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Best proximity points for semi-cyclic contraction pairs in regular cone metric spaces

M. Ahmadi Baseri* and H. Mazaheri

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Abstract
The aim of this paper is to establish some conditions which guarantee the existence of best proximity for semi-cyclic contraction pairs in regular cone metric spaces. We obtain best proximity points and prove convergence results for such maps in regular cone metric spaces.

Keywords: *Best proximity point; Regular cone metric; Semi-cyclic contraction pairs; Lower bound.*

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****Corresponding author***

1. Introduction and preliminaries

Let $X := (X, d)$ be a metric space and A and B be non-empty subsets of X, $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map and S, T be two self mappings on $A \cup B$. The pair (S, T) is called a semi-cyclic $\varphi-$contraction pair if $S(A) \subseteq B,\ T(B) \subseteq A$ and

$$d(Sx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for all $x \in A\ and\ y \in B$ [12]. When $S = T$, T is called a $\varphi-$contraction map [1]. A semi-cyclic contraction pair is a semi-cyclic $\varphi-$contraction pair with $\varphi(t) = (1 - k)t,\ k \in [0, 1)$. In this case the pair (S, T) satisfies for some $k \in (0, 1)$,

$$d(Sx, Ty) \leq kd(x, y) + (1 - k)d(A, B),$$

for all $x \in A\ and\ y \in B$ [3]. When $S = T$, T is called a cyclic contraction map. In 2006, Eldered and Veeramani obtained best proximity point results for cyclic contraction maps [2]. They raised a question and in 2009, Al-Thagafi and Shahzad answered it for cyclic $\varphi-$contraction maps [1]. Also, in 2012, Karapinar proved some theorems for generalized cyclic contraction maps [7].
In 2011, Gabeleh and Abkar proved a theorem on the existence and convergence of best proximity points for a semi-cyclic contraction pair (S, T) [3]. Thakur and Sharma [12], obtained best proximity point results for semi-cyclic $\varphi-$contraction pair in 2014.

On the other hand, Huang and Zhang [6] introduced cone metric spaces as a generalization of metric spaces. In cone metric spaces the distance between two members not necessary a real positive, it can be sequence, function, matrix and any arbitrary Banach space. Hence achieved results is important and has many applications in sciences. In 2007, Rezapour [10] prove best proximity results in cone metric spaces. In 2011, Haghi et al [4] obtained best proximity points for cyclic contraction maps. In 2014, Lee [9] prove cone metric version of existence and convergence for best proximity points. Also, In 2015, Kumar and Som [8] give best proximity theorems in regular cone metric spaces. In this paper, we establish some conditions which guarantee the existence of best proximity for semi-cyclic contraction pairs in regular cone metric spaces. Then, we prove existence and convergence results for semi-cyclic contraction pair (S, T) in regular cone metric spaces.

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To prove our results in the next section we recall some definitions and facts.

Definition 1.1. [6] Let E be a real Banach space. A subset P of E is called a cone if and only if

(P1) P is closed, non-empty and $P \neq \{0\}$;
(P2) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ implies $ax + by \in P$;
(P3) $x \in P$ and $-x \in P$ implies $x = 0$.

We define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

Definition 1.2. [6] Let X be a non-empty set and E be a Banach space. Suppose that a mapping $d : X \times X \to E$ satisfies:

(d1) $0 \preceq d(x, y)$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for every $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for every $x, y, z \in X$.

Then d is called a cone metric and (X, d) is called a cone metric space.

A map $f : P \to P$ is said to be increasing (strictly increasing) whenever $x \preceq y$ implies that $f(x) \preceq f(y)$ ($x \prec y$ implies that $f(x) \prec f(y)$).

A continuous function $f : P \to P$ has a maximum point at a if $f(x) \preceq f(a)$ for all $x \in P$. Similarly, the function has a minimum point at a if $f(a) \preceq f(x)$ for all $x \in P$. The value of the function at a maximum point is called the maximum value of the function and the value of the function at a minimum point is called the minimum value of the function.

A cone P is said to be normal if there is a number $M > 0$ such that for all $x, y \in E$

$$0 \preceq x \preceq y \quad implies \quad \|x\| \leq M\|y\|.$$

The least positive number M satisfying the above inequality is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \preceq x_2 \preceq \ldots \preceq y$ for some $y \in E$, then there is $x \in E$ such that $lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. Every regular cone is normal [11].

The following example shows that the category of regular cone metric spaces is bigger that the category of metric spaces.

Example 1.1. [5] Let $E = (L^1[0, 1], \| \cdot \|_1)$, $P = \{f \in E : f \succeq 0 \quad a.e.\}$, (X, ρ) be a metric space and $d : X \times X \to E$ be defined by $d(x, y) = f_{x,y}$, where $f_{x,y}(t) = \rho(x, y)t^2$. Then (X, d) is a regular cone metric space. In fact, if $\{f_n\}_{n\geq 1}$ is an increasing sequence and there is $g \in L^1$ such that $f_1 \preceq f_2 \preceq \ldots \preceq f_n \preceq \ldots \preceq g$ for all almost x, then $\{f_n\}_{n\geq 1}$ converges to a function f for all almost x. Then, $f_n \preceq f \preceq g$ (a.e.) for all $n \geq 1$. Thus $g - f_1 \in L^1$, $g - f_n \preceq g - f_1$ for all $n \geq 1$ and $lim_{n\to\infty} g - f_n = g - f$ (a.e.). Hence by the Lebesgue dominated convergence theorem, $f \in L^1$ and $lim_{n\to\infty} \parallel f_n - f \parallel_1 = 0$. So, the cone P is regular.

Let (X, d) be a cone metric space and A be a non-empty subset of X. We say that A is bounded whenever there is $e \gg 0$ such that $d(x, y) \preceq e$ for all $x, y \in A$.

Definition 1.3. [4] Let A and B be non-empty subsets of cone metric space (X, d). An element $p \in P$ is said to be a lower bound for $A \times B$ whenever

$$p \preceq d(a, b),$$

for all $(a, b) \in A \times B$. If $p \succeq q$ for all lower bound q for $A \times B$, then p is called the greatest lower bound for $A \times B$. We denote it by $d(A, B)$.
Clearly, $d(A, B)$ is a unique vector in P.

Let $\{x_n\}$ be a sequence in a cone metric space (X, d) and $x \in X$. If for every $c \in intP$, there is a natural number N such that for every $n > N$, $c - d(x_n, x) \in intP$, then $\{x_n\}$ converges to x with respect to P and is denoted by $\lim_{n\to\infty} x_n = x$.

Lemma 1.1. *[6] Let (X, d) be a cone metric space, P be a normal cone, $\{x_n\}$ and $\{y_n\}$ be sequences in X. Then* *(i) x_n converges to x with respect to P if and only if $d(x_n, x) \to 0$ as $n \to \infty$,*

(ii) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ with respect to P, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$,

(iii) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ with respect to P and $y_n - x_n \in P$ for every $n \in \mathbb{N}$, then $y - x \in P$.

2. Main results

Throughout this section, E is a normed space, (X, d) is regular cone metric space, $\preceq$ is the partial ordering with respect of P and A, B are non-empty subsets of X.

Sequences Construction Consider $x_0 \in A$, then $Sx_0 \in B$, so there exists $y_0 \in B$ such that $y_0 = Sx_0$. Now $Ty_0 \in A$, so there exists $x_1 \in A$ such that $x_1 = Ty_0$. Inductively, we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B, respectively by

$$x_{n+1} = Ty_n, \quad y_n = Sx_n \ \ for\ n \in \mathbb{N} \cup \{0\}. \tag{2.1}$$

Theorem 2.1. *Let $S, T : A \cup B \to A \cup B$ be maps such that $S(A) \subseteq B,\ T(B) \subseteq A$ and*

$$d(Sx, Ty) \preceq (k/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} + (1 - k)d(a, b), \tag{2.2}$$

for all $(a, b), (x, y) \in A \times B$, where $k \in (0, 1)$ is a constant. Then $d(A, B)$ exists.

proof. Let $d_n = d(x_n, Sx_n)$. By inequality (2.2),

$$d_{n+1} \preceq (k/3)\{d(y_n, x_{n+1}) + d_{n+1} + d(y_n, x_{n+1})\} + (1 - k)d(a, b).$$

Since

$$d(y_n, x_{n+1}) \preceq (k/3)\{2d_n + d(y_n, x_{n+1})\} + (1 - k)d(a, b),$$

hence

$$d(y_n, x_{n+1}) \preceq \frac{(2k/3)}{(1 - (k/3))} d_n + \frac{(1 - k)}{(1 - (k/3))} d(a, b).$$

Therefore

$$\begin{aligned} d_{n+1} &\preceq (2k/3)\frac{(2k/3)}{(1 - (k/3))} d_n + (2k/3)\frac{(1 - k)}{(1 - (k/3))} d(a, b) \\ &+ (k/3)d_{n+1} + (1 - k)d(a, b). \end{aligned}$$

Then

$$d_{n+1} \preceq \frac{(4k^2/9)}{(1 - (k/3))^2} d_n + \frac{(1 - k)(1 + k/3)}{(1 - (k/3))^2} d(a, b),$$

which implies that

$$d_{n+1} \preceq \alpha d_n + (1 - \alpha)d(a, b),$$

for all $(a, b) \in A \times B$, where $\alpha = (4k^2/9)/((1 - (k/3))^2) \in (0, 1)$. It follows that $d_{n+1} \preceq d_n$. By the regularity of P, there exists $p \in P$ such that $\lim_{n\to\infty} d_n = p$. Thus $p \preceq d(a, b)$ holds for any (a, b) in $A \times B$. Now if q is a lower bound for $A \times B$, then $q \preceq d_n$ for all $n \in \mathbb{N} \cup \{0\}$. So $q \preceq p$. Therefore, $d(A, B) = p$. □

Note that, the inequality (2.2) is equivalent to

$$d(Sx, Ty) \leq (k/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} + (1 - k)d(A, B)$$

in metric spaces.

Theorem 2.2. *Suppose that the conditions of Theorem 2.1 hold, for $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2.1). If $\{x_n\}$ and $\{y_n\}$ respectively have a convergent subsequence in A and B, then there exists $x \in A$ and $y \in B$ such that*

$$d(x, Sx) = d(A, B) = d(y, Ty).$$

proof. Set $d_n = d(x_n, Sx_n)$. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. The relation

$$p = d(A,B) \preceq d(Ty_{n_k}, y) \preceq d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k})$$

holds for each $k \geq 1$. Since

$$p = d(A,B) \preceq d(y_{n_k}, Ty_{n_k}) \preceq \alpha d_{n_k} + (1-\alpha)d(a,b),$$

for all $(a,b) \in A \times B$, where $\alpha = (2k/3)/(1-(k/3)) \in (0,1)$. It follows that $p = d(A,B) \preceq d(y_{n_k}, Ty_{n_k}) \preceq d_{n_k}$. Since $\{d(Sx_{n_k}, x_{n_k})\}$ is a subsequence of $\{d_n\}$, hence $\lim_{k\to\infty} d(Sx_{n_k}, x_{n_k}) = p$. Thus $\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = p$. So $d(Ty_{n_k}, y) \to p$ as $k \to \infty$. Now, for $k \geq 1$,

$$\begin{aligned} d(Ty, y_{n_k}) &\preceq (k/3)\{d(y, x_{n_k}) + d(Sx_{n_k}, x_{n_k}) + d(Ty, y)\} + (1-k)d(a,b) \\ &\preceq (k/3)\{2d(y, y_{n_k}) + 2d(y_{n_k}, x_{n_k}) + d(Ty, y_{n_k})\} + (1-k)d(a,b). \end{aligned}$$

Thus

$$p = d(A,B) \preceq d(Ty, y_{n_k}) \preceq \alpha\{d(y, y_{n_k}) + d(y_{n_k}, x_{n_k})\} + (1-k)d(a,b), \qquad (2.3)$$

for all $(a,b) \in A \times B$, where $\alpha = ((2k)/3)/(1-(k/3)) \in (0,1)$. Therefore, by relation (2.3), $d(Ty, y) = p = d(A,B)$. Similarly, it can be proved that $d(x, Sx) = d(A,B)$. □

Example 2.1. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : \ x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ be such that $d(x,y) = (|x-y|, \lambda|x-y|)$, where $\lambda \geq 0$ is a constant. Let $A = [0,1]$, $B = [-1,0]$. So $d(A,B) = 0$. Define $S, T : A \cup B \to A \cup B$ by

$$S(x) = \begin{cases} \frac{-x}{2}, & x \in A \\ \\ \frac{x}{2}, & x \in B, \end{cases} \qquad T(x) = \begin{cases} \frac{x}{2}, & x \in A \\ \\ \frac{-x}{2}, & x \in B. \end{cases}$$

then for all $(a,b), (x,y) \in A \times B$ and $k = 7/10$,

$$\begin{aligned} &(7/30)\{d(x,y) + d(Sx,x) + d(Ty,y)\} + (3/10)d(a,b) - d(Sx,Ty) \\ &\quad = (7/30)\{(|x-y|, \lambda|x-y|) + (3|x|/2, 3\lambda|x|/2) + (3|y|/2, 3\lambda|y|/2)\} \\ &\quad + (3/10)(|a-b|, \lambda|a-b|) - (1/2)(|x-y|, \lambda|x-y|) \\ &\quad = ((-8/30)|x-y| + 3|x|/2 + 3|y|/2 + (3/10)|a-b|, \lambda((-8/30)|x-y| + 3|x|/2 \\ &\quad + 3|y|/2 + (3/10)|a-b|)) \in P. \end{aligned}$$

Hence for all $(a,b), (x,y) \in A \times B$,

$$d(Sx, Ty) \preceq (7/30)\{d(x,y) + d(Sx,x) + d(Ty,y)\} + (3/10)d(a,b).$$

So $d(A,B) = 0$. Therefore $x = 0$ and $y = 0$ are best proximity points for S and T respectively.

Theorem 2.3. *Let $S, T : A \cup B \to A \cup B$ be maps such that $S(A) \subseteq B$, $T(B) \subseteq A$ and*

$$d(Sx, Ty) \preceq k \max\{d(x,y), (1/2)\{d(Sx,x) + d(Ty,y)\}\} + (1-k)d(a,b), \quad (2.4)$$

for all $(a,b), (x,y) \in A \times B$, where $k \in (0,1)$ is a constant. Then $d(A,B)$ exists.

proof. Assume that $\max\{d(x,y), (1/2)\{d(Sx,x) + d(Ty,y)\}\} = d(x,y)$. So $(1/2)\{d(Sx,x) + d(Ty,y)\} \preceq d(x,y)$. Set $d_n = d(x_n, Sx_n)$. Since

$$\begin{aligned} d_{n+1} &\preceq kd(y_n, x_{n+1}) + (1-k)d(a,b) \\ &\preceq k^2 d_n + (1-k^2)d(a,b), \end{aligned}$$

for all (a,b) in $A \times B$. It follows that $d_{n+1} \preceq d_n$.
Assume that $\max\{d(x,y), (1/2)\{d(Sx,x) + d(Ty,y)\}\} = (1/2)\{d(Sx,x) + d(Ty,y)\}$. So $d(x,y) \preceq (1/2)\{d(Sx,x) + d(Ty,y)\}$. Thus

$$d_{n+1} \preceq (k/2)\{d_{n+1} + d(y_n, x_{n+1}\} + (1-k)d(a,b).$$

Since

$$d(y_n, x_{n+1}) \preceq (k/2)\{d_n + d(y_n, x_{n+1})\} + (1-k)d(a,b),$$

hence

$$d(y_n, x_{n+1}) \preceq \frac{(k/2)}{1-(k/2)}d_n + \frac{(1-k)}{1-(k/2)}d(a,b).$$

Therefore

$$d_{n+1} \preceq \frac{(k^2/4)}{(1-(k/2))^2}d_n + \frac{(1-k)}{(1-(k/2))^2}d(a,b),$$

which implies that

$$d_{n+1} \preceq \alpha d_n + (1-\alpha)d(a,b),$$

for all $(a,b),(x,y) \in A \times B$, where $\alpha = (k^2/4)/((1-(k/2))^2) \in (0,1)$. It follows that $d_{n+1} \preceq d_n$. Next, the proof continues similar to the proof of Theorem 2.1. □

Note that, the inequality (2.4) is equivalent to

$$d(Sx, Ty) \leq k \max\{d(x,y), (1/2)\{d(Sx,x) + d(Ty,y)\}\} + (1-k)d(A,B)$$

in metric spaces.

Theorem 2.4. *Suppose that the conditions of Theorem 2.3 hold, for $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2.1). If $\{x_n\}$ and $\{y_n\}$ respectively have a convergent subsequence in A and B, then there exists $x \in A$ and $y \in B$ such that*

$$d(x, Sx) = d(A,B) = d(y, Ty).$$

proof. Set $d_n = d(x_n, Sx_n)$. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. The relation

$$p = d(A,B) \preceq d(Ty_{n_k}, y) \preceq d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k})$$

holds for each $k \geq 1$.
Assume that $\max\{d(x,y), (1/2)\{d(Sx,x) + d(Ty,y)\}\} = d(x,y)$. So $(1/2)\{d(Sx,x) + d(Ty,y)\} \preceq d(x,y)$. Thus

$$d(y_{n_k}, Ty_{n_k}) \preceq kd_{n_k} + (1-k)d(a,b),$$

for all $(a,b) \in A \times B$. It follows that $d(y_{n_k}, Ty_{n_k}) \preceq d_{n_k}$. Since $\{d(Sx_{n_k}, x_{n_k})\}$ is a subsequence of $\{d_n\}$, hence $\lim_{k\to\infty} d(Sx_{n_k}, x_{n_k}) = p$. Thus

$$\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = p.$$

So $d(Ty_{n_k}, y) \to p$ as $k \to \infty$. Now, for each $k \geq 1$

$$\begin{aligned} d(Ty, y_{n_k}) &\preceq kd(y, x_{n_k}) + (1-k)d(a,b) \\ &\preceq k\{d(y, y_{n_k}) + d(y_{n_k}, x_{n_k})\} + (1-k)d(a,b). \end{aligned}$$

i.e.

$$p = d(A,B) \preceq d(Ty, y_{n_k}) \preceq k\{d(y, y_{n_k}) + d_{n_k}\} + (1-k)d(a,b),$$

for all $(a,b) \in A \times B$. Letting $k \to \infty$, we have $d(Ty,y) = p = d(A,B)$.
Assume that $\max\{d(x,y), (1/2)\{d(Sx,x)+d(Ty,y)\}\} = (1/2)\{d(Sx,x)+d(Ty,y)\}$. So $(1/2)\{d(Sx,x)+d(Ty,y)\} \preceq d(x,y)$. Thus

$$d(y_{n_k}, Ty_{n_k}) \preceq (k/2)\{d_{n_k} + d(y_{n_k}, Ty_{n_k})\} + (1-k)d(a,b),$$

which implies that

$$d(y_{n_k}, Ty_{n_k}) \preceq \alpha d_{n_k} + (1-\alpha)d(a,b),$$

for all $(a,b) \in A\times B$, where $\alpha = (k/2)/(1-(k/2)) \in (0,1)$. It follows that, $d(y_{n_k}, Ty_{n_k}) \preceq d_{n_k}$. Since $\lim_{k\to\infty} d_{n_k} = p$, hence $d(y_{n_k}, Ty_{n_k}) \to p$ as $k \to \infty$. So $\lim_{k\to\infty} d(Ty_{n_k}, y) = p$. Now, for each $k \geq 1$

$$\begin{aligned} d(Ty, y_{n_k}) &\preceq (k/2)\{d(y_{n_k}, x_{n_k}) + (Ty, y)\} + (1-k)d(a,b) \\ &\preceq (k/2)\{d_{n_k} + d(Ty, y_{n_k}) + d(y_{n_k}, y)\} + (1-k)d(a,b). \end{aligned}$$

So

$$(Ty, y_{n_k}) \preceq \alpha\{d_{n_k} + d(y_{n_k}, y)\} + (1-\alpha)d(a,b),$$

for all $(a,b) \in A \times B$, where $\alpha = (k/2)/(1-(k/2)) \in (0,1)$. Letting $k \to \infty$, we have $d(Ty,y) = p = d(A,B)$.
Similarly, it can be proved that $d(x, Sx) = d(A,B)$. □

Example 2.2. Suppose that the conditions of Example 2.1 hold. So for all $(a,b),(x,y) \in A \times B$ and $k = 6/10$, $\max\{d(x,y),(1/2)\{d(Sx,x)+d(Ty,y)\}\} = d(x,y)$. Thus

$$\begin{aligned}&(6/10)d(x,y) + (4/10)d(a,b) - d(Sx,Ty)\\ &\quad = ((1/10)|x-y| + (4/10)|a-b|, \lambda((1/10)|x-y| + (4/10)|a-b|)) \in P.\end{aligned}$$

Hence for all $(a,b),(x,y) \in A \times B$,

$$d(Sx,Ty) \preceq (6/10)\max\{d(x,y),(1/2)\{d(Sx,x)+d(Ty,y)\}\} + (4/10)d(a,b).$$

So $d(A,B) = 0$. Therefore $x = 0$ and $y = 0$ are best proximity points for S and T respectively.

Theorem 2.5. *Let $\varphi : P \to P$ be a strictly increasing map, $S,T : A \cup B \to A \cup B$ be maps satisfying $S(A) \subseteq B, T(B) \subseteq A$ and*

$$d(Sx,Ty) \preceq d(x,y) - \varphi(d(x,y)) + \varphi(p), \qquad (2.5)$$

for all $(x,y) \in A \times B$, where p is a lower bound for $A \times B$. Then, $d(A,B) = p$.

proof. Let $d_n = d(x_n, Sx_n)$. Then, $d_{n+1} \preceq d_n$. By the regularity of P, there exists $q \in P$ such that $\lim_{n\to\infty} d_n = q$. Since φ be a strictly increasing map and p is a lower bound for $A \times B$. Hence $\varphi(p) \preceq \varphi(d(y_n, x_{n+1}))$. So

$$\varphi(d(y_n, x_{n+1})) - \varphi(p) \in P. \qquad (2.6)$$

By inequality (2.5),

$$d(y_n, x_{n+1}) - \varphi(d(y_n, x_{n+1})) + \varphi(p) - d_{n+1} \in P.$$

From (2.5) and (2.6),

$$d(y_n, x_{n+1}) - d_{n+1} \in P.$$

So $d_{n+1} \preceq d(y_n, x_{n+1})$. Since

$$\begin{aligned} d_{n+1} &\preceq d(y_n, x_{n+1})\\ &\preceq d_n - \varphi(d_n) + \varphi(p)\end{aligned}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} \varphi(d_n) = \varphi(p)$. Since $p \preceq d_n$. Hence, $\varphi(p) \preceq \varphi(q) \preceq \varphi(d_n)$. Therefore $\varphi(p) = \varphi(q)$. It implies that $p = q$ and so $d(A,B) = p$. □

Theorem 2.6. *Suppose that the conditions of Theorem 2.5 hold, for $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2.1). If $\{x_n\}$ and $\{y_n\}$ respectively have a convergent subsequence in A and B, then there exists $x \in A$ and $y \in B$ such that*

$$d(x,Sx) = d(A,B) = d(y,Ty).$$

proof. Set $d_n = d(x_n, Sx_n)$. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. The relation

$$p = d(A,B) \preceq d(Ty_{n_k}, y) \preceq d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k})$$

holds for each $k \geq 1$. Since

$$d(y_{n_k}, Ty_{n_k}) \preceq d_{n_k}.$$

Hence $\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = p$. Thus $d(Ty_{n_k}, y) \to p$ as $k \to \infty$. Now, for each $k \geq 1$

$$\begin{aligned} d(Ty, y_{n_k}) &\preceq d(y, x_{n_k})\\ &\preceq d(y, y_{n_k}) + d(y_{n_k}, x_{n_k}).\end{aligned}$$

Letting $k \to \infty$, we have $d(Ty,y) = p = d(A,B)$.
Similarly, it can be proved that $d(x,Sx) = d(A,B)$. □

Example 2.3. Suppose that the conditions of Example 2.1 hold. Define $\varphi(t_1, t_2) = ({t_1}^2/(1 + 2t_1), {t_2}^2/(1 + 2t_2))$ for $t_1, t_2 \geq 0$. Because p is a lower bound for $A \times B$. Then, $p = (0, 0)$. Put $t = |x - y|$. So for all $(a, b), (x, y) \in A \times B$,

$$\begin{aligned} &d(x, y) - \varphi(d(x, y)) + \varphi(p) - d(Sx, Ty) \\ &\quad = (t, \lambda t) + (t^2/(1 + 2t), \lambda^2 t^2/(1 + 2\lambda t) - (1/2)(t, \lambda t) \in P. \end{aligned}$$

Hence for all $(x, y) \in A \times B$,

$$d(Sx, Ty) \preceq d(x, y) - \varphi(d(x, y)) + \varphi(p).$$

So $d(A, B) = 0$. Therefore $x = 0$ and $y = 0$ are best proximity points for S and T respectively.

Theorem 2.7. *Let $\varphi : P \to P$ be a strictly increasing map, $S, T : A \cup B \to A \cup B$ be maps satisfying $S(A) \subseteq B, T(B) \subseteq A$ and*

$$\begin{aligned} d(Sx, Ty) &\preceq (1/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} \\ &- \varphi(d(x, y) + d(Sx, x) + d(Ty, y)) + \varphi(p), \end{aligned}$$

for all $(x, y) \in A \times B$, where p is a lower bound for $A \times B$. Then, $d(A, B) = p$.

proof. For a strictly increasing mapping $\varphi : P \to P$

$$\begin{aligned} \varphi(p) &\preceq \varphi(d(x, y)) \\ &\preceq \varphi(d(x, y) + d(Sx, x) + d(Ty, y)), \end{aligned}$$

for all $(x, y) \in A \times B$, so that

$$d(Sx, Ty) \preceq (1/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\}.$$

Thus we have

$$\begin{aligned} d(x_n, Sx_n) &\preceq (1/3)\{d(x_n, y_{n-1}) + d(Sx_n, x_n) + d(Ty_{n-1}, y_{n-1})\} \\ &= (2/3)d(x_n, y_{n-1}) + (1/3)d(Sx_n, x_n). \end{aligned}$$

Since

$$\begin{aligned} d(x_n, y_{n-1}) &\preceq (1/3)\{d(y_{n-1}, x_{n-1}) + d(y_{n-1}, x_n) + d(y_{n-1}, x_{n-1})\} \\ &= (2/3)d(y_{n-1}, x_{n-1}) + (1/3)d(y_{n-1}, x_n), \end{aligned}$$

hence

$$\begin{aligned} d(x_n, y_{n-1}) &\preceq d(y_{n-1}, x_{n-1}) \\ &= d(x_{n-1}, Sx_{n-1}). \end{aligned}$$

So

$$d(x_n, Sx_n) \preceq (2/3)d(x_{n-1}, Sx_{n-1}) + (1/3)d(Sx_n, x_n).$$

Therefore

$$d(x_n, Sx_n) \preceq d(x_{n-1}, Sx_{n-1}).$$

Let $d_n = d(x_n, Sx_n)$. Then $d_{n+1} \preceq d_n$ for $n \in \mathbb{N} \cup \{0\}$. By the regularity of P, there exists $q \in P$ such that $\lim_{n\to\infty} d_n = q$. Since

$$\begin{aligned} d_{n+1} &\preceq (1/3)\{d(x_{n+1}, y_n) + d_{n+1}\} - \varphi(2d_n + d(x_{n+1}, y_n)) + \varphi(p) \\ &\preceq (2/3)d_n + (1/3)d_{n+1} - \varphi(d_n) + \varphi(p). \end{aligned}$$

Hence

$$\varphi(d_n) - \varphi(p) \preceq (2/3)\{d_n - d_{n+1}\}.$$

Therefore $\lim_{n\to\infty} \varphi(d_n) = \varphi(p)$. Since $p \preceq d_n$. Hence, $p \preceq q$ and $\varphi(p) \preceq \varphi(q) \preceq \varphi(d_n)$. Thus, $\varphi(p) = \varphi(q)$. It implies that $p = q$ and so $d(A, B) = p$. □

Theorem 2.8. *Suppose that the conditions of Theorem 2.7 hold, for $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (2.1). If $\{x_n\}$ and $\{y_n\}$ respectively have a convergent subsequence in A and B, then there exists $x \in A$ and $y \in B$ such that*

$$d(x, Sx) = d(A, B) = d(y, Ty).$$

proof. Set $d_n = d(x_n, Sx_n)$. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. The relation

$$p = d(A, B) \preceq d(Ty_{n_k}, y) \preceq d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k}),$$

holds for each $k \geq 1$. Since

$$d(y_{n_k}, Ty_{n_k}) \preceq (2/3)d_{n_k} + (1/3)d(y_{n_k}, Ty_{n_k}).$$

Hence

$$d(y_{n_k}, Ty_{n_k}) \preceq d_{n_k}.$$

It follows that $\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = p$. Thus $d(Ty_{n_k}, y) \to p$ as $k \to \infty$. Now, for $k \geq 1$

$$\begin{aligned} d(Ty, y_{n_k}) &\preceq (1/3)\{d(y, x_{n_k}) + d(Sx_{n_k}, x_{n_k}) + d(Ty, y)\} \\ &\preceq (1/3)\{2d(y, y_{n_k}) + 2d(y_{n_k}, x_{n_k}) + d(Ty, y_{n_k})\}. \end{aligned}$$

Thus

$$d(Ty, y_{n_k}) \preceq \{d(y, y_{n_k}) + d(y_{n_k}, x_{n_k})\}.$$

Therefore, $d(Ty, y) = p = d(A, B)$.
Similarly, it can be proved that $d(x, Sx) = d(A, B)$. □

Example 2.4. Suppose that the conditions of Example 2.1 hold. Define $\varphi(t_1, t_2) = ({t_1}^2/(1 + 8t_1), {t_2}^2/(1 + 8t_2))$ for $t_1, t_2 \geq 0$. Because p is a lower bound for $A \times B$. Then, $p = (0, 0)$. So for all $(x, y) \in A \times B$,

$$(1/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} - \varphi(d(x, y) + d(Sx, x) + d(Ty, y)) + \varphi(p) \in P.$$

Hence for all $(x, y) \in A \times B$,

$$\begin{aligned} d(Sx, Ty) \quad &\preceq \quad (1/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} \\ &- \quad \varphi(d(x, y) + d(Sx, x) + d(Ty, y)) + \varphi(p). \end{aligned}$$

So $d(A, B) = 0$. Therefore $x = 0$ and $y = 0$ are best proximity points for S and T respectively.

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Affiliations

M. AHMADI BASERI
ADDRESS: Department of Mathematics, Yazd University, Yazd, Iran.
E-MAIL: m.ahmadi@stu.yazd.ac.ir
ORCID ID: orcid.org/0000-0003-4997-6576

H. MAZAHERI
ADDRESS: Department of Mathematics, Yazd University, Yazd, Iran.
E-MAIL: hmazaheri@yazd.ac.ir
ORCID ID: orcid.org/0000-0003-3450-3776