MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
6 (1) 37-42 (2018) ©MSAEN

New Aspects on Square Roots of a Real 2×2 Matrix and Their Geometric Applications

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(Communicated by Cihan ÖZGÜR)

Abstract
We present a new study on the square roots of real 2×2 matrices with a special view towards examples, some of them inspired by geometry.

Keywords: *real 2×2 matrix; square root.*

AMS Subject Classification (2010): *Primary: 15A24 ; Secondary: 15A18.*

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We begin with the following general matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_2(\mathbb{R})$ and ask: is there a matrix $B \in M_2(\mathbb{R})$ such that $B^2 = A$? Such a matrix B is called *square root* of A. We point out that the more complicated case of a real matrix of order 3 is discussed in [4]. Although the case we consider is also well studied (according to the bibliography of [3]) we add several examples and facts concerning this notion as well as a series of geometrical applications.

The Euclidean example We recall the *n-orthogonal group*: $O(n) = \{A \in M_n(\mathbb{R}) : A^t \cdot A = I_n\}$; is the *invariant group* of the Euclidean inner product $< \cdot, \cdot >$ (yielding the usual Euclidean norm $\| \cdot \|$). If $A \in O(n)$ then $(\det A^t) \cdot (\det A) = detI_n = 1$ implies that $\det A = \pm 1$. Hence, the orthogonal group splits into two components:

$$O(n) = SO(n) \sqcup O^-(n)$$

where $SO(n)$ contains the matrices from $O(n)$ having $detA = 1$ and $O^-(n)$ those with $detA = -1$; $\sqcup$ represents *the disjoint reunion* of sets. $SO(n)$ is a subgroup in $O(n)$ and is called *n-special orthogonal group*. $O^-(n)$ is not closed under product: $A_1, A_2 \in O^-(n)$ implies that $A_1 A_2 \in SO(n)$.

Since $M_1(\mathbb{R}) = \mathbb{R}$ we have that $O(1) = \{\pm 1\}$ with $SO(1) = \{1\}$ and $O^-(1) = \{-1\}$; we remark that $O(1)$ contains the integer unit roots! We know $O(2)$ as well:

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2), \quad S(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \in O^-(2), \quad t \in \mathbb{R}.$$

Hence, we have that:

$$S(t)^2 = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} = I_2$$

which means that any $S(t)$ is a root of the unit matrix I_2. We recall that from a geometrical point of view a square root of the unit matrix is called *almost product structure*, see for example [6].

Geometrical significance: $R(t)$ is the matrix *of rotation of angle t* in trigonometrical sense (i.e anticlockwise) around the origin and $S(t)$ is the matrix *of axial symmetry* with respect to $d_{t/2}$=line from plane $\mathbb{R}^2$ which contains the origin O and makes the oriented angle $t/2$ with Ox. We have that $S(t_2) \cdot S(t_1) = A(t_2 - t_1) \neq S(t_1) \cdot S(t_2)$. $\square$

Received : *09–11–2017,* ***Accepted :*** *11–01–2018*

We return to the general case of matrix A. We remind that A has *two invariants*:

$$TrA := a + d, \quad \det A := ad - bc.$$

Properties:
i) $Tr : M_2(\mathbb{R}) \to \mathbb{R}$ is a linear operator: $Tr(\alpha A_1 + \beta A_2) = \alpha TrA_1 + \beta TrA_2$,
ii) $\det : M_2(\mathbb{R}) \to \mathbb{R}$ is a multiplicative function: $\det(A_1A_2) = \det A_1 \det A_2$,
iii) *The characteristic equation of* A:

$$A^2 - TrA \cdot A + \det A \cdot I_2 = O_2.$$

The multiplicative property of the determinant yields:

The necessary condition for existence of square roots:

$$\exists B : B^2 = A \Rightarrow \det A \geq 0.$$

Hence we assume from now on that $\det A \geq 0$.

Revised Euclidean example 1: $TrS(t) = 0$, $detS(t) = -1$ which says that $S(t)$ does not admit roots. A root of order 4 of unit matrix is called *structure of electromagnetic type* according to [7, p. 3807]. □

We also have relationships between the invariants of A and B:

$$TrA = (TrB)^2 - 2\det B, \quad \det A = (\det B)^2. \tag{0}$$

Proof. It is enough to proof the first identity. We write *the characteristic equation of* B:

$$A - TrB \cdot B + \det B \cdot I_2 = O_2 \tag{1}$$

which gives:

$$A = TrB \cdot B - \det B \cdot I_2. \tag{2}$$

We square this relation:

$$A^2 = (TrB)^2 \cdot A - 2TrB \cdot \det B \cdot B + (\det B)^2 I_2$$

or:

$$A^2 - (TrB)^2 \cdot A + 2\det B[TrB \cdot B] - (\det B)^2 I_2 = O_2. \tag{3}$$

From (1) we have that:

$$TrB \cdot B = A + \det B \cdot I_2 \tag{4}$$

which is replaced in square brackets from (3):

$$A^2 - (TrB)^2 \cdot A + 2\det B[A + \det B \cdot I_2] - (\det B)^2 I_2 = A^2 - [(TrB)^2 - 2\det B] \cdot A + (\det B)^2 I_2 = O_2$$

and by comparing with the characteristic equation of A we obtain the conclusion. □

The relation (4) is fundamental for finding B and we have two cases:
Case I): $TrB = 0$ implies that: $A = -\det B \cdot I_2$.
Case II) $TrB \neq 0$ implies that:

$$B = \frac{1}{TrB}[A + \det B \cdot I_2]. \tag{5}$$

From the first relation (0) we have that:

$$(TrB)^2 = TrA + 2|\sqrt{\det A}| \tag{6}$$

hence, if $A \neq aI_2$, we obtain that:
II1) $TrA + 2\sqrt{\det A} \leq 0$ implies that A does not have roots,
II2) $TrA + 2\sqrt{\det A} > 0$ but $TrA - 2\sqrt{\det A} \leq 0$ implies that A has two roots:

$$B_{\pm} = \pm\frac{1}{\sqrt{TrA + 2\sqrt{\det A}}}[A + \sqrt{\det A} I_2] \tag{7}$$

II3) $TrA - 2\sqrt{\det A} > 0$ (which implies that $TrA + 2\sqrt{\det A} > 0$) implies that A has four roots:

$$B_{\pm}(\varepsilon) = \pm \frac{1}{\sqrt{TrA + 2\varepsilon\sqrt{\det A}}}[A + \varepsilon\sqrt{\det A}I_2], \quad \varepsilon = \pm 1. \tag{8}$$

Revised Euclidean example 2 For $A(t)$ we have:

$$TrR(t) = 2\cos t, \det R(t) = 1, TrR(t) + 2\sqrt{\det R(t)} = 4\cos^2\frac{t}{2}, TrR(t) - 2\sqrt{\det R(t)} = 2(\cos t - 1) \leq 0. \tag{9}$$

From Case II2, we get that $R(t)$ has two roots:

$$B_{\pm}(t) = \pm\frac{1}{2\cos\frac{t}{2}}\begin{pmatrix} \cos t + 1 & -\sin t \\ \sin t & \cos t + 1 \end{pmatrix} = \pm R\left(\frac{t}{2}\right). \tag{10}$$

The relation (10) can be considered the matrix version of the well-known Moivre's relation from complex algebra $(\mathbb{C}, +, \cdot)$:

$$(\cos t + i\sin t)^2 = \cos(2t) + i\sin(2t). \tag{11}$$

The group law of $SO(2)$ is: $R(t_1)\cdot R(t_2) = R(t_1+t_2) = R(t_2)\cdot R(t_1)$ which gives: $R(t)^2 = R(2t)$ and the fact that $SO(2)$ is a group isomorphic to the multiplicative group $(S^1, \cdot)$ of all unit complex numbers. $\square$

Inspired by characteristic equation of A we introduce *the characteristic polynomial of A*, namely $p_A \in \mathbb{R}[X]$:

$$p_A(X) = X^2 - TrA \cdot X + \det A. \tag{12}$$

We know that the possible real roots of p_A are called *eigenvalues of A* and are useful in studying the diagonalisation of A. So, if the eigenvalues exist and are *different*, we denote them by $\lambda_1 < \lambda_2$ and it follows that A admits *a diagonal form*:

$$A = S^{-1}\,\mathrm{diag}(\lambda_1, \lambda_2)S \tag{13}$$

with $S \in GL(2,\mathbb{R})$=2-*general linear group* i.e. the group of all real invertible matrices of order 2. Obviously, the condition of existence and inequality for $\lambda_{1,2}$ holds when *the discriminant* $\Delta(p_A)$ is strictly positive:

$$\Delta(p_A) := (TrA)^2 - 4\det A. \tag{14}$$

The relationship between $\Delta(p_A)$ and $\Delta(p_B)$ is given by:

Proposition *Let B be a square root of A. Then:*

$$\Delta(p_A) = (TrB)^2\Delta(p_B). \tag{15}$$

Thus, if $TrB \neq 0$ then A has different eigenvalues if and only if B has different eigenvalues.

Proof. The relation (15) is a direct consequence of (0). $\square$

Corollary *Suppose the matrix A with* $\det A > 0$ *has the root B with $TrB \neq 0$. Assume that A is diagonalisable with $S \in GL(2,\mathbb{R})$ and different eigenvalues $\lambda_1 < \lambda_2$. Then, $0 < \lambda_1 < \lambda_2$ and B is diagonalisable with the same matrix S having the eigenvalues $\{\sqrt{\lambda_1}, \sqrt{\lambda_2}\}$ or $\{-\sqrt{\lambda_1}, -\sqrt{\lambda_2}\}$ or $\{-\sqrt{\lambda_1}, \sqrt{\lambda_2}\}$ or $\{\sqrt{\lambda_1}, -\sqrt{\lambda_2}\}$. Equivalently, we are in case II3 with:*

$$B_{\pm}(\varepsilon) = \pm\frac{1}{\sqrt{\lambda_2} + \varepsilon\sqrt{\lambda_1}}[A + \varepsilon\sqrt{\lambda_1\lambda_2}I_2] = S \cdot \mathrm{diag}(\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}) \cdot S^{-1}. \tag{16}$$

Proof. Because $\det A > 0$ we have that λ_1 and λ_2 have the same sign. We suppose that $\lambda_1 < \lambda_2 < 0$. From (6) we have that $(TrB)^2 = \lambda_1 + \lambda_2 \pm 2\sqrt{\lambda_1\lambda_2} > 0$. It follows only the case with $+$ i.e. $-|\lambda_1| - |\lambda_2| + 2\sqrt{|\lambda_1||\lambda_2|} > 0$ but this is impossible because of the AM-GM inequality. Recall that the AM–GM inequality states that the arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list. $\square$

The golden example It is known that *the golden proportion* (or *the golden number*) is the positive root, $\phi = \frac{\sqrt{5}+1}{2}$, of the equation [6]:

$$x^2 - x - 1 = 0. \tag{17}$$

The negative root is $-\phi^{-1} = \frac{1-\sqrt{5}}{2}$. Let us consider the matrix:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \quad TrA = 6, \det A = 5. \tag{18}$$

A is diagonalisable, being symmetric, with $0 < \lambda_1 = 1 < \lambda_2 = 5$. We have that:

$$TrA + 2\varepsilon\sqrt{\det A} = 6 + 2\varepsilon\sqrt{5} = (\sqrt{5} + \varepsilon)^2 \tag{19}$$

We are in Case II3 and for example:

$$B_{\pm}(1) = \pm\frac{1}{\sqrt{5}+1}\begin{pmatrix} 3+\sqrt{5} & 2 \\ 2 & 3+\sqrt{5} \end{pmatrix} = \pm\frac{1}{2\phi}\begin{pmatrix} 2\phi^2 & 2 \\ 2 & 2\phi^2 \end{pmatrix} = \pm\begin{pmatrix} \phi & \phi^{-1} \\ \phi^{-1} & \phi \end{pmatrix}. \tag{20}$$

By analogy with the problem studied here, we call the matrix $A \in M_n(\mathbb{R})$ satisfying $A^2 - A - I_n = O_n$, as being *an almost golden structure*. In [6] we study the relationship between almost golden structures and almost product structures. □

We return to the case I given by $A = aI_2$ and we present the solution from [3, p. 491]. We have, irrespective of a's sign, an infinity of roots:

$$B_{\pm}(c,s) := \pm\begin{pmatrix} c & s \\ \frac{a-c^2}{s} & -c \end{pmatrix}, \quad c \in \mathbb{R}, s \in \mathbb{R}^*. \tag{21}$$

If $a = 0$ then we add the infinite set of *almost tangent structures*:

$$B_{\pm}(u) := \pm\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, \quad u \in \mathbb{R}. \tag{22}$$

If $a > 0$ then we add the infinite set:

$$B_{\pm}(u) := \begin{pmatrix} \pm\sqrt{a} & 0 \\ u & \mp\sqrt{a} \end{pmatrix}, \quad B_{\pm} := \begin{pmatrix} \pm\sqrt{a} & 0 \\ 0 & \pm\sqrt{a} \end{pmatrix}. \tag{23}$$

Revised Euclidean example 3 For $a = 1$ the family $B_+(c,s)$ becomes:

$$B(c,s) = \begin{pmatrix} c & s \\ \frac{1-c^2}{s} & -c \end{pmatrix} \tag{24}$$

which gives:

$$B(\cos t, \sin t) = S(t). \tag{25}$$

This way we obtain the matrices from $O^-(2)$. We consider now a right triangle with sides x, y and hypotenuse z. We have that:

$$S(t) = \frac{1}{z}\begin{pmatrix} x & y \\ y & -x \end{pmatrix}. \tag{26}$$

If $(x,y,z) \in (\mathbb{N}^*)^3$ then (x,y,z) is a *Pythagorean triple*. This example of almost product structures provided by Pythagorean triples appears on the Web page [1]. In [5] we gave a method for finding matrices $A \in M_3(\mathbb{R})$ which transforms a Pythagorean triple into another Pythagorean triple.

Open problem Do the matrices A which preserve Pythagorean triples admit roots? □

We return now to the given corollary: a symmetric matrix A with different and strictly positive eigenvalues is *positive definite*, [2]. Thus, A defines a new inner product on $\mathbb{R}^n$:

$$< \bar{x}, \bar{y} >_A := < \bar{x}, A\bar{y} > . \tag{27}$$

If A admits B as a root then:

$$< \bar{x}, \bar{y} >_A := < \bar{x}, B^2 \bar{y} > = < B^t \bar{x}, B \bar{y} > . \tag{28}$$

If B is also symmetric, which happens when $n = 2$, then:

$$< \bar{x}, \bar{y} >_A := < B\bar{x}, B\bar{y} > \tag{29}$$

hence:

$$\|\bar{x}\|_A^2 = \|B\bar{x}\|^2. \tag{30}$$

Thus, for nonzero vectors $\bar{x}, \bar{y} \in \mathbb{R}^n$, the angle $\varphi_A(\bar{x}, \bar{y})$ between them with respect to $< \cdot, \cdot >_A$ is given by:

$$\cos \varphi_A(\bar{x}, \bar{y}) = \cos \varphi(B\bar{x}, B\bar{y}). \tag{31}$$

Generalized golden example The matrix $A \in M_2(\mathbb{R})$ is called *bi-symmetric* if it has the form:

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{32}$$

Then $\det A = a^2 - b^2$ and, for having roots, we assume that $a > b$. We are in case II3 and obtain that:

$$B_{\pm}(\varepsilon) = \pm \frac{1}{2} \begin{pmatrix} \sqrt{a+b} + \varepsilon\sqrt{a-b} & \sqrt{a+b} - \varepsilon\sqrt{a-b} \\ \sqrt{a+b} - \varepsilon\sqrt{a-b} & \sqrt{a+b} + \varepsilon\sqrt{a-b} \end{pmatrix} \tag{33}$$

which gives us the result that any of its root is also bi-symmetric. The conversely: *If B is bi-symmetric then B^2 is bi-symmetric* is obvious from calculus. □

Hyperbolic example We consider:

$$A(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \tag{34}$$

A is a bi-symmetric matrix with $a > b$ and with formula (33) we obtain that:

$$B_{\pm}(1) = \pm \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}, \quad B_{\pm}(-1) = \pm \begin{pmatrix} \sinh \frac{t}{2} & \cosh \frac{t}{2} \\ \cosh \frac{t}{2} & \sinh \frac{t}{2} \end{pmatrix}. \tag{35}$$

□

Fibonacci example In [8, p. 24] is introduced the *Q-Fibonacci matrix*:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{36}$$

that has the natural powers:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{37}$$

Because of the golden example, we consider the matrix:

$$Q(n) = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n+1} \end{pmatrix}. \tag{38}$$

With relation (33) we have the roots:

$$Q_{\pm}(n, \varepsilon) = \pm \frac{1}{2} \begin{pmatrix} \sqrt{F_{n+2}} + \varepsilon\sqrt{F_{n-2}} & \sqrt{F_{n+2}} - \varepsilon\sqrt{F_{n-2}} \\ \sqrt{F_{n+2}} - \varepsilon\sqrt{F_{n-2}} & \sqrt{F_{n+2}} + \varepsilon\sqrt{F_{n-2}} \end{pmatrix}. \tag{39}$$

□

Almost complex example A root of the matrix $-I_2$ is called *almost complex structure*. According to (21) we have:

$$B_{\pm}(s, c) := \pm \begin{pmatrix} s & c \\ \frac{-1-s^2}{c} & -s \end{pmatrix}, \quad s \in \mathbb{R}, c \in \mathbb{R}^*. \tag{40}$$

An interesting particular case is:

$$B_{\pm}(\sinh t, \cosh t) := B(t) = \pm \begin{pmatrix} \sinh t & \cosh t \\ -\cosh t & -\sinh t \end{pmatrix}. \tag{41}$$

□

Acknowledgements The authors are extremely indebted to two anonymous referees for their extremely useful remarks and improvements.

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