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On the Some Properties of Circulant Matrices with Third Order Linear Recurrent Sequences

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Abstract
In this paper, firstly, we give the some fundamental properties of Van Der Laan numbers. After, we define the circulant matrices $C(Z)$ which entries are third order linear recurrent sequences. In addition, we compute eigenvalues, spectral norm and determinant of this matrix. Consequently, by using properties of this sequence, we obtain the eigenvalues, norms and determinants of circulant matrices with Cordonnier, Perrin and Van Der Laan numbers.

Keywords: *Third order linear recurrent sequence; Cordonnier numbers; Perrin numbers; Van Der Laan numbers; circulant matrix.*

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1. Introduction

Shannon et al., in [8], defined respectively Cordonnier sequence $\{P_n\}_{n\in\mathbb{N}}$ as

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1,\ P_1 = 1,\ P_2 = 1, \tag{1.1}$$

Perrin sequence $\{Q_n\}_{n\in\mathbb{N}}$ as

$$Q_{n+3} = Q_{n+1} + Q_n, \quad Q_0 = 3,\ Q_1 = 0,\ Q_2 = 2, \tag{1.2}$$

and Van Der Laan sequence $\{R_n\}_{n\in\mathbb{N}}$ as

$$R_{n+3} = R_{n+1} + R_n, \quad R_0 = 0,\ R_1 = 1,\ R_2 = 0. \tag{1.3}$$

The fact that the Cordonnier numbers and Perrin numbers are a linear combination of α^n, β^n and γ^n, that is,

$$P_n = \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.4}$$

$$Q_n = \alpha^n + \beta^n + \gamma^n. \tag{1.5}$$

Hence, the relations are hold

$$\alpha+\beta+\gamma = 0,\ \alpha\beta\gamma = 1,\ \alpha\beta+\beta\gamma+\alpha\gamma = -1, \tag{1.6}$$

where α, β and γ are roots of the common characteristic equation of (1.1), (1.2), (1.3).

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Elia, in [4], gave third order linear recurrent sequence $\{T_0, T_1, T_2, ...\}$ defined by the recurrence

$$T_{n+3} = pT_{n+2} + qT_{n+1} + rT_n, \quad T_0 = a,\ T_1 = b,\ T_2 = c. \tag{1.7}$$

Also, he studied Tribonacci cubic form and solved the integer representation problem for the Tribonacci cubic form.

Recently, it has been widely studied the some properties of the circulant matrices with special numbers. For instance, in [13], the authors defined the generalized k-Horadam numbers and computed the spectral norm, eigenvalues and the determinant of circulant matrix with this numbers. Solak, in [11], defined the $n \times n$ circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(mod(j-i,n))}$ and $b_{ij} \equiv L_{(mod(j-i,n))}$. Additionally, he investigated the upper and lower bounds of the matrices A and B, respectively. In [6], Ipek obtained the spectral norms of circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(mod(j-i,n))}$ and $b_{ij} \equiv L_{(mod(j-i,n))}$. Shen and Cen, in [9, 10], have found upper and lower bounds for the spectral norms of r-circulant matrices and obtained some bounds for the spectral norms of Kronecker and Hadamard products of these matrices. Also, they gave the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. In [7], it has been studied the norms, eigenvalues and determinants of some matrices related to different numbers. Yazlik [12] obtained upper and lower bounds for the spectral norm of an r-circulant matrices $H = C_r\left(H_{k,0}, H_{k,1}, H_{k,2}, ..., H_{k,n-1}\right)$ whose entries are the generalized k-Horadam numbers. Additionally, he find new formulas to calculate the eigenvalues and determinant of the matrix H. In [2], the authors defined the circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers and computed the determinants and inverses of these matrices. Bozkurt et al., in [1] introduced the skew-circulant matrices with tribonacci numbers and obtained some properties of circulant matrix with this numbers.

In the light of the above studies, in here, we present some properties of the Van Der Laan sequence as Binet formula, sum. After, we find eigenvalues, spectral norm and determinant of circulant matrix with the third order sequence. Finally, we give eigenvalues, norms and determinants of circulant matrices with Cordonnier, Perrin and Van Der Laan numbers by considering properties of circulant matrix with this third order sequence.

Now, we give some preliminaries about circulant matrix and the spectral norm of a matrix.

The circulant matrix $C = [c_{ij}] \in M_{n,n}(\mathbb{C})$ is defined by the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i \\ c_{n+j-i}, & j < i \end{cases}$$

The spectral norm of A, for a matrix $A = [a_{i,j}] \in M_{m,n}(\mathbb{C})$, is given by

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i\left(A^* A\right)},$$

where A^* is the conjugate transpose of matrix A.

Lemma 1.1. *[3] Let $A = circ(a_0, a_1, \cdots, a_{n-1})$ be a $n \times n$ circulant matrix. Then we have*

$$\lambda_j(A) = \sum_{k=0}^{n-1} a_k w^{-jk},$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \ldots n-1$.

Lemma 1.2. *[5] Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2$, where A^* is the conjugate transpose of the matrix A.*

2. Circulant Matrices with Third Order Linear Sequences

Firstly, since some of results of this paper concern about the spectral norm, eigenvalues and determinant of the circulant matrix entried by the Van Der Laan numbers, we need to introduce some properties of this sequence.

The Binet formula of Van Der Laan sequence, for every $n \in \mathbb{N}$, is obtained as

$$R_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{2.1}$$

where α, β and γ are roots of characteristic equation of (1.3).

Also, for $n \geq 1$, we have

$$\sum_{k=0}^{n} R_k = R_{n+5} - 1,$$

$$\sum_{k=0}^{n} R_k^2 = R_{n+2}^2 - R_{n-1}^2 - R_{n-3}^2 + 1.$$

If we take $p = 0, q = 1, r = 1$ in (1.7), we obtain as

$$Z_{n+3} = Z_{n+1} + Z_n, \tag{2.2}$$

where initial conditions $Z_0 = a,\ Z_1 = b,\ Z_2 = c$. Also, the Binet formula of Z_n sequence, for every $n \in \mathbb{N}$, is obtained as

$$Z_n = \frac{\left(\alpha^2 - 1\right) a + \alpha b + c}{(\alpha - \beta)(\alpha - \gamma)}\alpha^n + \frac{\left(\beta^2 - 1\right) a + \beta b + c}{(\beta - \alpha)(\beta - \gamma)}\beta^n + \frac{\left(\gamma^2 - 1\right) a + \gamma b + c}{(\gamma - \alpha)(\gamma - \beta)}\gamma^n, \tag{2.3}$$

where α, β and γ are roots of equation $x^3 - x - 1 = 0$.

Proposition 2.1. *For $n \geq 1$, the following relations are hold:*

$$\sum_{k=0}^{n} Z_k = Z_{n+5} - Z_4,$$

$$\sum_{k=0}^{n} Z_k^2 = Z_{n+2}^2 - Z_{n-1}^2 - Z_{n-3}^2 + T$$

where $T = 2a(a-c) - (b+c)^2$.

We formulate eigenvalues, spectral norms and determinants of the circulant matrices with the third order linear sequences. In order to do that, we can define *the circulant matrix* as follows.

Definition 2.1. $n \times n$ circulant matrix with third order linear sequence entries is defined by

$$C(Z) = \begin{bmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_{n-1} \\ Z_{n-1} & Z_0 & Z_1 & \cdots & Z_{n-2} \\ Z_{n-2} & Z_{n-1} & Z_0 & \cdots & Z_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_1 & Z_2 & Z_3 & \cdots & Z_0 \end{bmatrix}. \tag{2.4}$$

The following theorem gives us the eigenvalues of circulant matrix with third order linear recurrent sequence.

Theorem 2.1. *Let $C(Z) = circ(Z_0, Z_1, ..., Z_{n-1})$ be circulant matrix. Then the eigenvalues of $C(Z)$ are*

$$\lambda_j\left(C(Z)\right) = \frac{Z_n - a + (Z_{n+1} - b)\, w^{-j} + (Z_{n-1} - c + a)\, w^{-2j}}{w^{-3j} + w^{-2j} - 1},$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \ldots, n-1$.

Proof. By taking $X = (\alpha - \beta)(\alpha - \gamma)$, $Y = (\beta - \alpha)(\beta - \gamma)$, $W = (\gamma - \alpha)(\gamma - \beta)$, from Lemma 1.1, we have

$$\begin{aligned}
\lambda_j\left(C(Z)\right) &= \sum_{k=0}^{n-1} Z_k w^{-jk} \\
&= \sum_{k=0}^{n-1} \left(\begin{array}{c} \frac{\left(\alpha^2-1\right)a+\alpha b+c}{X}\alpha^k + \frac{\left(\beta^2-1\right)a+\beta b+c}{Y}\beta^k \\ +\frac{\left(\gamma^2-1\right)a+\gamma b+c}{W}\gamma^k \end{array} \right) w^{-jk} \\
&= \begin{array}{c} \frac{\left(\alpha^2-1\right)a+\alpha b+c}{X}\left(\frac{\left(\alpha w^{-j}\right)^n-1}{\alpha w^{-j}-1}\right) + \frac{\left(\beta^2-1\right)a+\beta b+c}{Y}\left(\frac{\left(\beta w^{-j}\right)^n-1}{\beta w^{-j}-1}\right) \\ +\frac{\left(\gamma^2-1\right)a+\gamma b+c}{W}\left(\frac{\left(\gamma w^{-j}\right)^n-1}{\gamma w^{-j}-1}\right) \end{array} \\
&= \begin{array}{c} \frac{(\alpha^n-1)\left[\left(\alpha^2-1\right)a+\alpha b+c\right]}{X(\alpha w^{-j}-1)} \\ +\frac{(\beta^n-1)\left[\left(\beta^2-1\right)a+\beta b+c\right]}{Y(\beta w^{-j}-1)} + \frac{(\gamma^n-1)\left[\left(\gamma^2-1\right)a+\gamma b+c\right]}{W(\gamma w^{-j}-1)} \end{array}
\end{aligned}$$

By considering (1.6) and (2.3), we obtain

$$\lambda_j\left(C(Z)\right)=\frac{Z_n-a+\left(Z_{n+1}-b\right)w^{-j}+\left(Z_{n-1}-c+a\right)w^{-2j}}{w^{-3j}+w^{-2j}-1}$$

which is desired. □

By using Theorem 2.1, we obtain eigenvalues of circulant matrices with Cordonnier, Perrin and Van Der Laan sequences as follows.

Corollary 2.1. *$i)$ Let $C(P)=circ(P_0,P_1,...,P_{n-1})$ be circulant matrix. If we take $a=b=c=1$ in Theorem 2.1, then the eigenvalues of $C(P)$ are*

$$\lambda_j\left(C(P)\right)=\frac{P_n-1+\left(P_{n+1}-1\right)w^{-j}+P_{n-1}w^{-2j}}{w^{-3j}+w^{-2j}-1},$$

where P_n is the nth Cordonnier number and $w=e^{\frac{2\pi i}{n}},\ i=\sqrt{-1},\ j=0,1,\ldots,n-1$.

$ii)$Let $C(Q)$ be circulant matrix with Perrin numbers. If we take $a=3,b=0,\ c=2$, the eigenvalues of $C(Q)$ are

$$\lambda_j\left(C(Q)\right)=\frac{Q_n-3+Q_{n+1}w^{-j}+\left(Q_{n-1}+1\right)w^{-2j}}{w^{-3j}+w^{-2j}-1},$$

where Q_n is the nth Perrin number and $w=e^{\frac{2\pi i}{n}},\ i=\sqrt{-1},\ j=0,1,\ldots,n-1$.

$iii)$Let $C(R)=circ(R_0,R_1,...,R_{n-1})$ be circulant matrix. If we take $a=c=0,\ b=1$ in Theorem 2.1, the eigenvalues of $C(R)$ are

$$\lambda_j\left(C(R)\right)=\frac{R_n+\left(R_{n+1}-1\right)w^{-j}+R_{n-1}w^{-2j}}{w^{-3j}+w^{-2j}-1},$$

where R_n is the nth Van Der Laan number and $j=0,1,\ldots,n-1,\ w=e^{\frac{2\pi i}{n}},\ i=\sqrt{-1}$.

Because matrix $C(Z)$ is normal matrix, we can write $\lambda_j\left(C(Z)C(Z)^*\right)=\left|\lambda_j\left(C(Z)\right)\right|^2$. Also, since the matrices $C(P)$, $C(Q)$ and $C(R)$ are normal matrices, we can write $\lambda_j\left(C(P)C(P)^*\right)=\left|\lambda_j\left(C(P)\right)\right|^2$, $\lambda_j\left(C(Q)C(Q)^*\right)=\left|\lambda_j\left(C(Q)\right)\right|^2$. And, similarly $\lambda_j\left(C(R)C(R)^*\right)=\left|\lambda_j\left(C(R)\right)\right|^2$. Then, we have the following theorem and corollary that deal with spectral norm.

Theorem 2.2. *Let $C(Z)$ be an $n\times n$ circulant matrix with the Z_n entries. Then, we have*

$$\left\|C(Z)\right\|_2=Z_{n+4}-Z_4.$$

Proof. From Lemma 1.2, we can write

$$\left\|C(Z)\right\|_2=\sqrt{\left(\max_{0\le j\le n-1}\left|\lambda_j\left(C(Z)\right)\right|^2\right)}.$$

In this last equality, for $j=0$, λ_0 becomes the maximum eigenvalue. Thus, $\left\|C(Z)\right\|_2=\left|\lambda_0\left(C(Z)\right)\right|$. Also, from the Theorem 2.1, we clearly obtain

$$\left\|C(Z)\right\|_2=Z_{n+4}-Z_4.$$

Hence proof is completed. □

For special cases of a,b,c in Theorem 2.2, we obtain following corollary for spectral norms of $C(P),C(Q)$ and $C(R)$.

Corollary 2.2. *$i)$ Let $C(P)$ be an $n\times n$ circulant matrix with the Cordonnier numbers entries. If we take $a=b=c=1$ in (2.2), we obtain Cordonnier sequence. And so, from Theorem 2.2, we obtain*

$$\left\|C(P)\right\|_2=P_{n+4}-2$$

where P_n is nth Cordonnier number.

$ii)$ Similarly to $i)$, if we take $a=3,b=0,c=2$ in (2.2), we obtain for $C(Q)$ matrix

$$\left\|C(Q)\right\|_2=Q_{n+4}-2.$$

where Q_n is nth Perrin number.

iii) Similarly, if we take $a=c=0, b=1$ in (2.2), we have

$$\|C(R)\|_2 = R_{n+4} - 1.$$

where R_n is nth Van Der Laan number.

Let $A=[a_{ij}]$ be an $n\times n$ matrix. Since row norm and column norm of the A matrix are $\|A\|_\infty = \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|a_{ij}|\right)$ and $\|A\|_1 = \max\limits_{1\leq j\leq n}\left(\sum\limits_{i=1}^{n}|a_{ij}|\right)$, it can be easily seen that $\|C(Z)\|_2 = \|C(Z)\|_1 = \|C(Z)\|_\infty$. Also, $\|C(P)\|_2 = \|C(P)\|_1 = \|C(P)\|_\infty$, $\|C(Q)\|_2 = \|C(Q)\|_1 = \|C(Q)\|_\infty$ and $\|C(R)\|_2 = \|C(R)\|_1 = \|C(R)\|_\infty$.

The following theorem gives us the determinant of $C(Z)$.

Theorem 2.3. *The determinant of the matrix $C(Z) = circ\,(Z_1, Z_2, \ldots, Z_n)$ is written by*

$$\det(C(Z)) = \frac{(Z_n - a)^n\,(1 - K^n - L^n + K^n L^n)}{(-1)^n\,(Q_{-n} - Q_n)},$$

where $K = \dfrac{b - Z_{n+1} - \sqrt{(Z_{n+1}-b)^2 - 4\,(Z_n - a)\,(Z_{n-1} - c + a)}}{2\,(Z_n - a)}$,

$$L = \frac{b - Z_{n+1} + \sqrt{(Z_{n+1}-b)^2 - 4\,(Z_n - a)\,(Z_{n-1} - c + a)}}{2\,(Z_n - a)}.$$

Proof. From Theorem 2.1, we have

$$\begin{aligned}\det(C(Z)) &= \prod_{j=0}^{n-1}\lambda_j\,(C(Z)) \\ &= \prod_{j=0}^{n-1}\frac{Z_n - a + (Z_{n+1} - b)\,w^{-j} + (Z_{n-1} - c + a)\,w^{-2j}}{w^{-3j} + w^{-2j} - 1}.\end{aligned}$$

By considering the equality

$$\prod_{k=0}^{n-1}\left(x - yw^{-k} + zw^{-2k}\right) = x^n\left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right),$$

we have

$$\det(C(Z)) = \frac{(Z_n - a)^n\,(1 - K^n - L^n + K^n L^n)}{(-1)^n\,(Q_{-n} - Q_n)}.$$

Hence proof is completed. □

By using in Theorem 2.3, we obtain determinants of circulant matrices with Cordonnier, Perrin and Van Der Laan sequences as follows.

Corollary 2.3. *i) Let $a=b=c=1$ in Theorem 2.3. Then, the determinant of the matrix $C(P) = circ\,(P_1, P_2, \ldots, P_n)$ is written by*

$$\det(C(P)) = \frac{(P_n - 1)^n\,(1 - K^n - L^n + K^n L^n)}{(-1)^n\,(Q_{-n} - Q_n)},$$

where $K = \dfrac{1 - P_{n+1} - \sqrt{(1 - P_{n+1})^2 - 4P_nP_{n-1} + 4P_{n-1}}}{2\,(P_n - 1)}$,

$$L = \frac{1 - P_{n+1} + \sqrt{(1 - P_{n+1})^2 - 4P_nP_{n-1} + 4P_{n-1}}}{2\,(P_n - 1)}.$$

$ii)$ If we take $a = 3, b = 0, c = 2$ in Theorem 2.3. Then, the determinant of the matrix $C(Q) = circ\,(Q_1, Q_2, \ldots, Q_n)$ is written by

$$\det(C(Q)) = \frac{(Q_n - 3)^n \left(1 - K^n - L^n + K^n L^n\right)}{(-1)^n \left(Q_{-n} - Q_n\right)},$$

where $K = \dfrac{-Q_{n+1} - \sqrt{Q_{n+1}^2 - 4Q_nQ_{n-1} - 4Q_n + 12Q_{n-1} + 12}}{2\,(Q_n - 3)}$ *and*

$L = \dfrac{-Q_{n+1} + \sqrt{Q_{n+1}^2 - 4Q_nQ_{n-1} - 4Q_n + 12Q_{n-1} + 12}}{2\,(Q_n - 3)}.$

$iii)$Let $a = c = 0, b = 1$ in Theorem 2.3. Then, the determinant of the matrix $C(R) = circ\,(R_1, R_2, \ldots, R_n)$ is written by

$$\det(C(R)) = \frac{R_n^n \left(1 - K^n - L^n + K^n L^n\right)}{(-1)^n \left(Q_{-n} - Q_n\right)},$$

where $K = \dfrac{1 - R_{n+1} - \sqrt{(1 - R_{n+1})^2 - 4R_nR_{n-1}}}{2R_n},$

$L = \dfrac{1 - R_{n+1} + \sqrt{(1 - R_{n+1})^2 - 4R_nR_{n-1}}}{2R_n}.$

References

[1] Bozkurt, D. and Dafonseca, C.M., The determinants of circulant and skew-circulant matrices with tribonacci numbers, *Mathematical Sciences and Applications E-Notes* 2 (2014), no. 2.

[2] Bozkurt, D. and Tam, T.Y., Determinants and inverses of circulant matrices with the Jacobsthal and Jacobsthal-Lucas numbers, *Applied Mathematics and Computation* 219 (2012), no.2, 544-551.

[3] Davis, P. J., Circulant Matrices, John Wiley&Sons, New York, 1979.

[4] Elia, M., Derived Sequences, the Tribonacci Recurrence and Cubic Forms, *The Fibonacci Quarterly* 37 (2001), 107-115.

[5] Horn, R. A. and Johnson, C. R., Matrix Analysis, Cambridge University Press, 1985.

[6] Ipek, A., On the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries. *Applied Mathematics and Computation* 217 (2011), no. 12, 6011-6012.

[7] Kocer, E.G., Circulant, negacyclic and semicirculant matrices with the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers, *Hacettepe J. Math., and Statistics* 36 (2007), no. 2, 133-142.

[8] Shanon, A.G., Horadam, A.F. and Anderson, P.G., Properties of Cordonnier, Perrin and Van Der Laan numbers, *International Journal of Mathematical Education in Science and Technology* 37 (2006), no. 7, 825-831.

[9] Shen, S.Q. and Cen, J.M., On the bounds for the norms of r-circulant matrices with the Fibonacci and Lucas numbers, *Applied Mathematics and Computation* 216 (2010), 2891-2897.

[10] Shen, S.Q. and Cen, J.M., On the determinants and inverses of circulant matrices with the Fibonacci and Lucas numbers, *Applied Mathematics and Computation* 217 (2011), 9790-9797.

[11] Solak, S., On the norms of circulant matrices with the Fibonacci and Lucas numbers, *Applied Mathematics and Computation* 160 (2005), 125-132.

[12] Yazlik, Y. and Taskara, N., On the norms of an r-circulant matrix with the generalized k-Horadam numbers, *Journal of Inequalities and Applications* 394 (2013), 505-512.

[13] Yazlik, Y. and Taskara, N., Spectral norm, eigenvalues and determinant of circulant matrix involving the generalized k-Horadam numbers, *Ars Combinatoria* 104 (2012), 505-512.

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