,X) + \alpha g(\widetilde{\mathrm{R}}(X,Y)Y,\xi)g(X,\xi). \tag{3.8}$$

From (3.1) with direct computation we get

$$\begin{aligned} g(\widetilde{\mathrm{R}}(X,Y)Y,X) &= g(\mathrm{R}(X,Y)Y,X) - \frac{\alpha}{\lambda}g(X,\xi)g(\mathrm{R}(X,Y)Y,\xi) + \frac{\alpha}{\lambda}g(\nabla_X\xi, X)g(\nabla_Y\xi, Y) - \frac{\alpha}{\lambda}g(\nabla_X\xi, Y)^2 \\ &\quad + \frac{\alpha^2}{\lambda^2}g(X,\xi)g(\nabla_Y\xi,\xi)g(\nabla_X\xi, Y) - \frac{\alpha^2}{\lambda^2}g(X,\xi)g(\nabla_X\xi,\xi)g(\nabla_Y\xi, Y), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \alpha g(X,\xi)g(\widetilde{\mathrm{R}}(X,Y)Y,\xi) &= \frac{\alpha}{\lambda}g(X,\xi)g(\mathrm{R}(X,Y)Y,\xi) - \frac{\alpha^2}{\lambda^2}g(X,\xi)g(\nabla_Y\xi,\xi)g(\nabla_X\xi, Y) \\ &\quad + \frac{\alpha^2}{\lambda^2}g(X,\xi)g(\nabla_X\xi,\xi)g(\nabla_Y\xi, Y). \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we find

$$\widetilde{g}(\widetilde{\mathrm{R}}(X,Y)Y,X) = g(\mathrm{R}(X,Y)Y,X) + \frac{\alpha}{\lambda}g(\nabla_Y\xi, Y)g(\nabla_X\xi, X) - \frac{\alpha}{\lambda}g(\nabla_X\xi, Y)^2. \tag{3.11}$$

On the other hand, we have

$$\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2 = 1 + \alpha g(X,\xi)^2 + \alpha g(Y,\xi)^2. \tag{3.12}$$

Finally, substituting (3.11) and (3.12) into (3.7) we find (3.6). □

Remark 3.4. Let $\{e_i\}_{i=1,\dots,m}$ such that $e_m = \frac{\xi}{|\xi|} = \sqrt{\frac{\alpha}{\lambda-1}}\xi$ be a local orthonormal frame on (M^m, g), then the set of vector fields $\{\widetilde{e}_i\}_{i=1,\dots,m}$ defined by

$$\widetilde{e}_i = e_i,\ \widetilde{e}_m = \frac{1}{\sqrt{\lambda}}e_m = \sqrt{\frac{\alpha}{\lambda(\lambda-1)}}\xi,\ i = 1,\cdots,m-1, \tag{3.13}$$

is a local orthonormal frame on $(M^m, \widetilde{g})$.

Theorem 3.5. *Given a Riemannian manifold* (M^m, g)*. If* Ricci *(resp.* $\widetilde{\mathrm{Ricci}}$*) denote the Ricci tensor of* (M^m, g) *(resp.,* $(M^m, \widetilde{g})$*). Then, we have*

$$\begin{aligned} \widetilde{\mathrm{Ricci}}(X) \ &= \ \mathrm{Ricci}(X) - \frac{\alpha}{\lambda}\mathrm{R}(X,\xi)\xi + \frac{\alpha}{\lambda}(\mathrm{div}\xi - \frac{\alpha}{\lambda}g(\nabla_\xi\xi,\xi))\nabla_X\xi - \frac{\alpha}{\lambda}(\mathrm{Ric}(X,\xi) + \frac{\alpha}{\lambda}\mathrm{div}\xi g(\nabla_X\xi,\xi) \\ &\quad -\frac{\alpha}{\lambda}g(\nabla_X\xi, \nabla_\xi\xi))\xi + \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\xi)\nabla_\xi\xi - \frac{\alpha}{\lambda}\nabla_{\nabla_X\xi}\xi, \end{aligned} \tag{3.14}$$

for all vector field X on M.

Proof. Given a local orthonormal frame $\{\widetilde{e}_i\}_{i=1,\dots,m}$ on $(M^m, \widetilde{g})$ defined by (3.13). Then, we have

$$\begin{aligned} \widetilde{\mathrm{Ricci}}(X) &= \sum_{i=1}^{m}\widetilde{\mathrm{R}}(X,\widetilde{e}_i)\widetilde{e}_i \\ &= \sum_{i=1}^{m-1}\widetilde{\mathrm{R}}(X,e_i)e_i + \frac{\alpha}{\lambda(\lambda-1)}\widetilde{\mathrm{R}}(X,\xi)\xi. \end{aligned} \tag{3.15}$$

From (3.1), with direct calculation we find

$$\sum_{i=1}^{m-1}\widetilde{\mathrm{R}}(X,e_i)e_i = \mathrm{Ricci}(X) + \frac{\alpha}{\lambda}\mathrm{div}\xi\nabla_X\xi - \frac{\alpha}{\lambda}\nabla_{\nabla_X\xi}\xi + \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\nabla_\xi\xi)\xi - \frac{\alpha^2}{\lambda^2}\mathrm{div}\xi g(\nabla_X\xi,\xi)\xi$$
$$- \frac{\alpha}{\lambda}\mathrm{Ric}(X,\xi)\xi - \frac{\alpha}{\lambda-1}\mathrm{R}(X,\xi)\xi - \frac{\alpha^2}{\lambda(\lambda-1)}g(\nabla_\xi\xi,\xi)\nabla_X\xi + \frac{\alpha^2}{\lambda(\lambda-1)}g(\nabla_X\xi,\xi)\nabla_\xi\xi, \tag{3.16}$$

$$\frac{\alpha}{\lambda(\lambda-1)}\widetilde{\mathrm{R}}(X,\xi)\xi = \frac{\alpha}{\lambda(\lambda-1)}\mathrm{R}(X,\xi)\xi + \frac{\alpha^2}{\lambda^2(\lambda-1)}g(\nabla_\xi\xi,\xi)\nabla_X\xi - \frac{\alpha^2}{\lambda^2(\lambda-1)}g(\nabla_X\xi,\xi)\nabla_\xi\xi. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we find (3.14). □

Corollary 3.6. *Given a Riemannian manifold (M^m, g). If ξ is unit vector field, then the Ricci tensor $\widetilde{\mathrm{Ricci}}$ of $(M^m,\widetilde{g})$ is expressed by:*

$$\widetilde{\mathrm{Ricci}}(X) = \mathrm{Ricci}(X) - \frac{\alpha}{1+\alpha}\mathrm{R}(X,\xi)\xi - \frac{\alpha}{1+\alpha}\mathrm{Ric}(X,\xi)\xi$$
$$+ \frac{\alpha}{1+\alpha}\mathrm{div}\xi\nabla_X\xi - \frac{\alpha}{1+\alpha}\nabla_{\nabla_X\xi}\xi,$$

for all vector field X on M.

Theorem 3.7. *Given a Riemannian manifold (M^m, g). If Ric (resp. $\widetilde{\mathrm{Ric}}$) denote the Ricci curvature of (M^m, g) (resp., $(M^m,\widetilde{g})$). Then, we have*

$$\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) - \frac{\alpha}{\lambda}g(\mathrm{R}(X,\xi)\xi,Y) + \frac{\alpha}{\lambda}g(\nabla_X\xi,Y)\mathrm{div}\xi - \frac{\alpha^2}{\lambda^2}g(\nabla_\xi\xi,\xi)g(\nabla_X\xi,Y)$$
$$+ \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\xi)g(\nabla_Y\xi,\xi) - \frac{\alpha}{\lambda}g(\nabla_X\xi,\nabla_Y\xi), \tag{3.18}$$

for all vector fields X and Y on M.

Proof. Given a local orthonormal frame $\{\widetilde{e}_i\}_{i=1,\dots,m}$ on $(M^m,\widetilde{g})$ defined by (3.13). Then, we have

$$\widetilde{\mathrm{Ric}}(X,Y) = \widetilde{g}(\widetilde{\mathrm{Ricci}}(X),Y)$$
$$= g(\widetilde{\mathrm{Ricci}}(X),Y) + \alpha g(\widetilde{\mathrm{Ricci}}(X),\xi)g(Y,\xi). \tag{3.19}$$

From the formula (3.14), with direct calculation we get:

$$g(\widetilde{\mathrm{Ricci}}(X),Y) = \mathrm{Ric}(X,Y) - \frac{\alpha}{\lambda}g(\mathrm{R}(X,\xi)\xi,Y) + \frac{\alpha}{\lambda}g(\nabla_X\xi,Y)\mathrm{div}\xi - \frac{\alpha^2}{\lambda^2}g(\nabla_\xi\xi,\xi)g(\nabla_X\xi,Y) - \frac{\alpha}{\lambda}\mathrm{Ric}(X,\xi)g(Y,\xi)$$
$$- \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\xi)g(Y,\xi)\mathrm{div}\xi + \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\nabla_\xi\xi)g(Y,\xi) + \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\xi)g(\nabla_Y\xi,\xi)$$
$$- \frac{\alpha}{\lambda}g(\nabla_X\xi,\nabla_Y\xi), \tag{3.20}$$

$$\alpha g(\widetilde{\mathrm{Ricci}}(X),\xi)g(Y,\xi) = \alpha\mathrm{Ric}(X,\xi)g(Y,\xi) + \frac{\alpha^2}{\lambda}g(\nabla_X\xi,\xi)g(Y,\xi)\mathrm{div}\xi - \frac{\alpha(\lambda-1)}{\lambda}\mathrm{Ric}(X,\xi)g(Y,\xi)$$
$$- \frac{\alpha^2}{\lambda^2}g(\nabla_X\xi,\nabla_\xi\xi)g(Y,\xi) - \frac{\alpha^2(\lambda-1)}{\lambda^2}g(\nabla_X\xi,\xi)g(Y,\xi)\mathrm{div}\xi. \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19), we find (3.18). □

Corollary 3.8. *Given a Riemannian manifold (M^m, g). If ξ is unit vector field, then the Ricci curvature $\widetilde{\mathrm{Ric}}$ of $(M^m,\widetilde{g})$ is expressed by:*

$$\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) - \frac{\alpha}{1+\alpha}g(\mathrm{R}(X,\xi)\xi,Y) + \frac{\alpha}{1+\alpha}g(\nabla_X\xi,Y)\mathrm{div}\xi - \frac{\alpha}{1+\alpha}g(\nabla_X\xi,\nabla_Y\xi),$$

for all vector fields X and Y on M.

Theorem 3.9. *Given a Riemannian manifold* (M^m, g)*. If* σ *(resp.,* $\widetilde{\sigma}$*) denote the scalar curvature of* (M^m, g) *(resp.,* $(M^m, \widetilde{g})$*). Then, we have*

$$\widetilde{\sigma} = \sigma - \frac{2\alpha}{\lambda}\mathrm{Ric}(\xi,\xi) + \frac{\alpha}{\lambda}(\mathrm{div}\xi)^2 - \frac{2\alpha^2}{\lambda^2}g(\nabla_\xi\xi,\xi)\mathrm{div}\xi + \frac{2\alpha^2}{\lambda^2}|\nabla_\xi\xi|^2 - \frac{\alpha}{\lambda}|\nabla\xi|^2.$$

Proof. Given a local orthonormal frame $\{\widetilde{e}_i\}_{i=1,\dots,m}$ on $(M^m, \widetilde{g})$ defined by (3.13). Then, we have

$$\begin{aligned}\widetilde{\sigma} &= \sum_{i=1}^{m}\widetilde{\mathrm{Ric}}(\widetilde{e}_i,\widetilde{e}_i)\\ &= \sum_{i=1}^{m-1}\widetilde{\mathrm{Ric}}(e_i,e_i) + \frac{\alpha}{\lambda(\lambda-1)}\widetilde{\mathrm{Ric}}(\xi,\xi). \end{aligned} \tag{3.22}$$

From the formula (3.18) and direct calculation we get,

$$\begin{aligned}\sum_{i=1}^{m-1}\widetilde{\mathrm{Ric}}(e_i,e_i) &= \sum_{i=1}^{m-1}\Big(\mathrm{Ric}(e_i,e_i) - \frac{\alpha}{\lambda}g(\mathrm{R}(e_i,\xi)\xi,e_i) + \frac{\alpha}{\lambda}\mathrm{div}\xi g(\nabla_{e_i}\xi,e_i) - \frac{\alpha^2}{\lambda^2}g(\nabla_\xi\xi,\xi)g(\nabla_{e_i}\xi,e_i)\\ &+ \frac{\alpha^2}{\lambda^2}g(\nabla_{e_i}\xi,\xi)g(\nabla_{e_i}\xi,\xi) - \frac{\alpha}{\lambda}g(\nabla_{e_i}\xi,\nabla_{e_i}\xi)\Big)\\ &= \sigma - \frac{\alpha(2\lambda-1)}{\lambda(\lambda-1)}\mathrm{Ric}(\xi,\xi) - \frac{\alpha^2(2\lambda-1)}{\lambda^2(\lambda-1)}g(\nabla_\xi\xi,\xi)\mathrm{div}\xi + \frac{\alpha}{\lambda}(\mathrm{div}\xi)^2\\ &+ \frac{\alpha^2(2\lambda-1)}{\lambda^2(\lambda-1)}|\nabla_\xi\xi|^2 - \frac{\alpha}{\lambda}|\nabla\xi|^2, \end{aligned} \tag{3.23}$$

$$\frac{\alpha}{\lambda(\lambda-1)}\widetilde{\mathrm{Ric}}(\xi,\xi) = \frac{\alpha}{\lambda(\lambda-1)}\mathrm{Ric}(\xi,\xi) + \frac{\alpha^2}{\lambda^2(\lambda-1)}\mathrm{div}\xi g(\nabla_\xi\xi,\xi) - \frac{\alpha^2}{\lambda^2(\lambda-1)}g(\nabla_\xi\xi,\nabla_\xi\xi). \tag{3.24}$$

By substituting (3.23) and (3.24) into (3.22), the proof is complete. □

Corollary 3.10. *Given a Riemannian manifold* (M^m, g)*. If* ξ *is unit vector field, then the scalar curvature* $\widetilde{\sigma}$ *of* $(M^m, \widetilde{g})$ *is expressed by:*

$$\widetilde{\sigma} = \sigma - \frac{2\alpha}{1+\alpha}\mathrm{Ric}(\xi,\xi) + \frac{\alpha}{1+\alpha}(\mathrm{div}\xi)^2 - \frac{\alpha}{1+\alpha}|\nabla\xi|^2.$$

In the following, we study the harmonicity and biharmonicity of a mapping between Riemannian manifolds, one of which is endowed with a Cheeger-Gromoll type deformation of the metric of this manifold.

4. Harmonicity Concerning the Cheeger-Gromoll Type Deformation of the Starting Metric or the Arrival Metric

4.1. Harmonicity of the Map $\phi : (M^m, \widetilde{g}) \to (N^n, h)$.

We study the harmonicity of the smooth map $\phi : (M^m, \widetilde{g}) \to (N^n, h)$, where $\widetilde{g}$ is the Cheeger-Gromoll type deformation of metric g and $\widetilde{\nabla}$ its Levi-Civita connection.

Theorem 4.1. *The map* $\phi : (M^m, \widetilde{g}) \to (N^n, h)$ *is harmonic if and only if*

$$\tau(\phi) = \frac{\alpha}{\lambda}(\mathrm{div}\xi - \frac{\alpha}{\lambda}g(\nabla_\xi\xi,\xi))d\phi(\xi) + \frac{\alpha}{\lambda}\nabla d\phi(\xi,\xi),$$

where $\tau(\phi)$ *is the tension field of* $\phi : (M^m, g) \to (N^n, h)$*.*

Proof. Given a local orthonormal frame $\{\widetilde{e}_i\}_{i=1,\dots,m}$ on $(M^m, \widetilde{g})$ defined by (3.13). We compute the tension field $\widetilde{\tau}(\phi)$ of the map $\phi : (M^m, \widetilde{g}) \to (N^n, h)$.

$$\widetilde{\tau}(\phi) = \sum_{i=1}^{m}\Big(\nabla^{\phi}_{\widetilde{e}_i}d\phi(\widetilde{e}_i) - d\phi(\widetilde{\nabla}_{\widetilde{e}_i}\widetilde{e}_i)\Big),$$

where ∇^{ϕ} is the pull-back connection induced by $\phi : (M^m, g) \to (N^n, h)$.
Using (2.3) and direct calculations, we obtain

$$\begin{aligned}\widetilde{\tau}(\phi) &= \sum_{i=1}^{m}\Big(\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\widetilde{\nabla}_{e_i} e_i)\Big) - \Big(\nabla^{\phi}_{e_m} d\phi(e_m) - d\phi(\widetilde{\nabla}_{e_m} e_m)\Big) + \Big(\frac{1}{\sqrt{\lambda}}\nabla^{\phi}_{e_m}\frac{1}{\sqrt{\lambda}} d\phi(e_m) - \frac{1}{\sqrt{\lambda}} d\phi(\widetilde{\nabla}_{e_m}\frac{1}{\sqrt{\lambda}} e_m)\Big)\\
&= \sum_{i=1}^{m}\nabla^{\phi}_{e_i} d\phi(e_i) - \sum_{i=1}^{m} d\phi\Big(\nabla_{e_i} e_i + \frac{\alpha}{\lambda} g(\nabla_{e_i}\xi, e_i)\xi\Big) - \frac{\lambda-1}{\lambda}\Big(\nabla^{\phi}_{e_m} d\phi(e_m) - d\phi\Big(\nabla_{e_m} e_m + \frac{\alpha}{\lambda} g(\nabla_{e_m}\xi, e_m)\xi\Big)\Big)\\
&= \tau(\phi) - \frac{\alpha}{\lambda}\mathrm{div}\xi\, d\phi(\xi) - \frac{\lambda-1}{\lambda}\nabla d\phi(e_m, e_m) + \frac{\alpha(\lambda-1)}{\lambda^2} g(\nabla_{e_m}\xi, e_m) d\phi(\xi)\\
&= \tau(\phi) - \frac{\alpha}{\lambda}\mathrm{div}\xi\, d\phi(\xi) - \frac{\alpha}{\lambda}\nabla d\phi(\xi, \xi) + \frac{\alpha^2}{\lambda^2} g(\nabla_{\xi}\xi, \xi) d\phi(\xi).\end{aligned}$$

Hence,

$$\widetilde{\tau}(\phi) = \tau(\phi) - \frac{\alpha}{\lambda}\big(\mathrm{div}\xi - \frac{\alpha}{\lambda} g(\nabla_{\xi}\xi, \xi)\big) d\phi(\xi) - \frac{\alpha}{\lambda}\nabla d\phi(\xi, \xi). \tag{4.1}$$

By setting (4.1) equal to zero, the proof is complete. □

As a direct consequence of (4.1), we find that the tension field of the identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$ is given by:

$$\widetilde{\tau}(I) = -\frac{\alpha}{\lambda}\big(\mathrm{div}\xi - \frac{\alpha}{\lambda} g(\nabla_{\xi}\xi, \xi)\big)\xi.$$

Corollary 4.2. *The identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$ is harmonic if and only if*

$$\mathrm{div}\xi = \frac{\alpha}{\lambda} g(\nabla_{\xi}\xi, \xi).$$

4.2. **Harmonicity of the Map $\phi : (M^m, g) \to (N^n, \widetilde{h})$.**
We study the harmonicity of the smooth map $\phi : (M^m, g) \to (N^n, \widetilde{h})$, where $\widetilde{h}$ is the Cheeger-Gromoll type deformation of metric h and $\widetilde{\nabla}$ its Levi-Civita connection.

Theorem 4.3. *The map $\phi : (M^m, g) \to (N^n, \widetilde{h})$ is harmonic if and only if*

$$\tau(\phi) = -\frac{\alpha}{\lambda}\mathrm{Tr}_g h(\nabla^{N}_{d\phi(*)}\xi, d\phi(*))\xi \circ \phi,$$

where $\tau(\phi)$ is the tension field of $\phi : (M^m, g) \to (N^n, h)$.

Proof. Let $\{e_i\}_{i=1,\cdots,m}$ be a local orthonormal frame on (M^m, g), we calculate the tension field $\widetilde{\tau}(\phi)$ of the map $\phi : (M^m, g) \to (N^n, \widetilde{h})$.

$$\begin{aligned}\widetilde{\tau}(\phi) &= \sum_{i=1}^{m}\Big(\widetilde{\nabla}^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)\Big)\\
&= \sum_{i=1}^{m}\Big(\widetilde{\nabla}^{N}_{d\phi(e_i)} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)\Big)\\
&= \sum_{i=1}^{m}\Big(\nabla^{N}_{d\phi(e_i)} d\phi(e_i) + \frac{\alpha}{\lambda} h(\nabla^{N}_{d\phi(e_i)}\xi, d\phi(e_i))\xi \circ \phi - d\phi(\nabla_{e_i} e_i)\Big)\\
&= \sum_{i=1}^{m}\Big(\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i) + \frac{\alpha}{\lambda} h(\nabla^{N}_{d\phi(e_i)}\xi, d\phi(e_i))\xi \circ \phi\Big).\end{aligned}$$

Hence,

$$\widetilde{\tau}(\phi) = \tau(\phi) + \frac{\alpha}{\lambda}\mathrm{Tr}_g h(\nabla^{N}_{d\phi(*)}\xi, d\phi(*))\xi \circ \phi. \tag{4.2}$$

By setting (4.2) equal to zero, the proof is complete. □

As a direct consequence of (4.2), we find that the tension field of the identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$ is given by:

$$\widetilde{\tau}(I) = \frac{\alpha}{\lambda}(\mathrm{div}\xi)\xi. \tag{4.3}$$

Corollary 4.4. *The identity map $I : (M^m, g) \to (M^m, \widetilde{g})$ is harmonic if and only if*

$$\mathrm{div}\xi = 0.$$

5. The Biharmonicity of the Identity Map

In the following, we consider ξ is a unit vector field.

5.1. The Biharmonicity of the Identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$.

Theorem 5.1. *The identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$ is a biharmonic if and only if*

$$\mathrm{div}\xi\bar{\Delta}\xi = \mathrm{div}\xi\,\mathrm{Ricci}(\xi) + \Delta(\mathrm{div}\xi)\xi + 2\nabla_{\mathrm{grad}(\mathrm{div}\xi)}\xi - \frac{\alpha}{1+\alpha}(\frac{1}{2}\xi(\mathrm{div}\xi)^2\xi + \xi\xi(\mathrm{div}\xi))\xi.$$

Proof. Let $\{\widetilde{e_i}\}_{i=1,\dots,m}$ be a local orthonormal frame on $(M^m, \widetilde{g})$ defined by (3.13). The bitension field of the identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$ is given by:

$$\widetilde{\tau}_2(I) = \Delta^I\widetilde{\tau}(I) - \mathrm{Tr}_{\widetilde{g}}\mathrm{R}(\widetilde{\tau}(I), dI)dI. \tag{5.1}$$

From (4.1), we have

$$\widetilde{\tau}(I) = -\frac{\alpha}{1+\alpha}(\mathrm{div}\xi)\xi.$$

First we calculate $\Delta^I\widetilde{\tau}(I)$, so using the Theorem 2.2, we find

$$\begin{aligned}\Delta^I\widetilde{\tau}(I) &= \frac{\alpha}{1+\alpha}\sum_{i=1}^{m}\Big(\nabla_{\widetilde{e_i}}\nabla_{\widetilde{e_i}}((\mathrm{div}\xi)\xi) - \nabla_{\widetilde{\nabla}_{\widetilde{e_i}}\widetilde{e_i}}((\mathrm{div}\xi)\xi)\Big)\\ &= \frac{\alpha}{1+\alpha}(\Delta(\mathrm{div}\xi)\xi - \mathrm{div}\xi\bar{\Delta}\xi + 2\nabla_{\mathrm{grad}(\mathrm{div}\xi)}\xi) - \frac{\alpha^2}{(1+\alpha)^2}(\frac{1}{2}\xi(\mathrm{div}\xi)^2 + \xi\xi(\mathrm{div}\xi))\xi.\end{aligned} \tag{5.2}$$

We also find,

$$\begin{aligned}-\mathrm{Tr}_{\widetilde{g}}\mathrm{R}(\widetilde{\tau}(I), dI)dI &= \frac{\alpha}{1+\alpha}\mathrm{div}\xi\sum_{i=1}^{m}\mathrm{R}(\xi, \widetilde{e_i})\widetilde{e_i}\\ &= \frac{\alpha}{1+\alpha}\mathrm{div}\xi\,\mathrm{Ricci}(\xi).\end{aligned} \tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1), we obtain

$$\widetilde{\tau}_2(I) = \frac{\alpha}{1+\alpha}(\mathrm{div}\xi\,\mathrm{Ricci}(\xi) + \Delta(\mathrm{div}\xi)\xi - \mathrm{div}\xi\bar{\Delta}\xi + 2\nabla_{\mathrm{grad}(\mathrm{div}\xi)}\xi) - \frac{\alpha^2}{(1+\alpha)^2}(\frac{1}{2}\xi(\mathrm{div}\xi)^2\xi + \xi\xi(\mathrm{div}\xi))\xi. \tag{5.4}$$

Finally, the proof is complete by setting (5.4) equal to zero. □

Theorem 5.2. *If $\mathrm{div}\xi$ is a non null constant on M, then the identity map $I : (M^m, \widetilde{g}) \to (M^m, g)$ is a proper biharmonic if and only if*

$$\bar{\Delta}\xi = \mathrm{Ricci}(\xi).$$

5.2. **The Biharmonicity of the Identity Map** $I : (M^m, g) \to (M^m, \widetilde{g})$.

Theorem 5.3. *The identity map* $I : (M^m, g) \to (M^m, \widetilde{g})$ *is a biharmonic if and only if*

$$\mathrm{div}\xi\bar{\Delta}\xi = \mathrm{div}\xi\,\mathrm{Ricci}(\xi) + \Delta(\mathrm{div}\xi)\xi + 2\nabla_{\mathrm{grad}(\mathrm{div}\xi)}\xi - \frac{\alpha}{1+\alpha}\mathrm{div}\xi(\mathrm{Ric}(\xi,\xi) - |\nabla\xi|^2)\xi.$$

Proof. Let $\{e_i\}_{i=1,\cdots,m}$ be a local orthonormal frame on (M^m, g). The bitension field of the identity map $I : (M^m, g) \to (M^m, \widetilde{g})$ is given by:

$$\widetilde{\tau}_2(I) = \widetilde{\Delta}^I\widetilde{\tau}(I) - \mathrm{Tr}_g\widetilde{\mathrm{R}}(\tau(I), dI)dI. \tag{5.5}$$

From (4.3), we have

$$\widetilde{\tau}(I) = \frac{\alpha}{1+\alpha}(\mathrm{div}\xi)\xi.$$

With similar calculations as before, we find

$$\begin{aligned}\widetilde{\Delta}^I\tau(I) &= \frac{-\alpha}{1+\alpha}\sum_{i=1}^{m}(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}((\mathrm{div}\xi)\xi) - \widetilde{\nabla}_{\nabla_{e_i}e_i}((\mathrm{div}\xi)\xi))\\ &= \frac{-\alpha}{1+\alpha}(\Delta(\mathrm{div}\xi)\xi - \mathrm{div}\xi\bar{\Delta}\xi + 2\nabla_{\mathrm{grad}(\mathrm{div}\xi)}\xi) - \frac{\alpha^2}{(1+\alpha)^2}\mathrm{div}\xi|\nabla\xi|^2\xi \end{aligned}\tag{5.6}$$

and

$$\begin{aligned}-\mathrm{Tr}_g\widetilde{\mathrm{R}}(\widetilde{\tau}(I), dI)dI &= \frac{-\alpha}{1+\alpha}\mathrm{div}\xi\sum_{i=1}^{m}\widetilde{\mathrm{R}}(\xi, e_i)e_i\\ &= \frac{-\alpha}{1+\alpha}\mathrm{div}\xi\,\mathrm{Ricci}(\xi) + \frac{\alpha^2}{(1+\alpha)^2}\mathrm{div}\xi\,\mathrm{Ric}(\xi,\xi)\xi.\end{aligned}\tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5), we find

$$\widetilde{\tau}_2(I) = \frac{-\alpha}{1+\alpha}(\mathrm{div}\xi\,\mathrm{Ricci}(\xi) + \Delta(\mathrm{div}\xi)\xi - \mathrm{div}\xi\bar{\Delta}\xi + 2\nabla_{\mathrm{grad}(\mathrm{div}\xi)}\xi) + \frac{\alpha^2}{(1+\alpha)^2}\mathrm{div}\xi(\mathrm{Ric}(\xi,\xi) - |\nabla\xi|^2)\xi. \tag{5.8}$$

Finally, the proof is complete by setting (5.8) equal to zero. □

Theorem 5.4. *If* $\mathrm{div}\xi$ *is a non null constant on* M*, then the identity map* $I : (M^m, g) \to (M^m, \widetilde{g})$ *is a proper biharmonic if and only if*

$$\bar{\Delta}\xi = \mathrm{Ricci}(\xi) - \frac{\alpha}{1+\alpha}(\mathrm{Ric}(\xi,\xi) - |\nabla\xi|^2)\xi.$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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