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Korovkin Type Theorem for Modified Bernstein Operators via A-Statistical Convergence and Power Summability Method

Tuğba Değirmenci and Merve İlkhan Kara*

Abstract

In this study, we investigate the approximation properties of modified Bernstein operators through the lens of A-statistical convergence and power summability methods. Our main objective is to establish a Korovkin type approximation theorem in this generalized setting. By incorporating statistical convergence, we aim to provide broader and more powerful approximation results that can be applied in various contexts where classical convergence criteria may fail or be insufficient.

Keywords: *A-statistical convergence, Bernstein operators, Korovkin type theorem, Power summability method*

AMS Subject Classification (2020): *40G10; 40C15*

1. Introduction and preliminaries

One of the most significant milestones in approximation theory is undoubtedly the renowned Weierstrass approximation theorem, which asserts that any continuous function on a closed interval can be uniformly approximated by polynomials. In the years that followed, numerous proofs of this theorem were developed. Among them, Bernstein's proof stands out due to its elegance and its introduction of the Bernstein operator, which laid the foundation for the theory of linear positive operators. The significance of linear positive operators soon became evident, as they provide a straightforward and constructive means to approximate functions.

In this context, Bohman established that if a sequence of linear positive operators L_n satisfies $L_n(1;x) \rightrightarrows 1$, $L_n(t;x) \rightrightarrows x$ and $L_n(t^2;x) \rightrightarrows x^2$, then for any continuous function f, it follows that $L_n(f(t);x) \rightrightarrows f(x)$. Subsequently, Korovkin extended this result to integral-type operators, leading to what is now known as Korovkin's theorem. The contributions of Bohman and Korovkin have significantly advanced the theory of linear positive operators.

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As a result, numerous researchers have since introduced and investigated various types of operators that fulfill the conditions of this theorem. Consequently, the theory of function approximation via linear positive operators remains a vibrant and continually evolving area of mathematical analysis.

The Bernstein polynomials are defined as

$$B_n(f,x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f(\frac{k}{n})$$

for every bounded function on $[0,1]$.

Let $C[0,1]$ be the Banach space of all real-valued continuous functions on $[0,1]$ endowed with the sup norm

$$\|f\|=\sup_{x\in[0,1]}|f(x)|.$$

We shall denote by $\mathbb{N}$ the set of all natural numbers. The sequence $x=(x_k)$ is said to be statistically convergent to L if for every $\varepsilon>0$, the set $K_\varepsilon=\{k\in\mathbb{N}:|x_k-L|\geq\varepsilon\}$ has natural density zero [1], i.e. for each $\varepsilon>0$,

$$\lim_n\frac{1}{n}|\{k\leq n:|x_k-L|\geq\varepsilon\}|=0.$$

In this case, we write $L=st-\lim x$. Note that every convergent sequence is statistically convergent, but not conversely. In what follows we will use the definition of the A-statistical convergence. Let $A=(a_{nj})$ be a summability matrix and $x=(x_j)$ be a sequence. If the series

$$(Ax)_n=\sum_j a_{nj}x_j$$

converges for every $n\in\mathbb{N}$, then we say that $((Ax)_n)$ is the A-transform of the sequence $x=(x_n)$. And if the $((Ax)_n)$ converges to a number L, we say that x is A-summable to L. The summability matrix A is regular whenever $\lim_j x_j=L$, then $\lim_n(Ax)_n=L$. A be a non-negative regular summability matrix. The sequence $x=(x_j)$ is said to be A-statistically convergent (see [2]) to real number a if for any $\varepsilon>0$

$$\lim_n\sum_{j:|x_j-a|\geq\varepsilon}a_{nj}=0.$$

In this case we write $st_A-limx=a$.

The A-statistical convergence is a generalization of the statistical convergence and it is proven in the Example given in paper [3]. The second summability method used in this paper is power summability method.

Let (p_j) be a real sequence with $p_0>0$ and $p_1,p_2,p_3,...\geq 0$ such that the corresponding power series $p(t)=\sum_{j=0}^{\infty}p_jt^j$ has radius of convergence R with $0<R\leq\infty$. If, for all $t\in(0,R)$,

$$\lim_{t\to R^-}\frac{1}{p(t)}\sum_{j=0}^{\infty}x_jp_jt^j=L$$

exists, then we say that $x=(x_j)$ is convergent in the sense of power summability method (see [4, 5]). Power summability method includes many known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and they are not matrix methods (see [6–8]). That power summability method is more effective than ordinary convergence, as shown in the example given in [9].

Note that the power summability method is regular if and only if

$$\lim_{t\to R^-}\frac{p_jt^j}{p(t)}=0$$

holds for each $j\in\{0,1,2,3,...\}$ ([10]). Throughout the paper we assume that power series method is regular.

The theory of Korovkin type theorems was studied in several function spaces, and further details the reader can find in those papers (see [11–15]).

In this paper, we will prove the Korovkin-type theorem for the modified Bernstein operators via A-statistical convergence and the power summability method. To this end, we define a new sequence of modified Bernstein operators tailored to preserve certain structural properties of the approximated functions. We then analyze their behavior under the A-statistical convergence scheme, which is governed by a regular summability matrix A, and combine this approach with power summability techniques to obtain stronger results concerning the convergence of the operators to the target function. The theoretical results presented in this paper extend and generalize classical Korovkin-type theorems, offering new insights into approximation theory in the context of summability methods.

2. Main results

In a recent paper, Usta [16] has defined the new modification of Bernstein type operators which fix a constant and preserves Korovkin's other test functions. The modification of the operator is defined for function $f \in C(0,1)$ as follows:

$$B_n^*(f,x) = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2 x^{k-1}(1-x)^{n-k-1} f(\frac{k}{n}). \tag{2.1}$$

It is observed that this new operator is positive and linear. The following lemma is required to prove our main results.

Lemma 2.1. *[16] Let $e_i = t^i$ for $i = 0,1,2$. Then the following equations hold:*

$$B_n^*(e_0,x) = 1$$

$$B_n^*(e_1,x) = \frac{n-2}{n}x + \frac{1}{n}$$

$$B_n^*(e_2,x) = \frac{n^2-7n+6}{n^2}x^2 + \frac{5n-6}{n^2}x + \frac{1}{n^2}.$$

Here and what follows

$$C_b[0,\infty) = \{f | f : [0,\infty) \to \mathbb{R} \text{ continuous and bounded}\}.$$

Let $A = (a_{jn})$ be a non negative regular summability matrix and the condition

$$\lim_{j\to\infty}\sum_{n=1}^{\infty} a_{jn}\frac{1}{n} = 0 \tag{2.2}$$

hold. We now define a new operator as a modification of the operator introduced in (2.1), originally proposed in [17] as follows:

$$B_j^{**}(f,x) = \sum_{n=1}^{\infty} a_{jn} B_n^*(f,x).$$

We first get the next approximation result.

Theorem 2.1. *Given any $f \in C_b[0,\infty)$*

$$\lim_{j\to\infty} B_j^{**}(f,x) = f(x)$$

holds uniformly on compact subsets of $[0,\infty)$.

Proof. Since the matrix $A = (a_{jn})$ is regular we have

$$\left|B_j^{**}(e_0,x) - e_0(x)\right| = \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \longrightarrow 0$$

for $j \to \infty$.

By using the inequality

$$|B_n^*(e_1,x) - x| = \left|-\frac{2}{n}x + \frac{1}{n}\right| \le \frac{2}{n}x + \frac{1}{n},$$

the regularity of the matrix $A = (a_{jn})$ with the condition (2.2), we obtain

$$\begin{aligned}
\left|B_j^{**}(e_1, x) - e_1(x)\right| &= \left|\sum_{n=1}^{\infty} a_{jn} B_n^*(e_1, x) - e_1(x)\right| \\
&= \left|\sum_{n=1}^{\infty} a_{jn} \left[B_n^*(e_1, x) - e_1(x)\right] + \sum_{n=1}^{\infty} a_{jn} e_1(x) - e_1(x)\right| \\
&= \left|\sum_{n=1}^{\infty} a_{jn} \left[B_n^*(e_1, x) - e_1(x)\right] + e_1(x) \left[\sum_{n=1}^{\infty} a_{jn} - 1\right]\right| \\
&\le \sum_{n=1}^{\infty} a_{jn} \left|B_n^*(e_1, x) - e_1(x)\right| + x \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \\
&\le \sum_{n=1}^{\infty} a_{jn} \left(\frac{2}{n} x + \frac{1}{n}\right) + x \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \\
&= 2x \sum_{n=1}^{\infty} \frac{a_{jn}}{n} + \sum_{n=1}^{\infty} \frac{a_{jn}}{n} + x \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \longrightarrow 0
\end{aligned}$$

for $j \to \infty$.

Similarly, using the inequality

$$\begin{aligned}
\left|B_n^*(e_2, x) - x^2\right| &= \left|-\frac{7}{n} x^2 + \frac{6}{n^2} x^2 + \frac{5}{n} x - \frac{6}{n^2} x + \frac{1}{n^2}\right| \\
&\le \frac{7}{n} x^2 + \frac{6}{n^2} x^2 + \frac{5}{n} x + \frac{6}{n^2} x + \frac{1}{n^2},
\end{aligned}$$

the regularity of the matrix $A = (a_{jn})$ with the condition (2.2), we obtain

$$\begin{aligned}
\left|B_j^{**}(e_2, x) - e_2(x)\right| &= \left|\sum_{n=1}^{\infty} a_{jn} B_n^*(e_2, x) - e_2(x)\right| \\
&= \left|\sum_{n=1}^{\infty} a_{jn} \left[B_n^*(e_2, x) - e_2(x)\right] + e_2(x) \left[\sum_{n=1}^{\infty} a_{jn} - 1\right]\right| \\
&\le \sum_{n=1}^{\infty} a_{jn} \left|B_n^*(e_2, x) - e_2(x)\right| + x^2 \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \\
&\le \sum_{n=1}^{\infty} a_{jn} \left(\frac{7}{n} x^2 + \frac{6}{n^2} x^2 + \frac{5}{n} x + \frac{6}{n^2} x + \frac{1}{n^2}\right) + x^2 \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \\
&= (7x^2 + 5x) \sum_{n=1}^{\infty} \frac{a_{jn}}{n} + (6x^2 + 6x + 1) \sum_{n=1}^{\infty} \frac{a_{jn}}{n^2} + x^2 \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \longrightarrow 0
\end{aligned}$$

for $j \to \infty$. □

Now we give the Korovkin type theorem for A-statistical convergence.

Theorem 2.2. *Let $A = (a_{jn})$ be a non negative regular summability matrix and*

$$st_A - \lim_n \|B_n^* e_i - e_i\| = 0$$

hold for $i = 0, 1, 2$. Then given any $f \in C[0,1]$

$$st_A - \lim_n \|B_n^* f - f\| = 0$$

holds, where $\|f\| = \sup_{t \in [0,1]} |f(t)|$.

Proof. The inequality

$$\|B_n^* e_2 - e_2\| \leq \sup_{x\in[0,1]} \left\{ \left| \frac{n^2-7n+6}{n^2} x^2 + \frac{5n-6}{n^2} x + \frac{1}{n^2} - x^2 \right| \right\}$$
$$\leq \frac{7n+6}{n^2} + \frac{5n+6}{n^2} + \frac{1}{n^2}$$

holds. If

$$M = \{n : \|B_n^* e_2 - e_2\| \geq \varepsilon\},$$
$$M_1 = \left\{ \frac{7n+6}{n^2} \geq \frac{\varepsilon}{3} \right\},$$
$$M_2 = \left\{ \frac{5n+6}{n^2} \geq \frac{\varepsilon}{3} \right\},$$
$$M_3 = \left\{ \frac{1}{n^2} \geq \frac{\varepsilon}{3} \right\}$$

then we have $M \subset M_1 \cup M_2 \cup M_3$. Hence we conclude that

$$st_A - \lim_n \|B_n^* e_2 - e_2\| = 0$$

holds. □

Now we will prove the Korovkin-type theorem for the modified Bernstein operators by the power summability method.

By the aid of sequence of operators B_n^* let define the power series $\sum_{n=0}^{\infty} B_n^*(f,x) p_n t^n$. For every $t \in (0,R)$, if the limit

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} B_n^*(f,x) p_n t^n$$

exists then we say that sequence of operators B_n^* converges in the sense of power series.

Theorem 2.3. *Let B_n^* be a sequence of positive linear operators from $C[0,1]$ into $B[0,1]$. For every $f \in C[0,R]$*

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} (B_n^* f - f) p_n t^n \right\| = 0 \tag{2.3}$$

if and only if

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} (B_n^* e_i - e_i) p_n t^n \right\| = 0 \tag{2.4}$$

for $i = 0, 1, 2$.

Proof. From the equality (2.3) we have the equality (2.4).

For the converse assume that the equality (2.4) holds. Let $f \in C[0,1]$. For all $t \in [0,1]$, there exists a real number $M > 0$ such that $|f(t)| \leq M$ holds. Hence, for $t, x \in [0,1]$ we have

$$|f(t) - f(x)| \leq 2M. \tag{2.5}$$

Since f is continuous, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-x| < \delta$ implies

$$|f(t) - f(x)| \leq \varepsilon. \tag{2.6}$$

Let $\Psi(t,x) = (t-x)^2$. $|t-x| \geq \delta$ implies that

$$|f(t) - f(x)| \leq \frac{2M}{\delta^2} \Psi(t,x). \tag{2.7}$$

By using the inequalities (2.5)-(2.7), we have

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}\Psi(t,x).$$

Hence we write

$$-\varepsilon - \frac{2M}{\delta^2}\Psi(t,x) < f(t) - f(x) < \varepsilon + \frac{2M}{\delta^2}\Psi(t,x).$$

Since $B_n^*(1,x)$ is monoton and linear

$$B_n^*(1,x)\left(-\varepsilon - \frac{2M}{\delta^2}\Psi(t,x)\right) < B_n^*(1,x)\left(f(t) - f(x)\right) < B_n^*(1,x)\left(\varepsilon + \frac{2M}{\delta^2}\Psi(t,x)\right)$$

and so we have

$$-\varepsilon B_n^*(1,x) - \frac{2M}{\delta^2}B_n^*(\Psi(t),x) < B_n^*(f,x) - f(x)B_n^*(1,x) < \varepsilon B_n^*(1,x) + \frac{2M}{\delta^2}B_n^*(\Psi(t),x). \tag{2.8}$$

By using the following equality with the inequality (2.8)

$$B_n^*(f,x) - f(x) = B_n^*(f,x) - f(x)B_n^*(1,x) + f(x)\left(B_n^*(1,x) - 1\right)$$

we obtain

$$B_n^*(f,x) - f(x) < \varepsilon B_n^*(1,x) + \frac{2M}{\delta^2}B_n^*(\Psi(t),x) + f(x)\left(B_n^*(1,x) - 1\right). \tag{2.9}$$

From the equality

$$B_n^*(\Psi(t),x) = B_n^*((t-x)^2,x) = B_n^*((x^2 - 2xt + t^2),x) = x^2B_n^*(1,x) - 2xB_n^*(t,x) + B_n^*(t^2,x)$$

and the inequality (2.9) we have

$$\begin{aligned} B_n^*(f,x) - f(x) &< \frac{2M}{\delta^2}\left\{x^2\left(B_n^*(1,x) - 1\right) - 2x\left(B_n^*(t,x) - x\right) + \left(B_n^*(t^2,x) - x^2\right)\right\} \\ &\quad + \varepsilon B_n^*(1,x) + f(x)\left(B_n^*(1,x) - 1\right) \\ &= \varepsilon + \varepsilon\left(B_n^*(1,x) - 1\right) + f(x)\left(B_n^*(1,x) - 1\right) \\ &\quad + \frac{2M}{\delta^2}\left\{x^2\left(B_n^*(1,x) - 1\right) - 2x\left(B_n^*(t,x) - x\right) + \left(B_n^*(t^2,x) - x^2\right)\right\}. \end{aligned}$$

It follows that

$$|B_n^*(f,x) - f(x)| \le \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}\right)|B_n^*(1,x) - 1| + \frac{4M}{\delta^2}|B_n^*(t,x) - x| + \frac{2M}{\delta^2}\left|B_n^*(t^2,x) - x^2\right|$$

holds. From the last inequality and the linearity of the operator $B_n^*(f,x)$ we conclude that

$$\begin{aligned} \frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(f,x) - f(x))p_nt^n\right\| &\le \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}\right)\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(1,x) - 1)p_nt^n\right\| \\ &\quad + \frac{4M}{\delta^2}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(t,x) - x)p_nt^n\right\| \\ &\quad + \frac{2M}{\delta^2}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(t^2,x) - x^2)p_nt^n\right\|. \end{aligned}$$

This completes the proof. □

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Affiliations

TUĞBA DEĞIRMENCI
ADDRESS: Düzce University, Dept. of Mathematics, 81620, Düzce-Türkiye
E-MAIL: tugbatutuncu@hotmail.com
ORCID ID: 0009-0004-5106-6517

MERVE İLKHAN KARA
ADDRESS: Düzce University, Dept. of Mathematics, 81620, Düzce-Türkiye
E-MAIL: merveilkhan@gmail.com
ORCID ID: 0000-0002-0831-1474