MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.1658156

13 (2) 92-105 (2025) - Research Article

ISSN: 2147-6268

Serret-Frenet Formula of Quaternionic Framed Curves

Zülal Derin Yaqub* and Mehmet Ali Güngör

Abstract

In this study, we present an approach by introducing the quaternionic structure of framed curves. Furthermore, we derive Serret-Frenet formulas and give specific results for quaternionic framed curves. Initially, we focus on the moving frame and its curvatures corresponding to the frame T, N, B along the quaternionic framed base curve in three-dimensional Euclidean space $\mathbb{R}^3$. Then, we establish the Serret-Frenet type formulas of quaternionic framed curves. We then generalize these formulas and the definition of quaternionic framed curves to four-dimensional Euclidean space, highlighting the relationship between the curvatures in both 3-dimensional and four-dimensional Euclidean spaces. In addition, the theorems are supported by examples, demonstrating the applicability of the proposed results.

Keywords: *Framed curves, Serret-Frenet formulas, Singular point, Quaternion*

AMS Subject Classification (2020): *53A04; 58K05; 53A40; 11R52*

1. Introduction

The theory of curves is one of the most fundamental subjects of differential geometry and has been studied for many years. Different frame constructions for curves have also been investigated, as first introduced in [1]. Furthermore, associated curves and their geometric properties have been examined in the Frenet frame context, offering valuable insights into curve behavior [2]. In the theory of curves, the Serret-Frenet formulas of a curve are given by defining the constant quantities of curvatures and torsions of the curve. The harmonic curvatures of the Frenet curve have been researched by K. Arslan and H.H. Hacısalihoğlu [3]. The generalized helix concept, which is an important curve making a constant angle with a constant vector with a fixed direction, has been studied in [4]. Various types of curves, such as rectifying, osculating, and normal curves, have been studied with the help of the frameworks of the curves, known as Serret-Frenet elements, where their changes can be observed instantly. A study on these curves can be found in [5]. The important point here is that by defining Frenet elements with the help of regular curves, many properties of the curve, such as the geometric structure of the relevant space, are investigated. However, if space curves contain singular points, constructing the Frenet frame becomes impossible to examine the structure of these curves. As it is known, the fact that the tangent vectors are zero at singular points creates some difficulties. Because the principal normal and binormal vectors cannot be normalized by known

Received : *14-03-2025,* ***Accepted :*** *14-06-2025,* ***Available online :*** *20-06-2025*

*(**Cite as** "Z. Derin Yaqub, M. A. Güngör, Serret-Frenet Formula of Quaternionic Framed Curves, Math. Sci. Appl. E-Notes, 13(2) (2025), 92-105")*

methods. However, thanks to the recently defined generalized tangent, principal normal vector, special curves with singular points can also be studied at non-singular points [6]. Framed curves, defined as space curves equipped with a moving frame that may include singular points, have been described to be both a generalization of Legendre curves and a generalization of regular curves with linearly independent conditions, and also Serret-Frenet type formulas of framed curves have been given by Honda and Takahashi [7]. In addition, spinors in terms of framed Manheim curves and framed Bertrand curves are discussed in [8, 9]. Framed curves have been studied in recent years, both as special curve couples in Lie groups [10] and as surfaces formed in Euclidean space [11], making important contributions to this field. Apart from these framed curves have been studied in the Euclidean space [12, 13]. Some geometric properties of framed curves in all four dimensions have been investigated [14, 15].

During the study of curves, quaternionic curves have been studied. The properties of smooth quaternionic curves in spatial and non-spatial quaternion spaces were studied by Baharatti and Nagaraj [16]. In addition, Serret-Frenet formulas are also given in this study. The Serret-Frenet formulas are given by Tuna for quaternionic curves in semi-Euclidean space [17]. In recent years, studies on quaternionic curves in differential geometry [18, 19] have come to the fore. For basic notions of differential geometry, we refer to [20]. In order to strengthen the theoretical background and provide a broader context, we have incorporated relevant studies on quaternionic frames and curves [21–25].

In this study, we derive the well-known Serret-Frenet formulas of differential geometry in terms of framed curves using quaternion algebra. We begin by presenting the fundamental definitions and algebraic properties of quaternions, along with the concept of framed curves in three-dimensional Euclidean space. Next, we introduce a notion of quaternionic framed curves in three-dimensional Euclidean space. We then rigorously prove the Serret-Frenet-type formulas for quaternionic framed curves and provide their matrix representations. Following this, we extend these formulas and the definition of quaternionic framed curves to four-dimensional Euclidean space, establishing a relationship between the curvatures in both 3-dimensional and 4-dimensional Euclidean spaces. This innovative framework, encompassing the definitions of quaternionic framed curves and their associated theorems offers a fresh perspective on the geometry of curves. Classical methods in elementary differential geometry do not provide a direct approach for relating a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ to its corresponding a curve $\tilde{\boldsymbol{\gamma}}$ in $\mathbb{R}^4$. However, through the use of quaternions, we are able to achieve this connection. The study is further substantiated by examples in both three-dimensional and four-dimensional spaces, demonstrating the practical application of the developed theory.

2. Preliminary

In this section, some definitions and theorems that are useful in our study are given. The set of real quaternions is defined as:

$$H = \left\{ q : q = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,\ a_0, a_1, a_2, a_3 \in \mathbb{R},\ \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^4 \right\}.$$

Here, $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are quaternion base elements and satisfy the following multiplication rules:

$$\begin{aligned} &{\mathbf{e}_0}^2 = 1,\ {\mathbf{e}_1}^2 = {\mathbf{e}_2}^2 = {\mathbf{e}_3}^2 = \mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_3 = -1, \\ &\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3,\ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1,\ \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \\ &\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3,\ \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1,\ \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2. \end{aligned}$$

Here, the symbol $\times$ denotes the quaternionic product. A quaternion q with the scalar part is denoted by S_q and the vector part is denoted by $\boldsymbol{V}_q$. Here $S_q = a_0$ and $\boldsymbol{V}_q = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ are written as $q = S_q + \boldsymbol{V}_q$. Then the quaternionic products of two quaternions q and p are given by:

$$q \times p = S_q S_p - \langle \boldsymbol{V}_q, \boldsymbol{V}_p \rangle + S_q \boldsymbol{V}_p + S_p \boldsymbol{V}_q + \boldsymbol{V}_q \wedge \boldsymbol{V}_p.$$

Here the symbols $\langle , \rangle$ and $\wedge$ denote the Euclidean dot product and the vector product in $\mathbb{R}^3$, respectively. The conjugate of a quaternion $q \in H$ is defined as $\alpha q = S_q - \boldsymbol{V}_q$. Accordingly, the product of the quaternion and its conjugate is written as follows:

$$q \times \alpha q = (a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times (a_0\mathbf{e}_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3) = {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2.$$

Also, the properties related the conjugate of the quaternion q and p for $a, b \in \mathbb{R}$ are given by:

$$\begin{aligned} \alpha(aq + bp) &= a\alpha(q) + b\alpha(p), \\ \alpha(q \times p) &= \alpha(p) \times \alpha(q), \\ \alpha(\alpha q) &= q. \end{aligned}$$

The inner product for real quaternions $p, q \in H$ is defined as:

$$h : H \times H \to \mathbb{R},$$
$$(p, q) \to h(p, q) = \frac{1}{2}(p \times \alpha q + q \times \alpha p)$$

where h is real-valued, symmetric, and bilinear. The norm of the quaternion $q \in H$ defined by:

$$\| \| : H \to \mathbb{R},$$
$$q \to \|q\|,$$
$$\|q\|^2 = h(q, q) = {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2.$$

If the quaternion $q \in H$ satisfies the equality

$$q + \alpha q = 0, \tag{2.1}$$

then it is called a spatial quaternion. If for the quaternion $q \in H$

$$q - \alpha q = 0$$

then it is called a temporal quaternion. In general, the quaternion $q \in H$ can be written as

$$q = \frac{1}{2}(q + \alpha q + q - \alpha q) = \frac{1}{2}(q + \alpha q) + \frac{1}{2}(q - \alpha q).$$

On the first hand, Honda and Takahashi stated that a framed curve is a smooth space curve with a moving frame with singular points [7]. In order to construct a framed curve, a structure known as a "Stiefel Manifold" in the literature is defined as follows:

$$\begin{aligned}\Delta_{n-1} &= \{\boldsymbol{v} = (v_1, v_2, ..., v_{n-1}) \in \mathbb{R}^n \times ... \times \mathbb{R}^n : \langle v_i, v_j\rangle = \delta_{ij},\ i, j = 1, ..., n-1\} \\ &= \left\{\boldsymbol{v} = (v_1, v_2, ..., v_{n-1}) \in S^{n-1} \times ... \times S^{n-1} : \langle v_i, v_j\rangle = 0,\ i \neq j,\ i, j = 1, ..., n-1\right\}.\end{aligned}$$

This manifold is $\frac{n(n-1)}{2}$-dimensional and smooth. If taken as $\boldsymbol{v} = (v_1, v_2, ..., v_{n-1}) \in \Delta_{n-1}$, the unit vector $\boldsymbol{\mu} \in \mathbb{R}^n$ can be defined as $\boldsymbol{\mu} = v_1 \wedge v_2 ... \wedge v_{n-1}$. It can be seen from here that $(\boldsymbol{v}, \boldsymbol{\mu}) \in \Delta_{n-1}$ and $\det(\boldsymbol{v}, \boldsymbol{\mu}) = 1$.

Definition 2.1. $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ is called a framed curve if $h(\dot{\boldsymbol{\gamma}}(t), \nu_i(t)) = 0$ for all $t \in I$ and $i = 1, 2, ...n-1$.$\boldsymbol{\gamma} : I \to \mathbb{R}^n$ is said that a framed base curve if there exists $\boldsymbol{\nu} : I \to \Delta_{n-1}$ such that $(\boldsymbol{\gamma}, v_1, v_2, ..., v_{n-1})$ is a framed curve [7].

Along the framed base curve $\gamma(t)$, the moving frame $\{\boldsymbol{\mu}(t), \boldsymbol{v}(t)\}$ can be defined and the Serret-Frenet type formulas are given as

$$\begin{bmatrix} \dot{\boldsymbol{v}}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{bmatrix} = A(t) \begin{bmatrix} \boldsymbol{v}(t) \\ \boldsymbol{\mu}(t) \end{bmatrix}.$$

Here $i, j = 1, 2, ..., n$ is for each $A(t) = (\sigma_{ij}(t)) \in o(n)$ and $o(n)$ is the set of anti-symmetric matrices. Also the smooth function $\sigma : I \to \mathbb{R}$ given as

$$\dot{\boldsymbol{\gamma}}(t) = \sigma(t)\, \boldsymbol{\mu}(t).$$

In addition, the necessary and sufficient condition for the point t to be a singular point of the curve γ is that it is $\sigma(t) = 0$. Here the functions $(\sigma_{ij}(t), \sigma(t))$ are called the curvatures of the framed curve. Now we will give the definition of framed curves, especially in the case of $n = 3$. The definition of framed curves in $\mathbb{R}^3 \times \Delta_2$ is given as follows:

Let be $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ a framed curve and $\boldsymbol{\mu} = v_1 \wedge v_2$. In this case, the following conditions are satisfied

$$\langle \dot{\boldsymbol{\gamma}}(t), v_i(t)\rangle = 0,\ \forall t \in I,\ i = 1, 2.$$

Here it is expressed as

$$\Delta_2 = \left\{\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle v_1, v_2\rangle = 0, \langle v_1, v_1\rangle = 1, \langle v_2, v_2\rangle = 1\right\}.$$

The Serret-Frenet type formulas are given by

$$\{\boldsymbol{\mu}(t), \boldsymbol{v}(t)\} \in \Delta_2\ ,\ \begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{\boldsymbol{\mu}}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & l(t) & m(t) \\ -l(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \boldsymbol{\mu}(t) \end{bmatrix}$$

where the curvatures are $l(t) = \langle \dot{v}_1(t), v_2(t)\rangle$, $m(t) = \langle \dot{v}_1(t), \boldsymbol{\mu}(t)\rangle$,
$n(t) = \langle \dot{v}_2(t), \boldsymbol{\mu}(t)\rangle$ and $\sigma(t) = \langle \dot{\gamma}(t), \boldsymbol{\mu}(t)\rangle$ [7].

3. Serret-Frenet formulas for quaternionic framed curves

In this section, the definition of a spatial quaternionic framed curve in 3-dimensional Euclidean space $\mathbb{R}^3$ is given. Then, the Serret-Frenet type formula is obtained for these quaternionic framed curves. Also, the relationship between their curvatures is presented. Serret-Frenet type formula for spatial quaternionic framed curve and framed curvatures is supported with an example and illustrated.

Definition 3.1. We define $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to H \times \Delta_2$ is a quaternionic framed curve in three dimensional Euclidean space $\mathbb{R}^3$ if $h\left(\dot{\boldsymbol{\gamma}}(t), \nu_i(t)\right) = 0$ for all $t \in I$ and $i = 1, 2$. In other words, we say that $\boldsymbol{\gamma} : I \to H$ is a quaternionic framed base curve in $\mathbb{R}^3$ if there exists $\boldsymbol{\nu} : I \to \Delta_2$ such that $(\boldsymbol{\gamma}, v_1, v_2)$ is a quaternionic framed curve.

Theorem 3.1. *Let $(\boldsymbol{\gamma}, v_1, v_2) : I \to H \times \Delta_2$ be a quaternionic framed curve in three dimensional Euclidean space $\mathbb{R}^3$ and $\boldsymbol{\mu} = v_1 \wedge v_2$ with the space of spatial quaternions $\{\gamma \in H : \gamma + \alpha(\gamma) = 0\}$ which the quaternionic framed base curve $\boldsymbol{\gamma} : I \subset \mathbb{R} \to H$ defining by $t \to \boldsymbol{\gamma}(t) = \sum_{i=1}^{3} \boldsymbol{\gamma}_i(t)\, e_i$ for all $t \in I = [0, 1] \subset \mathbb{R}$. Then, Serret-Frenet type formula of the quaternionic framed curve $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ at point $\boldsymbol{\gamma}(t)$ is given by*

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{bmatrix} = \begin{bmatrix} 0 & l(t) & m(t) \\ -l(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \boldsymbol{\mu}(t) \end{bmatrix}$$

where the curvatures of the quaternionic framed curve are $l(t) = h\left(\dot{v}_1(t), v_2(t)\right)$, $m(t) = h\left(\dot{v}_1(t), \boldsymbol{\mu}(t)\right)$, $n(t) = h\left(\dot{v}_2(t), \boldsymbol{\mu}(t)\right)$ and $\sigma(t) = h\left(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t)\right)$.

Proof. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be the quaternionic framed curve and $\boldsymbol{\mu}(t) = v_1(t) \wedge v_2(t)$ for all $t \in I = [0, 1] \subset \mathbb{R}$. Since

$$\|\boldsymbol{\mu}(t)\|^2 = h\left(\boldsymbol{\mu}(t), \boldsymbol{\mu}(t)\right) = \frac{1}{2}\left(\boldsymbol{\mu}(t) \times \alpha\left(\boldsymbol{\mu}(t)\right) + \boldsymbol{\mu}(t) \times \alpha\left(\boldsymbol{\mu}(t)\right)\right) = \boldsymbol{\mu}(t) \times \alpha\left(\boldsymbol{\mu}(t)\right) = 1.$$

$\boldsymbol{\mu}(t)$ has unit length. Taking the derivative of both sides of the above equation is obtained

$$\dot{\boldsymbol{\mu}}(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \times (\alpha\dot{\boldsymbol{\mu}})(t) = 0. \tag{3.1}$$

We get from the equation (3.1) as follows:

$$h\left(\dot{\boldsymbol{\mu}}(t), \boldsymbol{\mu}(t)\right) = \frac{1}{2}\left(\dot{\boldsymbol{\mu}}(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \otimes \alpha\dot{\boldsymbol{\mu}}(t)\right) = 0. \tag{3.2}$$

In this case it is $\dot{\boldsymbol{\mu}} \perp \boldsymbol{\mu}$. This shows the meaning of quaternionic orthogonality. If the equation (3.1) is arranged according to the conjugate rules, then we have $(\dot{\boldsymbol{\mu}} \times \alpha\boldsymbol{\mu}) + \alpha\left(\boldsymbol{\mu} \times \alpha\dot{\boldsymbol{\mu}}\right) = 0$. Hence, $\dot{\boldsymbol{\mu}} \times \alpha\boldsymbol{\mu}$ is a spatial quaternion from equation (2.1). Since $\boldsymbol{\gamma}$ is a framed curve, there exists a smooth function $\sigma : I \to \mathbb{R}$ such that $\dot{\boldsymbol{\gamma}}(t) = \sigma(t)\,\boldsymbol{\mu}(t)$ and unit vector $\boldsymbol{\mu}$. Accordingly $\dot{\boldsymbol{\mu}}(t) \in Sp\left\{\boldsymbol{\mu}(t), v_1(t), v_2(t)\right\}$ and since $\dot{\boldsymbol{\mu}}$ is a spatial quaternion it can be written as follows:

$$\dot{\boldsymbol{\mu}}(t) = a_{31}(t)v_1(t) + a_{32}(t)v_2(t) + a_{33}(t)\boldsymbol{\mu}(t), \tag{3.3}$$

where $a_{31}, a_{32}, a_{33} : I \to \mathbb{R}$ are smooth functions. Right now, we compute quaternionic inner product that is $h-$ inner product of the vectors $v_1, v_2, \boldsymbol{\mu}$ respectively, as follows:

$$h\left(v_1(t), \boldsymbol{\mu}(t)\right) = \frac{1}{2}\left(v_1(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \times \alpha v_1(t)\right) = \frac{1}{2}\left(v_2(t) + \alpha v_2(t)\right) = 0,$$

$$h\left(v_2(t), \boldsymbol{\mu}(t)\right) = \frac{1}{2}\left(v_2(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \times \alpha v_2(t)\right) = \frac{1}{2}\left(\alpha v_1(t) + v_1(t)\right) = 0.$$

It is seen that $\boldsymbol{\mu}$,v_1 and v_2 are orthogonal quaternions. Also v_1 and v_2 are the unit spatial quaternions. It is $a_{33}(t) = 0$ from the equation (3.2) and the last two equations. Thus the equation (3.3) becomes as follows:

$$\dot{\boldsymbol{\mu}}(t) = a_{31}(t)v_1(t) + a_{32}(t)v_2(t). \tag{3.4}$$

On the other hand, since it is $\dot{v}_1(t) \in Sp\left\{v_1(t), v_2(t), \boldsymbol{\mu}(t)\right\}$ a combination of the vectors, v_1, v_2 and $\boldsymbol{\mu}$ can be written as

$$\dot{v}_1(t) = a_{11}(t)\, v_1(t) + a_{12}(t)\, v_2(t) + a_{13}(t)\, \mu(t). \tag{3.5}$$

From here, by $h-$ inner producting with the vectors v_1, v_2 and $\boldsymbol{\mu}$ respectively, for the vector $\dot{v}_1$ we have

$$h\left(\dot{v}_1\left(t\right), v_1\left(t\right)\right) = a_{11}(t) h\left(v_1\left(t\right), v_1\left(t\right)\right) + a_{12}(t) h\left(v_2\left(t\right), v_1\left(t\right)\right) + a_{13}(t) h\left(\boldsymbol{\mu}\left(t\right), v_1\left(t\right)\right), \tag{3.6}$$

$$h\left(\dot{v}_1\left(t\right), v_2\left(t\right)\right) = a_{11}(t) h\left(v_1\left(t\right), v_2\left(t\right)\right) + a_{12}(t) h\left(v_2\left(t\right), v_2\left(t\right)\right) + a_{13}(t) h\left(\boldsymbol{\mu}\left(t\right), v_2\left(t\right)\right), \tag{3.7}$$

$$h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = a_{11}(t) h\left(v_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) + a_{12}(t) h\left(v_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) + a_{13}(t) h\left(\boldsymbol{\mu}\left(t\right), \boldsymbol{\mu}\left(t\right)\right).$$

In addition, we obtain by the simple computing with inner products in the equation (3.6) as follows:

$$h\left(v_1\left(t\right), v_1\left(t\right)\right) = \frac{1}{2}\left(v_1\left(t\right) \times \alpha v_1\left(t\right) + v_1\left(t\right) \times \alpha v_1\left(t\right)\right) = \frac{1}{2} 2\left(v_1\left(t\right) \times -v_1\left(t\right)\right) = 1$$

and similarly, we get the following equations as

$$\begin{aligned} h\left(v_2\left(t\right), v_1\left(t\right)\right) &= \tfrac{1}{2}\left(v_2\left(t\right) \times \alpha v_1\left(t\right) + v_1\left(t\right) \times \alpha v_2\left(t\right)\right) = \left(v_2\left(t\right) \times -v_1\left(t\right)\right) + \left(v_1\left(t\right) \times -v_2\left(t\right)\right) \\ &= \tfrac{1}{2}\left(-\left\langle v_2\left(t\right), -v_1\left(t\right)\right\rangle + v_2\left(t\right) \wedge v_1\left(t\right) + \left(-\left\langle v_1\left(t\right), -v_2\left(t\right)\right\rangle\right) + v_1\left(t\right) \wedge v_2\left(t\right)\right) \\ &= \tfrac{1}{2}\left(\alpha \boldsymbol{\mu} + \boldsymbol{\mu}\right) = 0, \end{aligned}$$

$$\begin{aligned} h\left(\boldsymbol{\mu}\left(t\right), v_1\left(t\right)\right) &= \tfrac{1}{2}\left(\boldsymbol{\mu}\left(t\right) \times \alpha v_1\left(t\right) + v_1\left(t\right) \times \alpha \boldsymbol{\mu}\left(t\right)\right) = \tfrac{1}{2}\left(\left(\boldsymbol{\mu}\left(t\right) \times -v_1\left(t\right)\right) + \left(v_1\left(t\right) \times -\boldsymbol{\mu}\left(t\right)\right)\right) \\ &= \tfrac{1}{2}\left(-\left\langle \boldsymbol{\mu}, -v_1\left(t\right)\right\rangle + \boldsymbol{\mu}\left(t\right) \wedge -v_1\left(t\right) + \left(-\left\langle v_1\left(t\right), -\boldsymbol{\mu}\left(t\right)\right\rangle\right) + v_1\left(t\right) \wedge -\boldsymbol{\mu}\left(t\right)\right) \\ &= \tfrac{1}{2}\left(\alpha v_2 + v_2\right) = 0. \end{aligned}$$

It is seen that $\boldsymbol{\mu} \perp v_1$ and $v_2 \perp v_1$ are orthogonal unit quaternions. Here is

$$\begin{aligned} \boldsymbol{\mu} \times v_1 &= -v_1 \times \boldsymbol{\mu} = v_2, \\ \boldsymbol{\mu} \times v_2 &= -v_2 \times \boldsymbol{\mu} = -v_1 \end{aligned}$$

and by using quaternionic inner product we get

$$\begin{aligned} h\left(\dot{v}_1\left(t\right), v_1\left(t\right)\right) &= a_{11}(t), \\ h\left(\dot{v}_1\left(t\right), v_2\left(t\right)\right) &= a_{12}(t), \\ h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) &= a_{13}(t). \end{aligned}$$

Also, by taking the derivative of both sides of the expression $h\left(v_1\left(t\right), v_1\left(t\right)\right) = 1$ for each $t \in I$ we get

$$h\left(\dot{v}_1\left(t\right), v_1\left(t\right)\right) + h\left(v_1\left(t\right), \dot{v}_1\left(t\right)\right) = 0.$$

The last equation gives $h\left(v_1\left(t\right), \dot{v}_1\left(t\right)\right) = 0$. In this case, it is $\dot{v}_1 \perp v_1$. In this case, it is obtained as $a_{11}(t) = 0$. Therefore the equation (3.5) can be written as

$$\dot{v}_1\left(t\right) = a_{12}(t) v_2\left(t\right) + a_{13}(t) \boldsymbol{\mu}\left(t\right). \tag{3.8}$$

The $h-$ inner products in the equation (3.7) are obtained as follows:

$$h\left(v_2\left(t\right), v_2\left(t\right)\right) = \frac{1}{2}\left(v_2\left(t\right) \times \alpha v_2\left(t\right) + v_2\left(t\right) \times \alpha v_2\left(t\right)\right) = 1,$$

and

$$h\left(\boldsymbol{\mu}\left(t\right), v_2\left(t\right)\right) = \frac{1}{2}\left(\left(\boldsymbol{\mu}\left(t\right) \times \alpha v_2\left(t\right)\right) + \left(v_2\left(t\right) \times \alpha \boldsymbol{\mu}\left(t\right)\right)\right) = 0.$$

Hence, the vector v_2 is unit and $\boldsymbol{\mu} \perp v_1$. Now we take the derivative of both sides of the equation $h\left(v_2\left(t\right), v_1\left(t\right)\right) = 0$ and by writing $h\left(\dot{v}_1\left(t\right), v_2\left(t\right)\right) = l\left(t\right)$ we obtain the below equations: $a_{12} = l\left(t\right)$ and $h\left(v_1\left(t\right), \dot{v}_2\left(t\right)\right) = -l\left(t\right)$. On the other hand, by derivativing of both sides of the equation $h\left(v_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = 0$ for each $t \in I$ we obtain

$$h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) + h\left(v_1\left(t\right), \dot{\boldsymbol{\mu}}\left(t\right)\right) = 0$$

and this results as $h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) + h\left(v_1\left(t\right), \dot{\boldsymbol{\mu}}\left(t\right)\right) = 0$ with by writing $m\left(t\right) := h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right)$. Hence also $a_{13}(t) = m\left(t\right)$. Then, using the given values, the equation (3.5) can be re-written as

$$\dot{v}_1\left(t\right) = l\left(t\right) v_2\left(t\right) + m\left(t\right) \boldsymbol{\mu}\left(t\right). \tag{3.9}$$

Now, considering that is $\dot{v}_2\left(t\right) = Sp\left\{v_1\left(t\right), v_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right\}$ the following equation can be written as

$$\dot{v}_2\left(t\right) = a_{21}(t) v_1\left(t\right) + a_{22}(t) v_2\left(t\right) + a_{23}(t) \boldsymbol{\mu}\left(t\right).$$

From here, by $h-$ inner product with v_1, v_2 and $\boldsymbol{\mu}$ respectively, we have for the quaternion v_1

$$h\left(\dot{v}_2\left(t\right), v_1\left(t\right)\right) = a_{21}(t)h\left(v_1\left(t\right), v_1\left(t\right)\right) + a_{22}(t)h\left(v_2\left(t\right), v_1\left(t\right)\right) + a_{23}(t)h\left(\boldsymbol{\mu}\left(t\right), v_1\left(t\right)\right)$$

where $v_1\left(t\right) \times v_1\left(t\right) = -1$, $v_2\left(t\right) \times v_1\left(t\right) = 0$ and $\mu\left(t\right) \times v_1\left(t\right) = 0$ from the above equation can be written as $-l\left(t\right) = a_{21}(t)$.
Similarly, $h-$ inner product for the quaternion v_2 it can be obtained as

$$h\left(\dot{v}_2\left(t\right), v_2\left(t\right)\right) = a_{21}(t)h\left(v_1\left(t\right), v_2\left(t\right)\right) + a_{22}(t)h\left(v_2\left(t\right), v_2\left(t\right)\right) + a_{23}(t)h\left(\mu\left(t\right), v_2\left(t\right)\right).$$

Hence, it can be seen that $h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = a_{23}(t)$. Also, by taking the derivative of both sides in the equation $h\left(v_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = a_{23}(t)$ for each $t \in I$ we can write the following equation:

$$h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) + h\left(v_2\left(t\right), \dot{\boldsymbol{\mu}}\left(t\right)\right) = 0.$$

By writing $n\left(t\right) := a_{23}(t) = h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right)$ in the last equation we have

$$h\left(v_2\left(t\right), \dot{\boldsymbol{\mu}}\left(t\right)\right) = -n\left(t\right)$$

and hence the equation (3.8) yields

$$\dot{v}_2\left(t\right) = -l\left(t\right)v_1\left(t\right) + n\left(t\right)\boldsymbol{\mu}\left(t\right). \tag{3.10}$$

To summarize, by considering these equations (3.4), (3.9) and (3.10) we get

$$\begin{aligned} \dot{\boldsymbol{\mu}}(t) &= -m(t)v_1(t) - n(t)v_2(t), \\ \dot{v}_1(t) &= l(t)v_2(t) + m(t)\boldsymbol{\mu}(t), \\ \dot{v}_2(t) &= -l(t)v_1\left(t\right) + n(t)\boldsymbol{\mu}(t) \ . \end{aligned}$$

Hence,

$$\begin{bmatrix} \dot{v}_1\left(t\right) \\ \dot{v}_2\left(t\right) \\ \dot{\boldsymbol{\mu}}\left(t\right) \end{bmatrix} = \begin{bmatrix} 0 & l\left(t\right) & m\left(t\right) \\ -l\left(t\right) & 0 & n\left(t\right) \\ -m\left(t\right) & -n\left(t\right) & 0 \end{bmatrix} \begin{bmatrix} v_1\left(t\right) \\ v_2\left(t\right) \\ \boldsymbol{\mu}\left(t\right) \end{bmatrix}. \tag{3.11}$$

□

Now the formulas given in equation (3.11) can be better understood with the following example.

Example 3.1.

$$\boldsymbol{\gamma}\left(t\right) = \left(3t^2, t^3, -t^3\right),$$

$$v_1\left(t\right) = \frac{1}{\sqrt{t^2+4}}\left(t, 0, 2\right),$$

$$v_2\left(t\right) = \frac{1}{\sqrt{t^2+4}\sqrt{2t^2\left(t^2+2\right)}}\left(2t^2, -4t - t^3, -t^3\right)$$

since $h\left(\dot{\boldsymbol{\gamma}}\left(t\right), v_1\left(t\right)\right) = 0$, $h\left(\dot{\boldsymbol{\gamma}}\left(t\right), v_2\left(t\right)\right) = 0$ and $h\left(v_1\left(t\right), v_2\left(t\right)\right) = 0$ for all $t \in I$ then $(\boldsymbol{\gamma}, v_1, v_2) : I \to H \times \Delta_2$ is quaternionic framed curve and $\boldsymbol{\mu} = v_1 \wedge v_2$ therefore we can calculate the vector $\boldsymbol{\mu}$ as perpendicular both the vector v_1 and the vector v_2 which

$$\boldsymbol{\mu}\left(t\right) = \frac{1}{\sqrt{t^2+2}}\left(2, t, -t\right).$$

It is clear from here that the point $t = 0$ is a singular point of the curve. In other words, since there exists $\boldsymbol{\mu}$ for the spatial quaternionic framed fundamental curve, Serret-Frenet type formulas and framed curvatures can be given in a quaternionic sense. Here, the curvatures are obtained as follows:

$$\begin{aligned} l\left(t\right) &= h\left(\dot{v}_1\left(t\right), v_2\left(t\right)\right) = \tfrac{\sqrt{2}t^2}{(t^2+4)\sqrt{t^2(t^2+2)}}, \\ m\left(t\right) &= h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = \tfrac{2}{\sqrt{t^2+2}\sqrt{t^2+4}}, \\ n\left(t\right) &= h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = \tfrac{2\sqrt{2}t}{\sqrt{t^2+2}\sqrt{(t^2+4)t^2(t^2+2)}}, \\ \sigma(t) &= 6t\sqrt{t^2+2}. \end{aligned}$$

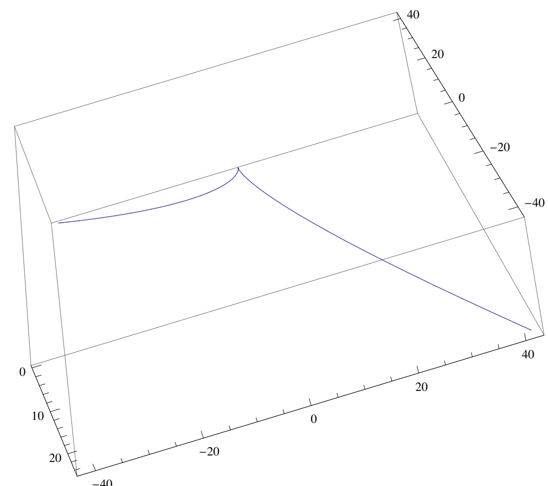

Figure 1. $\boldsymbol{\gamma}(t) = \left(3t^2, t^3, -t^3\right)$ quaternionic framed base curve.

Example 3.2. The curve $\boldsymbol{\gamma}(t) = \left(\dfrac{t^5}{5}, \sin t^4, \cos t^4\right)$ is a quaternionic Frenet-type framed curve. Indeed, the first derivative of the curve is taken, and the part that provides the singularity is separate,d and the other vector unit is united to obtain as

$$\sigma(t) = t^3\sqrt{16+t^2},$$

$$\boldsymbol{\mu}(t) = \frac{1}{\sqrt{16t^6+t^8}}\left(t^4, 4t^3\cos t^4, -4t^3\sin t^4\right)$$

framed elements for each $t \in I$. Therefore, the frame can be established over this vector.

$$v_1(t) = \frac{1}{\sqrt{\sin t^4 + \cos t^4}}\left(0, \sin t^4, \cos t^4\right),$$

$$v_2(t) = \frac{1}{\sqrt{\sin t^4 + \cos t^4}}\left(4t^3, -t^4\cos t^4, t^4\sin t^4\right)$$

since $h(\dot{\boldsymbol{\gamma}}(t), v_1(t)) = 0$, $h(\dot{\boldsymbol{\gamma}}(t), v_2(t)) = 0$ and $h(v_1(t), v_2(t)) = 0$ for all $t \in I$ then $(\boldsymbol{\gamma}, v_1, v_2) : I \to H \times \Delta_2$ is quaternionic framed curve. Also the vector $\boldsymbol{\mu}$, since $\boldsymbol{\mu} = v_1 \wedge v_2$ as perpendicular both the vector v_1 and the vector v_2. It is clear from here that the point $t = 0$ is a singular point of the curve. In other words, since there exists $\boldsymbol{\mu}$ for the spatial quaternionic framed fundamental curve, Serret-Frenet type formulas and framed curvatures can be given in a quaternionic sense. Here, the curvatures are obtained as follows:

$$l(t) = h(\dot{v}_1(t), v_2(t)) = -\frac{4t^7}{\sqrt{t^8+16t^6}},$$

$$m(t) = h(\dot{v}_1(t), \boldsymbol{\mu}(t)) = \frac{16t^6}{\sqrt{t^8+16t^6}},$$

$$n(t) = h(\dot{v}_2(t), \boldsymbol{\mu}(t)) = -\frac{4}{t^2+16},$$

$$\sigma(t) = h(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t)) = \frac{t^8+16t^6}{t^3\sqrt{16+t^2}} = t^3\sqrt{16+t^2}.$$

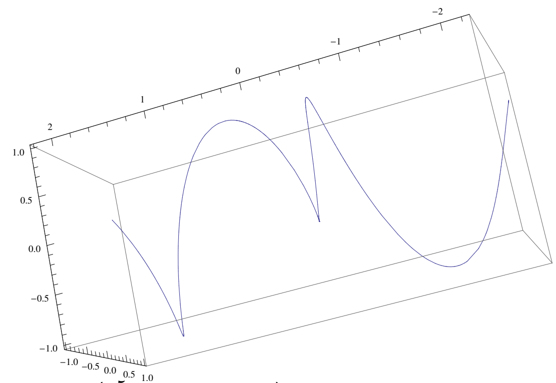

Figure 2. $\boldsymbol{\gamma}(t) = \left(\dfrac{t^5}{5}, \sin t^4, \cos t^4\right)$ quaternionic framed base curve

4. Quaternionic framed curves for $\mathbb{R}^4$

In this section, the relationship between a four-dimensional matrix and the curvatures of Serret-Frenet type formulas for quaternionic framed curves in four-dimensional Euclidean space is given. Then, in the light of the information obtained here, Serret-Frenet type formulas and framed curvatures in the quaternionic sense are reinforced with an example.

Theorem 4.1. *Let $\mathbb{R}^4$ denote the four dimensional Euclidean space and let $(\tilde{\boldsymbol{\gamma}}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3) : I = [0,1] \subset \mathbb{R} \to H \times \Delta_3$ be a quaternionic framed curve. Then the quaternionic framed basis curve is defined by $t \to \tilde{\boldsymbol{\gamma}}(t) = \sum_{i=0}^{3} \tilde{\boldsymbol{\gamma}}_i(t)\, \boldsymbol{e}_i,\ \boldsymbol{e}_0 = 1$ where $t \in I$ is any parameter. In this case there exist curvature functions $L, M, (-l+N), N, (-n-M), (m+L), \tilde{\sigma}(t) : I \to \mathbb{R}$ such that the Serret-Frenet type formulas at point $\tilde{\boldsymbol{\gamma}}(t)$ of the quaternionic frame curve $(\tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\nu}})$ are given as follows:*

$$\begin{bmatrix} \dot{\tilde{v}}_1 \\ \dot{\tilde{v}}_2 \\ \dot{\tilde{v}}_3 \\ \dot{\tilde{\boldsymbol{\mu}}} \end{bmatrix} = \begin{bmatrix} 0 & L & M & -l+N \\ -L & 0 & N & -n-M \\ -M & -N & 0 & m+L \\ l-N & n+M & -m-L & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{\boldsymbol{\mu}} \end{bmatrix}$$

where curvatures are
$L(t) = h\left(\dot{\tilde{v}}_1(t), \tilde{v}_2(t)\right)$, $M(t) = h\left(\dot{\tilde{v}}_1(t), \tilde{v}_3(t)\right)$, $(-l+N)(t) = h\left(\dot{\tilde{v}}_1(t), \boldsymbol{\mu}(t)\right)$, $N(t) = h\left(\dot{\tilde{v}}_2(t), \tilde{v}_3(t)\right)$, $(-n-M)(t) = h\left(\dot{\tilde{v}}_2(t), \tilde{\boldsymbol{\mu}}(t)\right)$, $(m+L)(t) = h\left(\dot{\tilde{v}}_3(t), \tilde{\boldsymbol{\mu}}(t)\right)$, $\tilde{\sigma}(t) = h\left(\dot{\tilde{\boldsymbol{\gamma}}}(t), \tilde{\boldsymbol{\mu}}(t)\right)$.

Proof. Let $(\tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\nu}})$ be a quaternionic framed curve and $\tilde{\boldsymbol{\mu}}(t) = \tilde{v}_1(t) \wedge \tilde{v}_2(t) \wedge \tilde{v}_3(t)$ unit vector for each $t \in I$ in $\mathbb{R}^4$. On the other hand, there is the smooth function $\tilde{\sigma} : I \to \mathbb{R}$ that satisfies the condition $\dot{\tilde{\boldsymbol{\gamma}}}(t) = \tilde{\sigma}(t)\, \tilde{\boldsymbol{\mu}}(t)$. Also, since $\tilde{\boldsymbol{\mu}}$ is the unit vector,

$$\|\tilde{\boldsymbol{\mu}}(t)\|^2 = h\left(\tilde{\boldsymbol{\mu}}(t), \tilde{\boldsymbol{\mu}}(t)\right) = \frac{1}{2}\left(\tilde{\boldsymbol{\mu}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \tilde{\boldsymbol{\mu}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t)\right) = \tilde{\boldsymbol{\mu}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) = 1$$

can be written. From here, by taking the derivative of both sides in the last expression, it becomes

$$\dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \tilde{\boldsymbol{\mu}}(t) \times \alpha\dot{\tilde{\boldsymbol{\mu}}}(t) = 0.$$

In this case $h\left(\dot{\tilde{\boldsymbol{\mu}}}(t), \tilde{\boldsymbol{\mu}}(t)\right) = \frac{1}{2}\left(\dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \tilde{\boldsymbol{\mu}}(t) \times \alpha\dot{\tilde{\boldsymbol{\mu}}}(t)\right) = 0$. Then $\dot{\tilde{\boldsymbol{\mu}}}(t)$ and $\tilde{\boldsymbol{\mu}}(t)$ are orthogonal. Since

$$\dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \tilde{\boldsymbol{\mu}}(t) \times \alpha\dot{\tilde{\boldsymbol{\mu}}}(t) = 0$$

then, according to conjugate properties we can write

$$\dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \alpha\left(\tilde{\boldsymbol{\mu}}(t) \times \alpha\dot{\tilde{\boldsymbol{\mu}}}(t)\right) = 0.$$

Therefore $\dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t)$ is spatial quaternion. Also since $\boldsymbol{\mu}$ are spatial and unit then

$$\|\boldsymbol{\mu}(t)\| = \boldsymbol{\mu}(t) \times \alpha\boldsymbol{\mu}(t) = \dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) \times \alpha\left(\dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t)\right) = \dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) \times \tilde{\boldsymbol{\mu}}(t) \times \alpha\dot{\tilde{\boldsymbol{\mu}}}(t) = \dot{\tilde{\boldsymbol{\mu}}}(t) \times \alpha\dot{\tilde{\boldsymbol{\mu}}}(t) = 1.$$

On the other hand since $\dot{\tilde{\boldsymbol{\mu}}}(t) \in Sp\left\{\tilde{v}_1(t), \tilde{v}_2(t), \tilde{v}_3(t)\right\}$ we get

$$\dot{\tilde{\boldsymbol{\mu}}}(t) = a_{41}(t)\tilde{v}_1(t) + a_{42}(t)\tilde{v}_2(t) + a_{43}(t)\tilde{v}_3(t) + a_{44}(t)\tilde{\boldsymbol{\mu}}(t). \tag{4.1}$$

Here, by using quaternionic inner product with $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ and $\tilde{\boldsymbol{\mu}}$ respectively,

$$h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{v}_1\right) = a_{41}, \quad h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{v}_2\right) = a_{42}, \quad h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{v}_3\right) = a_{43}, \quad h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{\mu}\right) = a_{44}.$$

Also, since $h\left(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\mu}}\right) = 1$ then $h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{\boldsymbol{\mu}}\right) + h\left(\tilde{\boldsymbol{\mu}}, \dot{\tilde{\boldsymbol{\mu}}}\right) = 0$. Thus, considering equation (4.1), it is seen that it is $a_{44}(t) = 0$. On the other hand, considering equation (4.1), we can write following equations

$$\begin{aligned}
\dot{\tilde{v}}_1(t) &= a_{11}(t)\tilde{v}_1(t) + a_{12}(t)\tilde{v}_2(t) + a_{13}(t)\tilde{v}_3(t) + a_{14}(t)\tilde{\mu}(t), \\
\dot{\tilde{v}}_2(t) &= a_{21}(t)\tilde{v}_1(t) + a_{22}(t)\tilde{v}_2(t) + a_{23}(t)\tilde{v}_3(t) + a_{24}(t)\tilde{\mu}(t), \\
\dot{\tilde{v}}_3(t) &= a_{31}(t)\tilde{v}_1(t) + a_{32}(t)\tilde{v}_2(t) + a_{33}(t)\tilde{v}_3(t) + a_{34}(t)\tilde{\mu}(t).
\end{aligned}$$

By using quaternionic inner product with $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ and $\tilde{\boldsymbol{\mu}}$ respectively,

$$\begin{aligned}
&h\left(\dot{\tilde{v}}_1, \tilde{v}_1\right) = a_{11} \; h\left(\dot{\tilde{v}}_2, \tilde{v}_1\right) = a_{21} \; h\left(\dot{\tilde{v}}_3, \tilde{v}_1\right) = a_{31}, \\
&h\left(\dot{\tilde{v}}_1, \tilde{v}_2\right) = a_{12} \; h\left(\dot{\tilde{v}}_2, \tilde{v}_2\right) = a_{22} \; h\left(\dot{\tilde{v}}_3, \tilde{v}_2\right) = a_{32}, \\
&h\left(\dot{\tilde{v}}_1, \tilde{v}_3\right) = a_{13} \; h\left(\dot{\tilde{v}}_2, \tilde{v}_3\right) = a_{23} \; h\left(\dot{\tilde{v}}_3, \tilde{v}_3\right) = a_{33}, \\
&h\left(\dot{\tilde{v}}_1, \tilde{\boldsymbol{\mu}}\right) = a_{14} \; h\left(\dot{\tilde{v}}_2, \tilde{\boldsymbol{\mu}}\right) = a_{24} \; h\left(\dot{\tilde{v}}_3, \tilde{\boldsymbol{\mu}}\right) = a_{34}.
\end{aligned}$$

Considering that there are $h\left(\tilde{v}_1, \tilde{v}_1\right) = 1$, $h\left(\tilde{v}_2, \tilde{v}_2\right) = 1$ and $h\left(\tilde{v}_1, \tilde{v}_2\right) = 0$ then we get $a_{11} = a_{22} = 0$. It is also seen that since $h\left(\tilde{v}_1, \tilde{v}_2\right) = 0$ then $a_{12} + a_{21} = 0$. Similarly, it can also be shown that $a_{13} + a_{31} = 0$. Suppose now that $a_{12} = L$ and $a_{13} = M$ then we have $a_{21} = -L$ and $a_{31} = -M$. Similarly, in the case that $a_{23} = N$, $a_{14} = F$, $a_{24} = G$ and $a_{34} = H$ we have $a_{32} = -N$, $a_{41} = -F$, $a_{42} = -G$ and $a_{43} = -H$. Using the given values, we get

$$\begin{aligned}
\dot{\tilde{\boldsymbol{\mu}}}(t) &= -F(t)\tilde{v}_1(t) - G(t)\tilde{v}_2(t) - H(t)\tilde{v}_3(t), \\
\dot{\tilde{v}}_1(t) &= L(t)\tilde{v}_2(t) + M(t)\tilde{v}_3(t) + F(t)\tilde{\boldsymbol{\mu}}(t) \\
\dot{\tilde{v}}_2(t) &= -L(t)\tilde{v}_1(t) + N(t)\tilde{v}_3(t) + G(t)\tilde{\boldsymbol{\mu}}(t), \\
\dot{\tilde{v}}_3(t) &= -M(t)\tilde{v}_1(t) - N(t)\tilde{v}_2(t) + H(t)\tilde{\boldsymbol{\mu}}(t).
\end{aligned} \tag{4.2}$$

On the other hand since $h\left(\tilde{v}_1, \tilde{\boldsymbol{\mu}}\right) = \frac{1}{2}\left(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_1\right) = 0$ and

$$\dot{\tilde{\boldsymbol{\mu}}} \times \alpha\tilde{\boldsymbol{\mu}} + \alpha\left(\dot{\tilde{\boldsymbol{\mu}}} \times \alpha\tilde{\boldsymbol{\mu}}\right) = 0$$

then

$$-F(t)\tilde{v}_1(t) - G(t)\tilde{v}_2(t) - H(t)\tilde{v}_3(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \tilde{\boldsymbol{\mu}}(t) \times \alpha\left(-F(t)\tilde{v}_1(t) - G(t)\tilde{v}_2(t) - H(t)\tilde{v}_3(t)\right) = 0.$$

If the above equation is arranged for each $t \in I$, then we get

$$-F\left(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_1\right) - G\left(\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_2\right) - H\left(\tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_3\right) = 0.$$

If F, G, and H are different from zero, then the following qualities are written

$$\begin{aligned}
\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} + \alpha\left(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}\right) &= 0, \\
\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} + \alpha\left(\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}\right) &= 0, \\
\tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}} + \alpha\left(\tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}}\right) &= 0.
\end{aligned} \tag{4.3}$$

Thus $\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}, \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}, \tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}}$ become spatial quaternions. Also, since $\boldsymbol{\mu}$ is spatial and unit,

$$\|\boldsymbol{\mu}\|^2 = \boldsymbol{\mu} \times \alpha\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} \times \alpha\left(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}\right) = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} \times \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_1 = \tilde{v}_1 \times \alpha\tilde{v}_1 = 1.$$

Since $\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}$ along the curve $\tilde{\boldsymbol{\gamma}}$, the vector $\tilde{v}_1$ can be written as

$$\boldsymbol{\mu} \times \tilde{\boldsymbol{\mu}} = \tilde{v}_1. \tag{4.4}$$

On the other hand, from equation (4.3) we get

$$\|v_1\|^2 = v_1 \times \alpha v_1 = \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} \times \alpha\left(\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}\right) = \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} \times \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_2 = \tilde{v}_2 \times \alpha\tilde{v}_2 = 1.$$

From the last equation, since $v_1 = \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}$ along the curve and $\tilde{\boldsymbol{\gamma}}$ the vector $\tilde{v}_3$ can be written as

$$\|v_2\|^2 = v_2 \times \alpha v_2 = \tilde{v}_3 \times \alpha\tilde{\mu} \times \alpha\left(\tilde{v}_3 \times \alpha\tilde{\mu}\right) = \tilde{v}_3 \times \alpha\tilde{\mu} \times \tilde{\mu} \times \alpha\tilde{v}_3 = 1.$$

Hence,

$$v_2 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_3$$

can be written. Taking the derivative of both sides of equation (4.4), we get

$$(-mv_1 - nv_2) \times \tilde{\boldsymbol{\mu}} + \boldsymbol{\mu} \times (-F\tilde{v}_1 - G\tilde{v}_2 - H\tilde{v}_3) = \dot{\tilde{v}}_1.$$

By making the necessary arrangements to the above equation we obtain

$$\begin{aligned} -m + H = L &\Rightarrow H = L + m, \\ -n - G = M &\Rightarrow G = -n - M. \end{aligned}$$

On the other hand, by using the equation $v_1 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_2$ and by differentiating we get

$$(\dot{v}_1 \times \tilde{\boldsymbol{\mu}} + v_1) \times \dot{\tilde{\boldsymbol{\mu}}} = \dot{\tilde{v}}_2.$$

This leads to the following equation:

$$(lv_2 + m\boldsymbol{\mu}) \times \tilde{\boldsymbol{\mu}} + v_1 \times (-F\tilde{v}_1 - G\tilde{v}_2 - H\tilde{v}_3) = -L\tilde{v}_1 + N\tilde{v}_3 + G\tilde{\boldsymbol{\mu}}.$$

By using the equations, $v_2 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_3$, $\boldsymbol{\mu} \times \tilde{\boldsymbol{\mu}} = \tilde{v}_1$, $v_1 \times \boldsymbol{\mu} = -v_2$ and $v_1 \times v_2 = \boldsymbol{\mu}$ in the above equation

$$\begin{aligned} l + F = N &\Rightarrow F = -l + N, \\ m - H = -L &\Rightarrow H = m + L \end{aligned}$$

are obtained. By differentiating $v_2 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_3$ we get

$$\dot{v}_2 \times \tilde{\boldsymbol{\mu}} + v_2 \times \dot{\tilde{\boldsymbol{\mu}}} = \dot{\tilde{v}}_3.$$

Considering the equations (4.2), (3.10), the above equation can be written as

$$(-lv_1 + n\boldsymbol{\mu}) \times \tilde{\boldsymbol{\mu}} + v_2 \times (-F\tilde{v}_1 - G\tilde{v}_2 - H\tilde{v}_3) = -M\tilde{v}_1 - N\tilde{v}_3 + H\tilde{\mu}.$$

If we use the $\tilde{v}_2$ following

$$\begin{aligned} v_1 \times \tilde{\boldsymbol{\mu}} &= \tilde{v}_2, \\ \boldsymbol{\mu} \times \tilde{\boldsymbol{\mu}} &= \tilde{v}_1, \\ \tilde{v}_3 &= v_2 \times \tilde{\boldsymbol{\mu}}. \end{aligned}$$

Then we get the following relations

$$\begin{aligned} -l - F = -n &\Rightarrow F = -l + N, \\ n + G = -M &\Rightarrow G = -M - n. \end{aligned}$$

Thus, the matrix representation of the Serret-Frenet formula is given by

$$\begin{bmatrix} \dot{\tilde{v}}_1 \\ \dot{\tilde{v}}_2 \\ \dot{\tilde{v}}_3 \\ \dot{\tilde{\boldsymbol{\mu}}} \end{bmatrix} = \begin{bmatrix} 0 & L & M & -l+N \\ -L & 0 & N & -n-M \\ -M & -N & 0 & m+L \\ l-N & n+M & -m-L & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{\boldsymbol{\mu}} \end{bmatrix}. \tag{4.5}$$

Therefore, the proof is completed. □

Here, a geometric interpretation can be provided. The quaternionic framed basis curve $\tilde{\boldsymbol{\gamma}}$ is selected such that its unit vector $\boldsymbol{\mu}$ is defined by the relation $\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}$. In this context, the curvatures of a quaternionic framed curve are associated with those in three-dimensional and four-dimensional Euclidean spaces. Consequently, the third curvature of $\tilde{\boldsymbol{\gamma}}$ is determined as the sum of the negative of the first curvature of $\boldsymbol{\gamma}$ and the positive of the fourth curvature of $\tilde{\boldsymbol{\gamma}}$. Similarly, the fifth curvature of $\tilde{\boldsymbol{\gamma}}$ is the sum of the negative sign of the second curvature of $\tilde{\boldsymbol{\gamma}}$ and the third curvature of $\boldsymbol{\gamma}$. Furthermore, the sixth curvature of $\tilde{\boldsymbol{\gamma}}$ is the sum of the first curvature of $\tilde{\boldsymbol{\gamma}}$ and the second curvature of $\boldsymbol{\gamma}$ (See, 4.5).

Example 4.1. $\tilde{\boldsymbol{\gamma}}(t) = \left(\frac{1}{5}\sin(5t) - t\cos(5t), \frac{1}{5}\cos(5t) + t\sin(5t), 2t\cos(2t) - \sin(2t), 2t\sin(2t) + \cos(2t)\right)$

$$\tilde{v}_1(t) = (-\cos(5t), \sin(5t), 0, 0),$$

$$\tilde{v}_2(t) = (0, 0, \cos(2t), \sin(2t)),$$

$$\tilde{v}_3(t) = \frac{5}{\sqrt{41}}\left(\frac{4}{5}\sin(5t), \frac{4}{5}, \cos(5t), \sin(2t), -\cos(2t)\right)$$

and for all $t \in I$ since $h\left(\dot{\tilde{\boldsymbol{\gamma}}}(t), \tilde{v}_1(t)\right) = 0$, $h\left(\dot{\tilde{\boldsymbol{\gamma}}}(t), \tilde{v}_2(t)\right) = 0$, $h(\tilde{v}_1(t), \tilde{v}_2(t)) = 0,\ h(\tilde{v}_1(t), \tilde{v}_3(t)) = 0,\ h(\tilde{v}_2(t), \tilde{v}_3(t)) = 0$, then $(\tilde{\boldsymbol{\gamma}}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3) : I \to H \times \Delta_3$ is a quaternionic framed curve and there exists $\tilde{\boldsymbol{\mu}}(t) = \tilde{v}_1(t) \wedge \tilde{v}_2(t) \wedge \tilde{v}_3(t)$ such that

$$\begin{aligned}
\tilde{\boldsymbol{\mu}}(t) &= \frac{1}{\sqrt{41}}(5\sin(5t), 5\cos(5t), -4\sin(2t), 4\cos(2t)),\\
\tilde{\sigma}(t) &= \sqrt{(41)}t.
\end{aligned}$$

Here, the quaternions are expressed in vector form. The unit quaternion $\tilde{\boldsymbol{\mu}}$ is computed as orthogonal to the quaternion $\tilde{v}_1$, the quaternion $\tilde{v}_2$ and the quaternion $\tilde{v}_3$. It is clear from here that point $t = 0$ is a singular point of the curve $\tilde{\gamma}$. In other words, since there exists $\tilde{\boldsymbol{\mu}}$ for the quaternionic framed fundamental curve $\tilde{\gamma}$, Serret-Frenet type formulas and framed curvatures can be given in a quaternionic sense. Hence, the curvatures of the curve $\tilde{\gamma}$ are

$$\begin{aligned}
L(t) &= h\left(\dot{\tilde{v}}_1(t), \tilde{v}_2(t)\right) = 0,\\
M(t) &= h\left(\dot{\tilde{v}}_1(t), \tilde{v}_3(t)\right) = \frac{20\sqrt{41}}{41},\\
(N - l)(t) &= h\left(\dot{\tilde{v}}_1(t), \tilde{\boldsymbol{\mu}}(t)\right) = \frac{25}{\sqrt{41}},\\
N(t) &= h\left(\dot{\tilde{v}}_2(t), \tilde{v}_3(t)\right) = -\frac{10\sqrt{41}}{41},\\
(-n - M)(t) &= h\left(\dot{\tilde{v}}_2(t), \tilde{\boldsymbol{\mu}}(t)\right) = \frac{8\sqrt{41}}{41},\\
(m + L)(t) &= h\left(\dot{\tilde{v}}_3(t), \tilde{\boldsymbol{\mu}}(t)\right) = 0,\\
\tilde{\sigma}(t) &= h\left(\dot{\tilde{\gamma}}(t), \tilde{\boldsymbol{\mu}}(t)\right) = \sqrt{41}t.
\end{aligned}$$

Also, in this example we can find the spatial quaternionic framed curve γ related to curve $\tilde{\gamma}$ and its Frenet elements and curvatures. Using the proof of Theorem 4.1, we get

$$\begin{aligned}
\boldsymbol{\gamma}(t) &= \frac{\sqrt{41}}{82}(10\cos(t), -\sin(4t) + 2\sin(2t), -\cos(4t) + 2\cos(2t))\\
v_1 &= -\frac{\sqrt{41}}{41}(-4, 5\sin(3t), 5\cos(3t)),\\
v_2 &= (0, \cos(3t), -\sin(3t)),\\
\boldsymbol{\mu}(t) &= (5, 4\sin(3t), 4\cos(3t)),\\
\alpha(t) &= \sin(t).
\end{aligned}$$

Thus, the curvatures of the spatial quaternionic framed curve γ are

$$l(t) = h(\dot{v}_1(t), v_2(t)) = -\frac{15\sqrt{41}}{41},$$

$$m(t) = h(\dot{v}_1(t), \boldsymbol{\mu}(t)) = 0,$$

$$n(t) = h(\dot{v}_2(t), \boldsymbol{\mu}(t)) = -\frac{12\sqrt{41}}{41}$$

$$\sigma(t) = h(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t)) = \sin(\mathrm{t}).$$

Based on this example, the theorem has been verified, and corresponding relations have been successfully obtained.

5. Results

As is known, a framed curve is a smooth space curve with a moving frame with singular points. In this study, the quaternionic structure of framed curves is discussed. Firstly, quaternionic framed curves are defined in three-dimensional Euclidean space with the help of framed curve definitions, which are defined in [7]. Then, Serret-Frenet formulas and curvatures for quaternionic framed curves in three-dimensional Euclidean space $\mathbb{R}^3$ are given in Theorem 3.1. In addition, these formulas is supported by examples. It is seen in the matrix expression given in (3.11) that the first curvature of $\boldsymbol{\gamma}$ is equal to the principal curvature given for regular curves and the third curvature of $\boldsymbol{\gamma}$ is equal to the torsion given for regular curves. Then, for non-spatial quaternionic framed curves in 4-dimensional Euclidean space, Theorem 4.1 and Serret-Frenet type new formulas and curvatures are obtained as in equation (4.5). Matrix representations of these new formulas are given. If we make some geometric deductions from equation (4.5), Frenet elements and Serret-Frenet formulas for quaternionic framed base curve $\tilde{\gamma}$ are obtained by using Serret-Frenet formulas for quaternionic framed base curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ (See 4.5). Here, the quaternionic framed basis curve $\tilde{\gamma}$ is chosen such that the unit vector $\boldsymbol{\mu}$ of the curve $\tilde{\gamma}$ is given by the relation $\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}$. Thus, the 3th curvature of $\tilde{\gamma}$ is the sum of the negative sign of the 1st curvature of $\boldsymbol{\gamma}$ and the positive sign of 4th curvature of $\tilde{\gamma}$, and the 5th curvature of $\tilde{\gamma}$ is the sum of the negative sign of 2th curvature of $\tilde{\gamma}$ and 3nd curvature of $\boldsymbol{\gamma}$, and the 6th curvature of $\tilde{\gamma}$ is the sum of the 1th curvature of $\tilde{\gamma}$ and 2nd curvature of $\boldsymbol{\gamma}$(See, 4.5). Finally, the theoretical framework is illustrated through examples, offering a quaternion-based perspective on framed curves.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions. I wish to extend my thanks to Prof. Hans-Peter Schröcker from the Department of Geometry and Surveying at the University of Innsbruck, Austria, for his kind assistance and valuable contributions to this work. Additionally, I would like to express my gratitude to Assoc. Prof. Ayşe Zeynep Azak from Sakarya University and Dr. Arzu Sürekçi from Sakarya University of Applied Sciences for their support and contributions.

Author's contributions: All authors read and approved the final manuscript. **Z. D. Y.** conducted the main research, prepared the examples, and wrote the manuscript. **M. A. G.** contributed significantly to the development and proof of the theorems and provided continuous academic guidance throughout the writing process.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] Bishop, R.L.: *There is more than one way to frame a curve*. The American Mathematical Monthly. **82**(3), 246-251 (1975).

[2] Choi, J.H., Kim, Y.H.: *Associated curves of a Frenet curve and their applications*. Applied Mathematics and Computation. **218**(18), 9116-9124 (2012).

[3] Arslan, K., Hacısalihoğlu, H.H.: *On harmonic curvatures of a Frenet curve*. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics Journal. **49**, 015-023 (2000).

[4] Romero-Fuster, M.C., Sanabria-Codesal, E.: *Generalized helices, twistings and flattenings of curves in n-space*. Matemática Contemporânea. **17**, 267-280 (1999).

[5] Deshmukh, S., Chen, B.Y., Alshamari, S.: *On rectifying curves in Euclidean 3-space*. Turkish Journal of Mathematics. **42**, 609-620 (2018).

[6] Fukunaga, T., Takahashi, M.: *Existence conditions of framed curves for smooth curves*. Journal of Geometry. **108**, 763-774 (2017).

[7] Fukunaga, T., Takahashi, M.: *Framed curves in the Euclidean space*. Advances in Geometry. **16**, 265-276 (2016).

[8] Doğan Yazıcı, B., İşbilir, Z., Tosun, M.: *Spinor representation of framed Mannheim curves* . Turkish Journal of Mathematics. **46**, 2690-2700 (2022).

[9] İşbilir, Z., Doğan Yazıcı, B., Tosun, M.: *The Spinor representations of framed Bertrand curves*. Filomat. **37**(9), 2831-2843 (2023).

[10] Doğan Yazıcı, B., Okuyucu, O. Z., Tosun, M.: *On special singular curve couples of framed curves in 3D Lie groups*. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics. **72**(3), 710–720 (2023).

[11] Yıldız, Ö. G., Akyiğit, M., Tosun, M.: *On the trajectory ruled surfaces of framed base curves in the Euclidean space*. Mathematical Methods in the Applied Sciences. **44**(9), 7463–7470 (2020).

[12] Doğan Yazıcı, B., Özkaldı Karakuş, S., Tosun, M.: *Framed normal curves in Euclidean space*. TBILISI-MATHEMATICS, Sciendo. 27-37 (2020).

[13] Okuyucu, O. Z., Canbirdi, C.: *Framed slant helices in Euclidean 3-space*. Advances in Difference Equations. **2021** (504), 1-14 (2021).

[14] Doğan Yazıcı, B., Özkaldı Karakuş, S., Tosun, M.: *Characterizations of framed curves in four-dimensional Euclidean space*. Universal Journal of Mathematics and Applications. **4**(4), 125-131 (2021).

[15] Akyiğit M., Yıldız Ö. G.: *On the framed normal curves in Euclidean 4-space*. Fundamental Journal of Mathematics and Applications. **4**(4), 258-263 (2021).

[16] Bharathi, K., Nagaraj, M.: *Quaternion valued function of a real variable Serret-Frenet formula*. Indian Journal of Pure and Applied Mathematics. **18**(6), 507-511 (1987).

[17] Tuna, A.: *Yarı Öklid uzayındaki kuaterniyonik eğriler için Serret-Frenet formülleri [Ph D Thesis]*. Süleyman Demirel Üniversitesi, Isparta, 2002.

[18] Güngör, M.A., Tosun, M.: *Some characterizations of quaternionic rectifying curves*. Differential Geometry-Dynamical Systems. **13**, 89-100 (2011).

[19] Soyfidan, T., Güngör, M. A.: *On the quaternionic curves in the semi-Euclidean space E_2^4*. Caspian Journal of Mathematical Sciences. **7**(1), 36-45 (2018).

[20] Hacısalihoğlu, H. H.: *Differential Geometry I*. Ankara University, Ankara, 2000.

[21] Kahraman Aksoyak, F.: *A new type quaternionic frame in $\mathbb{R}^4$*. International Journal of Geometric Methods in Modern Physics. **16**(6), 1950084 (2019).

[22] Çöken Ceylan, A., Tuna, A.: *On the quaternionic inclined curves in the semi-Euclidean space E_4^2*. Applied Mathematics and Computation. **155**(2), 373–389 (2004).

[23] Cansu, G., Yaylı, Y., Gök, I.: *A new quaternion valued frame of curves with an Application*. Filomat. **35**(1), 315–330 (2021).

[24] Gök, İ., Okuyucu, O. Z., Kahraman, F., Hacısalihoğlu, H. H.: *On the quaternionic B_2 slant helices in Euclidean 4-space E^4*. Advances in Applied Clifford Algebras. **21**, 707–719 (2011).

[25] Kahraman, F., Gök, .İ., Hacısalihoğlu, H.H.: *On the quaternionic B_2 slant helices in the semi-Euclidean space E_4^2*. Applied Mathematics and Computation. **218**(11), 6391-6400 (2012).

Affiliations

ZÜLAL DERIN YAQUB
ADDRESS: Sakarya University, Dept. of Math., 54050, Sakarya-Türkiye
E-MAIL: zulal.derin1@ogr.sakarya.edu.tr
ORCID ID: 0000-0001-9712-0428

MEHMET ALI GÜNGÖR
ADDRESS: Sakarya University, Dept. of Math., 54050, Sakarya-Türkiye
E-MAIL: agungor@sakarya.edu.tr
ORCID ID: 0000-0003-1863-3183