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A Fixed point theorem for extended large contraction mappings

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Abstract

In this paper, we introduce a new large contraction via (c)-comparison function in the setting of complete metric space. We investigate the existence of a fixed point for such contractions.

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1. Introduction and Preliminaries

Theorem 1. *Let (X, d) be a complete metric spaces and $T : X \to X$ be a contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $0 < \lambda < 1$. Then T has a unique fixed point.

In last decades, this perfectly formulated theorem has been improved, extended and generalized in several aspect. Among them, we pay attention to the interesting results of Burton [1]. He introduce the large contraction to extend the Banach contraction principle, in his interesting paper.

Definition 1. *Let (X, d) be a metric space and assume that $T : X \to X$. T is said to be a large contraction, if for $x, y \in X$, with $x \neq y$, we have $d(Tx, Ty) < d(x, y)$, and if $\forall \epsilon > 0$, $\exists \delta < 1$ such that*

$$[x, y \in X, \quad d(x, y) \geq \epsilon] \Longrightarrow d(Tx, Ty) \leq \delta d(x, y).$$

Burton [1] observed that in the context of a complete metric space, every large contraction posses a unique fixed point.

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Theorem 2 (Burton)**.** *Let (X, d) be a complete metric space and T be a large contraction. Suppose there is an $x \in X$ and an $L > 0$, such that $d(x, T^n x) \leq L$ for all $n \geq 1$. Then T has a unique fixed point in X.*

In this paper, we extend the definition of large contraction by using an auxiliary function, namely, (c)-comparison function. Then, we investigate the existence and uniqueness of a fixed point for operators that form extended large contractions.

2. Main results

In this paper, in the case of complete metric spaces, we establish some results of the existence and uniqueness of fixed points for large contraction mapping T defined by a (c)-comparison functions.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

We start our results by the generalization of the definition of large contraction.

Definition 2. *Let (X, d) be a metric space and assume that $T : X \to X$. T is said to be a extended large contraction, if for $x, y \in X$, with $x \neq y$, we have $d(Tx, Ty) < d(x, y)$, and if $\forall \epsilon > 0$, there exists $\psi \in \Psi$ such that*

$$[x, y \in X, \quad d(x, y) \geq \epsilon] \Longrightarrow d(Tx, Ty) \leq \psi(d(x, y)). \tag{2.1}$$

Theorem 3. *Let (X, d) be a complete metric space and T be a extended large contraction. Suppose there is an $x_0 \in X$ and an $L > 0$, such that $d(x_0, T^n x_0) \leq L$ for all $n \geq 1$. Then T has a unique fixed point in X.*

Proof. By assumption of the theorem, there is $x_0 \in X$ and an $L > 0$, such that

$$d(x_0, T^n x_0) \leq L, \tag{2.2}$$

for all $n \geq 1$. For this $x_0 \in (X, d)$, we set a sequence $\{x_n\}$ as $x_{n+1} = T^n(x_0)$ and n is a positive integer. If $\{x_n\}$ is a Cauchy sequence, by a routine way, we conclude that the limit of the sequence $\{x_n\}$ forms a fixed point of T.

On what follows, we shall show that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Thus, there exist

$$\epsilon > 0, \ \ N_k \uparrow \infty, \ \ n_k > N_k, \ \ m_k > n_k, \text{ with}$$

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{2.3}$$

First of all, observe that

From (2.1), it follows that for all $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \psi(d(x_n, x_{n-1})). \tag{2.4}$$

Recursively, we get

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)), \text{ for all } n \geq 1. \tag{2.5}$$

On account of (2.5) and (2.3), for every $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} d\left(x_{m_k}, x_{n_k}\right) = d\left(Tx_{m_k-1}, x_{n_k-1}\right) &\leq \psi(d\left(x_{m_k-2}, x_{n_k-2}\right)) \\ &\leq \psi^2(d\left(x_{m_k-3}, x_{n_k-3}\right)) \\ &\cdots \\ &\leq \psi^{m-n}(d\left(x_0, x_{m_k-n_k}\right)) \qquad (2.6) \\ &\leq \psi^{m-n}(L), \qquad (2.7) \end{aligned}$$

since ψ in nondecreasing and $d(x_0, T^n x_0) \leq L$ for all $n \geq 1$.

Combining the above inequality with (2.3), we find that

$$d\left(x_{m_k}, x_{n_k}\right) \leq \psi^{m-n}(d\left(x_0, x_{m_k-n_k}\right)) \leq \psi^{m-n}(L). \qquad (2.8)$$

Since $\delta < 1$, $m, n \longrightarrow \infty$ in (2.8) we get a contraction which complete the proof. □

Remark 1. *It is clear that if take $\psi(t) = \delta t$, then we deduce Theorem 2 of Burton [1].*

References

[1] T. A. Burton, Integral equations, implicit relations and fixed points. Proc. Amer. Math. Soc. 124 (1996); 2383-2390.