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Oscillation Results for a Class of Fourth-Order Nonlinear Differential Equations with Positive and Negative Coefficients

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Abstract
We are interested in oscillation of the fourth-order nonlinear differential equations of the form

$$(r_1(t)(x(t)+p(t)x(\sigma(t)))'')''+\sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t)))-\sum_{i=1}^{\ell} h_i(t)H(x(\rho_i(t)))=0$$

under the assumption that

$$\int_0^\infty \frac{t}{r_1(t)}dt<\infty$$

for different ranges of $p(t)$.

Keywords: *Oscillation; fourth order nonlinear differential equation.*

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1. Introduction

Over the past few years, there has been a strong concern in the study of the oscillatory behavior of solutions of delay differential equations with positive and negative coefficients of the first and second orders; see, e.g., [1, 5-8, 10-11]. In this paper, we consider the nonlinear fourth-order delay differential equations of the form

$$\left(r_1(t)\left(x(t)+p(t)x(\sigma(t))\right)''\right)''+\sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t)))-\sum_{i=1}^{\ell} h_i(t)H(x(\rho_i(t)))=0 \quad (1.1)$$

where $r_1,\sigma,q_i,\tau_i,h_i,\rho_i$ are continuous and positive on $[0,\infty)$,$i\in\{1,2,...,\ell\}$ $p\in C([0,\infty),\mathbb{R})$, $G,H\in C(\mathbb{R},\mathbb{R})$ with $dG(d)>0$ and $bH(b)>0$ for $d,b\neq 0$, H is bounded, G is nondecreasing. Further $\sigma(t)\leq t$, $\lim\limits_{t\to\infty}\sigma(t)=\infty$, and σ_1 is a positive constant such that $\sigma_1<\sigma(t)\leq t$. And $\tau\in C([0,\infty),\mathbb{R})$ such that $\tau(t)\leq\tau_i(t)\leq t$ for $i\in\{1,2,...,\ell\}$, and $\lim\limits_{t\to\infty}\tau(t)=\infty$ and $\rho\in C([0,\infty),\mathbb{R})$ such that $\rho(t)\leq\rho_i(t)\leq t$ for $i\in\{1,2,...,\ell\}$, $\lim\limits_{t\to\infty}\rho(t)=\infty$.

The main object of our work is to investigate the oscillatory and asymptotic behaviors of the solutions of (1.1)

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under the assumption that

$$(H_1) \qquad \int\limits_0^\infty \frac{t}{r_1(t)} dt < \infty.$$

If $\sigma(t) = t - \tau$ and $\sum\limits_{i=1}^{\ell} q_i(t)G(x(\tau_i(t))) - \sum\limits_{i=1}^{\ell} h_i(t)H(x(\rho_i(t))) = q(t)G(x(t-\alpha))$, then (1.1) reduced to

$$(r_1(t)(x(t) + p(t)x(t-\tau))'')'' + q(t)G(x(t-\alpha)) = 0 \tag{1.2}$$

where $G \in C(\mathbb{R}, \mathbb{R})$ with $dG(d) > 0, d \neq 0, G$ is nondecreasing, $\tau, \alpha > 0$ are constants. In[4], Parhi and Tripathy studied Eq. (1.2) under the assumption (H_1).

Tripathy et al.[2] considered nonlinear fourth-order neutral delay differential equations of the form

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0 \tag{1.3}$$

where G, H has the same properties with us and $\tau, \alpha, \beta > 0$ are constants.

They studied (1.3) under the same assumption in addition to

$$(H_2) \qquad \int\limits_0^\infty \frac{s}{r_1(s)} \int\limits_s^\infty th(t)dtds < \infty.$$

Since Eq. (1.1) is more general than Eqs. (1.2) and (1.3), it is worth studying. Not only the present work is more illustrative than [2] but also some of results are generalized and improved. A solution of (1.1) is understood as a function $x \in C([-\eta, \infty), \mathbb{R})$ such that $(x(t) + p(t)x(\sigma(t)))$ and $(r_1(t)(x(t) + p(t)x(\sigma(t)))'')$ are twice continuosly differentiable, and (1.1) is satisfied for $t \geq 0$, where $\eta = \max\{\sigma, \tau_i, \rho_i\}$,

$$\sup\{x(t) : t \geq t_0\} > 0$$

for every $t \geq t_0$. A solution $x(t)$ of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative, and it is called nonoscillatory otherwise.

2. Preliminaries

We begin with the following results frequently used in what follows:

Lemma 2.1. *[4] Let (H_1) hold. If $f(t)$ is an eventually positive twice continuously differentiable function such that $r_1(t)f''(t)$ is twice continuously differentiable and*

$$(r_1(t)f''(t))'' \leq 0, \quad \not\equiv 0$$

for large t, where $r_1 \in C([0, \infty), (0, \infty))$, then one of the following cases holds for large t:
(a) $f'(t) > 0, f''(t) > 0$ and $(r_1(t)f''(t))' > 0$,
(b) $f'(t) > 0, f''(t) < 0$ and $(r_1(t)f''(t))' > 0$,
(c) $f'(t) > 0, f''(t) < 0$ and $(r_1(t)f''(t))' < 0$,
(d) $f'(t) < 0, f''(t) > 0$ and $(r_1(t)f''(t))' > 0$.

Lemma 2.2. *[4] Assume that the conditions of Lemma 2.1 are satisfied. Then*

(i) the following inequalities hold for large t in the case (c) of Lemma 2.1

$$f'(t) \geq -(r_1(t)f''(t))'R(t), f'(t) \geq -r_1(t)f''(t) \int\limits_t^\infty \frac{ds}{r_1(s)},$$

$$f(t) \geq ktf'(t) \text{ and } f(t) \geq -k(r_1(t)f''(t))'tR(t),$$

where $k > 0$ is a constant and

$$R(t) = \int_t^\infty \frac{s-t}{r_1(s)} ds,$$

and

(ii) $f(t) \geq r_1(t) f''(t) R(t)$ *for large* t *in the case (d) of Lemma 2.1.*

Lemma 2.3. *[4] If the conditions of Lemma 2.1 are satisfied, then there exist constants* $k_1 > 0$ *and* $k_2 > 0$ *such that* $k_1 R(t) \leq f(t) \leq k_2 t$ *for large* t*.*

Lemma 2.4. *[4] Let* (H_1) *hold. Suppose that* $z(t)$ *is a real-valued twice continuously differentiable function on* $[0, \infty)$ *such that* $(r_1(t) z''(t))'' \leq 0, \quad \not\equiv 0$ *for large* t*. If* $z(t) > 0$ *eventually, then one of the following cases holds for large* t*:*

(a) $z'(t) > 0, z''(t) > 0$ *and* $(r_1(t) z''(t))' > 0$,
(b) $z'(t) > 0, z''(t) < 0$ *and* $(r_1(t) z''(t))' > 0$,
(c) $z'(t) > 0, z''(t) < 0$ *and* $(r_1(t) z''(t))' < 0$,
(d) $z'(t) < 0, z''(t) > 0$ *and* $(r_1(t) z''(t))' > 0$.

If $z(t) < 0$ *for large* t*, then either one of the cases (b)-(d) holds or one of the following cases holds for large* t*:*

(e) $z'(t) < 0, z''(t) < 0$ *and* $(r_1(t) z''(t))' > 0$,
(f) $z'(t) < 0, z''(t) < 0$ *and* $(r_1(t) z''(t))' < 0$.

Lemma 2.5. *[3] Let* $p, x, z \in C([0, \infty), \mathbb{R})$ *be such that*

$$z(t) = x(t) + p(t) x(t - \tau)$$

for $t \geq \tau \geq 0, x(t) > 0$ *for* $t \geq t_1 > \tau$ $\liminf_{t \to \infty} x(t) = 0$*, and* $\lim_{t \to \infty} z(t) = L$ *exists. Also let* $p(t)$ *satisfy one of the following conditions:*

(i) $0 \leq p(t) \leq p_1 < 1$,

(ii) $1 < p_2 \leq p(t) \leq p_3$,

(iii) $p_4 \leq p(t) \leq 0$,

where p_i *is a constant,* $1 \leq i \leq 4$*. Then* $L = 0$

3. Main Results

In this section, sufficient conditions are established for the unbounded oscillation and asymptotic behavior of the solutions of (1.1) under the assumption (H_1). For our aim, we need the following assumptions:

(H_3) $\lambda > 0$ such that $G(d) + G(b) \geq \lambda G(d + b)$ for $d, b > 0, d, b \in \mathbb{R}$,
(H_4) $G(db) = G(d) G(b), d, b \in \mathbb{R}$,
(H_5) $G(-d) = -G(d)$, and $H(-d) = -H(d), d \in \mathbb{R}$,
(H_6) $\int_{\sigma_1}^\infty Q(t) dt = \infty, Q(t) = \min\{q(t), q(\sigma(t))\}, t \geq \sigma_1$,
(H_7) $\int_{t_0}^\infty b(t) Q(t) \sum_{i=1}^{\ell} G(R(\tau_i(t))) dt = \infty$, where $b(t) = \min\{R^\gamma(t), R^\gamma(\sigma(t))\}$,

$\gamma > 1, t_0 \geq \eta > 0$,
$(H_8) \int\limits_{t_0}^{\infty} R^{\gamma}(t) \sum\limits_{i=1}^{\ell} G(R(\tau_i(t)))q_i(t)dt = \infty, \gamma > 1, t_0 \geq \eta > 0.$

Remark 3.1. Since

$$R(t) < \int\limits_t^{\infty} \frac{s}{r_1(s)} ds,$$

we conclude that $R(t) \to 0$ as $t \to \infty$ in view of (H_1).

Remark 3.2. Assumption (H_4) implies that $G(-d) = -G(d)$. Indeed,
$G(1)G(1) = G(1)$ and $G(1) > 0$ imply that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ implies that $(G(-1))^2 = 1$. Since $G(-1) < 0$, we conclude that $G(-1) = -1$. Hence,

$$G(-d) = G(-1)G(d) = -G(d).$$

Moreover, $G(xy) = G(x)G(y)$ for every $x, y \in \mathbb{R}$ such that $G(db) = G(d)G(b)$ for $d > 0$ and $b > 0$ and $G(-d) = -G(d)$. In addition, the prototype of G satisfying (H_3), (H_4) and (H_5) is

$$G(u) = (a + b|u|^{\gamma})|u|^{\mu} \operatorname{sgn} u,$$

where $a \geq 0, b > 0, \gamma \geq 0$ and $\mu \geq 0$ are such that $a + b = 1$.

Theorem 3.1. *Let $0 \leq p(t) \leq a < 1$ or $1 < p(t) \leq a < \infty$. Suppose that (H_1)-(H_7) hold. Then every solution of Eq. (1.1) with $\sigma(t) = t - \sigma_1$ either oscillates or tends to zero as $t \to \infty$.*

Proof. Due to Remark 3.1, we have $b(t) \to 0$ as $t \to \infty$. So (H_7) implies that

$$\int\limits_{t_0}^{\infty} Q(t) \sum_{i=1}^{\ell} G(R(\tau_i(t)))dt = \infty. \tag{3.1}$$

We assume that $x(t)$ is a nonoscillatory solution of (1.1). Then $x(t) > 0$ or $x(t) < 0$ for $t \geq t_0 > \rho$. Let $x(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = x(t) + p(t)x(\sigma(t)) \tag{3.2}$$

$$K(t) = \int\limits_t^{\infty} \frac{s-t}{r_1(s)} \int\limits_s^{\infty} \sum_{i=1}^{\ell} (\theta - s)h_i(\theta)H(x(\rho_i(\theta)))d\theta ds \tag{3.3}$$

and

$$v(t) = z(t) - K(t) = x(t) + p(t)x(\sigma(t)) - K(t) \tag{3.4}$$

we have

$$(r_1(t)v''(t))'' = -\sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t))) \leq 0, \quad \not\equiv 0 \tag{3.5}$$

for $t \geq t_0 + \sigma_1$. Therefore, $v(t), v'(t), (r_1(t)v''(t))$, and $(r_1(t)v''(t))'$ are monotonic on $[t_1, \infty), t_1 \geq t_0 + \sigma_1$. In what follows, we have two cases, $v(t) > 0$ or < 0 for $t \geq t_1$. Assume that we have the first case. By Lemma 2.1, any of the cases (a), (b), (c), and (d) holds. Suppose that any of the cases (a), (b), and (d) holds. By using (H_3), (H_4), and (H_6), Eq. (1.1) can be represented as

$$0 = (r_1(t)v''(t))'' + \sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t))) + G(a)(r_1(\sigma(t))v''(\sigma(t)))''$$
$$+G(a)\sum_{i=1}^{\ell} q_i(\sigma(t))G(x(\tau_i(\sigma(t))))$$
$$\geq (r_1(t)v''(t))'' + G(a)(r_1(\sigma(t))v''(\sigma(t)))'' + \lambda Q(t)\sum_{i=1}^{\ell} G(x(\tau_i(t)) + ax(\tau_i(\sigma(t))))$$
$$\geq (r_1(t)v''(t))'' + G(a)(r_1(\sigma(t))v''(\sigma(t)))'' + \lambda Q(t)\sum_{i=1}^{\ell} G(z(\tau_i(t)))$$

for $t \geq t_2 > t_1$, where we have used the fact that $z(t) \leq x(t) + ax(\sigma(t))$. From (3.3), $K(t) > 0, K'(t) < 0$, and thus, $\lim_{t\to\infty} K(t)$ exists due to (H_2). Also, the inequality $v(t) > 0$ for $t \geq t_1$ implies that $v(t) > z(t)$ for $t \geq t_2$ and, thus, the last inequality yields

$$(r_1(t)v''(t))'' + G(a)(r_1(\sigma(t))v''(\sigma(t)))'' + \lambda Q(t)\sum_{i=1}^{\ell} G(v(\tau_i(t))) \leq 0,$$

for $t \geq t_2$, i.e.,

$$(r_1(t)v''(t))'' + G(a)(r_1(\sigma(t))v''(\sigma(t)))'' + \lambda G(k_1)Q(t)\sum_{i=1}^{\ell} G(R(\tau_i(t))) \leq 0$$

due to (H_4) and Lemma 2.3, for $t \geq t_3 > t_2$. Integrating this inequality from t_3 to ∞, we get

$$\lambda G(k_1)\int_{t_3}^{\infty} Q(t)\sum_{i=1}^{\ell} G(R(\tau_i(t)))dt < \infty$$

but this contradicts (3.1). Further, we suppose that the case (c) holds. By using Lemmas 2.2 and 2.3, we have

$$k(-r_1(t)v''(t))'tR(t) \leq v(t) \leq k_2 t$$

for $t \geq t_4 > t_3$. Hence,

$$-[((-r_1(t)v''(t))')^{1-\gamma}]' = (\gamma-1)((-r_1(t)v''(t))')^{-\gamma}(-r_1(t)v''(t))''$$
$$\geq (\gamma-1)L^{\gamma}R^{\gamma}(t)\sum_{i=1}^{\ell} q_i(\sigma(t))G(x(\tau_i(\sigma(t)))), \tag{3.6}$$

where $L = \frac{k}{k_2} > 0$. Therefore, the inequality

$$-[((-r_1(t)v''(t))')^{1-\gamma}]' - G(a)[((-r_1(\sigma(t))v''(\sigma(t)))')^{1-\gamma}]'$$
$$\geq (\gamma-1)L^{\gamma}\left[R^{\gamma}(t)\sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t))) + G(a)R^{\gamma}(\sigma(t))\sum_{i=1}^{\ell} q_i(\sigma(t))G(x(\tau_i(\sigma(t))))\right]$$
$$\geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)\sum_{i=1}^{\ell} G(z(\tau_i(t))) \geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)\sum_{i=1}^{\ell} G(v(\tau_i(t)))$$
$$\geq \lambda(\gamma-1)L^{\gamma}G(k_1)b(t)Q(t)\sum_{i=1}^{\ell} G(R(\tau_i(t)))$$

implies that

$$\lambda(\gamma-1)L^{\gamma}G(k_1)\int_{t_4}^{\infty} b(t)Q(t)\sum_{i=1}^{\ell} G(R(\tau_i(t)))dt < \infty,$$

which contradicts (H_7). Therefore, the latter holds. Consequently, the inequality $z(t) < K(t)$, where $K(t)$ is bounded, implies that $x(t)$ is bounded. It follows from Lemma 2.4 that any of the cases (b)-(f) is realized for $t \geq t_2 > t_1$. In the cases (e) and (f) of Lemma 2.4, we get $\lim_{t\to\infty} v(t) = -\infty$, which contradicts the facts that $x(t)$ is bounded and $\lim_{t\to\infty} v(t)$ exists. Keep in view either the case (b) or the case (c), where $-\infty < \lim_{t\to\infty} v(t) \leq 0$. Hereby,

$$0 \geq \lim_{t\to\infty} v(t) = \limsup_{t\to\infty}[z(t) - K(t)] \geq \limsup_{t\to\infty}[x(t) - K(t)]$$
$$\geq \limsup_{t\to\infty} x(t) - \lim_{t\to\infty} K(t) = \limsup_{t\to\infty} x(t)$$

implies that $\lim\limits_{t\to\infty} x(t) = 0$. We may note that $\lim\limits_{t\to\infty} K(t) = 0$. At last, let the case (d) of Lemma2.4 hold. Then $\lim\limits_{t\to\infty} (r_1(t)v''(t))'$ exists. Hence, integrating (3.5) from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} \sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t)))dt < \infty,$$

i.e.,

$$\int_{t_2}^{\infty} Q(t) \sum_{i=1}^{\ell} G(x(\tau_i(t)))dt < \infty. \tag{3.7}$$

If $\liminf\limits_{t\to\infty} x(t) > 0$, then inequality (3.7) implies that

$$\int_{t_2}^{\infty} Q(t)dt < \infty,$$

which contradicts (H_6) due to Remark 3.1. Therefore, $\liminf\limits_{t\to\infty} x(t) = 0$. Since $\lim\limits_{t\to\infty} v(t)$ exists, by using Lemma 2.5, we get

$$\lim_{t\to\infty} v(t) = 0 = \lim_{t\to\infty} z(t).$$

Even, $z(t) \geq x(t)$ implies that $\lim\limits_{t\to\infty} x(t) = 0$. If $x(t) < 0$ for $t \geq t_0$, then we set $y(t) = -x(t)$ for $t \geq t_0$ and

$$(r_1(t)(y(t) + p(t)y(\sigma(t)))'')'' + \sum_{i=1}^{\ell} q_i(t)G(y(\tau_i(t))) - \sum_{i=1}^{\ell} h_i(t)H(y(\rho_i(t))) = 0$$

Thus Theorem 3.1 is proved. □

Remark 3.3. It follows from Theorem 3.1 that $x(t)$ is bounded in the case where $v(t) < 0$ for $t \geq t_1$, which further converges to zero as $t \to \infty$. However, this fact is not required in the other case. Hence, the following theorem has been proved.

Theorem 3.2. *Let $0 \leq p(t) \leq a < \infty$. Suppose that $(H_1) - (H_7)$ hold. Then every unbounded solution of (1.1) oscillates.*

Theorem 3.3. *Let $0 \leq p(t) \leq a < 1$. If $(H_1), (H_2), (H_4), (H_5)$, and (H_8) hold, then every unbounded solution of (1.1) oscillates.*

Proof. Since $R(t) \to 0$ as $t \to \infty$. (H_8) implies that

$$\int_{t_0}^{\infty} \sum_{i=1}^{\ell} G(R(\tau_i(t)))q_i(t)dt < \infty \tag{3.8}$$

and, hence,

$$\int_{t_0}^{\infty} \sum_{i=1}^{\ell} q_i(t)dt < \infty. \tag{3.9}$$

Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t)$ is unbounded and $x(t) > 0$ for $t \geq t_0 > 0$. The case $x(t) < 0$ for $t \geq t_0 > 0$ is similar. We set $z(t)$, $K(t)$, and $v(t)$ as in (3.2), (3.3), and (3.4), respectively, to obtain (3.5) for $t \geq t_0 + \sigma_1$. Consequently, each of $v(t), v'(t), (r_1(t)v''(t))$, and $(r_1(t)v''(t))'$ is of constant sign on $[t_1, \infty), t \geq t_0 + \sigma_1$. Assume that $v(t) > 0$ for $t \geq t_1$. Then Lemma 2.1 holds. If any of the cases (a) or (b) holds, then

$$0 < v'(t) = z'(t) - K'(t)$$

implies that $z'(t) > 0$ or < 0 for $t \geq t_1$. Pay attention to $z(t)$ is unbounded because $x(t)$ is unbounded. Thus, $z'(t) < 0$ is not true. Ultimately, $z'(t) > 0$ and we obtain

$$(1 - p(t))z(t) < z(t) - p(t)z(\sigma(t)) = x(t) - p(t)p(\sigma(t))x(\sigma(\sigma(t))) < x(t).$$

This means that

$$x(t) > (1 - a)z(t) > (1 - a)v(t)$$

for $t \geq t_2 > t_1$. Hence, (3.5) yields

$$G((1 - a)v(\tau_i(t))q_i(t) \leq -(r_1(t)v''(t))'',$$

i.e.,

$$G(k_1(1 - a))G(R(\tau_i(t)))q_i(t) \leq -(r_1(t)v''(t))'' \tag{3.10}$$

due to Lemma 2.3 and (H_4). Integrating (3.10) from t_2 to ∞, we conclude that

$$\int_{t_2}^{\infty} \sum_{i=1}^{\ell} G(R(\tau_i(t)))q_i(t)dt < \infty,$$

which contradicts (3.8). For the case (c) of Lemma 2.1, we proceed as in the proof of Theorem 3.1 to obtain (3.6). By using inequality (3.6), we obtain

$$-[((-r_1(t)v''(t))')^{1-\gamma}]' \geq (\gamma - 1)L^{\gamma}G((1 - a)k_1)R^{\gamma}(t) \sum_{i=1}^{\ell} q_i(t)G(R(\tau_i(t)))$$

for $t \geq t_2$. Integrating the last inequality from t_2 to ∞, we find

$$\int_{t_2}^{\infty} R^{\gamma}(t) \sum_{i=1}^{\ell} q_i(t)G(R(\tau_i(t)))dt < \infty$$

which contradicts (H_8). In the case (d) of Lemma 2.1, $\lim_{t\to\infty} v(t)$ exists, i.e., $\lim_{t\to\infty} z(t)$ exists in contradiction with our hypothesis. Due to Remark 3.3, the case $v(t) < 0$ is not executed. Thus, Theorem 3.3 is proved. □

Theorem 3.4. *Let $-1 < a \leq p(t) \leq 0$. If $(H_1), (H_2), (H_5)$ and (H_8) hold, then every solution of Eq. (1.1) with $\sigma(t) = t - \sigma_1$ is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) > 0$ for $t \geq t_0 > 0$.Setting $z(t), K(t)$, and $v(t)$ as in (3.2), (3.3) and (3.4) we obtain (3.5) for $t \geq t_0 + \sigma_1$ and, therefore, $v(t)$ is monotone on $[t_1, \infty), t_1 \geq t_0 + \sigma_1$. Let $v(t) > 0$ for $t \geq t_1$. Assume that one of the cases (a), (b) and (d) of Lemma 2.1 holds for $t \geq t_1$. From Lemma 2.3, we conclude that $x(t) \geq v(t) \geq k_1R(t)$ for $t \geq t_2 > t_1$ and, hence, (3.5) yields

$$\int_{t_3}^{\infty} \sum_{i=1}^{\ell} q_i(t)G(R(\tau_i(t)))dt < \infty, \quad t_3 > t_2 + \sigma_1,$$

which contradicts (3.8). Now consider the case (c). Proceeding as in the proof of Theorem 3.1 we have (3.6). Further, $x(t) \geq v(t) \geq k_1R(t)$ for $t \geq t_2$ by Lemma 2.3. Consequently, for $t \geq t_3 > t_2 + \alpha$,

$$-[((-r_1(t)v''(t))')^{1-\gamma}]' \geq (\gamma - 1)L^{\gamma}G(k_1)R^{\gamma}(t) \sum_{i=1}^{\ell} q_i(t)G(R(\tau_i(t))).$$

Integrating above inequality from t_3 to ∞, we obtain

$$\int\limits_{t_3}^{\infty} R^{\gamma}(t) \sum_{i=1}^{\ell} q_i(t)G(R(\tau_i(t)))dt < \infty$$

in contradiction with (H_8).
If $v(t) < 0$ for $t \geq t_1$, then $x(t)$ is ultimately bounded. Thus, $z(t)$ is bounded and the same is true for $v(t)$. In what follows, none of the cases (e) and (f) of Lemma 2.4 is executed. In the case (b)[or (c)], we have

$$-\infty < \lim_{t\to\infty} v(t) \leq 0.$$

In view of the fact that $\lim_{t\to\infty} K(t) = 0$, we obtain

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} z(t).$$

Hence,
$0 \geq \lim_{t\to\infty} v(t) = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty}[x(t) + p(t)x(\sigma(t))] \geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(ax(\sigma(t)))$

$$= \limsup_{t\to\infty} x(t) + a \limsup_{t\to\infty} x(\sigma(t)) = (1+a) \limsup_{t\to\infty} x(t)$$

implies that $\limsup_{t\to\infty} x(t) = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$. Let the case (d) hold. Since

$$\lim_{t\to\infty} (r_1(t)v''(t))'$$

exists, (3.5) implies that

$$\int_{t_2}^{\infty} \sum_{i=1}^{\ell} q_i(t)G(x(\tau_i(t)))dt < \infty. \tag{3.11}$$

If $\liminf_{t\to\infty} x(t) > 0$, then it follows from (3.11) that

$$\int\limits_{t_2}^{\infty} \sum_{i=1}^{\ell} q_i(t)dt < \infty,$$

which contradicts (3.9). Therefore, $\liminf_{t\to\infty} x(t) = 0$. In view of Lemma 2.5, we assert that

$$\lim_{t\to\infty} v(t) = 0 = \lim_{t\to\infty} z(t).$$

Following the above proof, we can see that $\limsup_{t\to\infty} x(t) = 0$ and, hence, $\lim_{t\to\infty} x(t) = 0$.
If $x(t) < 0$ for $t \geq t_0$, then, acting as above, we obtain $\liminf_{t\to\infty} x(t) = 0$. This means that $\lim_{t\to\infty} x(t) = 0$.
Thus, Theorem 3.4 is proved. □

Theorem 3.5. *Let $-\infty < p(t) \leq 0$. If $(H_1), (H_2), (H_5)$ and (H_8) hold, then every unbounded solution of (1.1) with $\sigma(t) = t - \sigma_1$ is oscillatory.*

The proof of this theorem is quite similar the proof of Theorem 3.4. Hence, the details are omitted.

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