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η-Ricci Bourguignon Soliton on Para-Sasakian Manifolds with SSNM-Connection

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Abstract

The purpose of the present paper is to study η-Ricci-Bourguignon soliton on para-Sasakian manifold under some curvature conditions. We introduce here a new semi-symmetric non metric connection (briefly, SSNM-connection) on para-Sasakian manifold. We have obtained characterizations of para-Sasakian manifold based on both η-Ricci-Bourguignon soliton and the T_θ-curvature tensor with the SSNM-connection, where the T_θ-curvature tensor is the generalization of conformal, concircular, conharmonic, projective, pseudo projective and M- projective curvature tensors. Moreover, we investigate T_θ-Ricci symmetric para-Sasakian manifold admitting η-Ricci-Bourguignon soliton with respect to SSNM-connection.

Keywords: *η-Ricci-Bourguignon soliton; η-Einstein manifold; para-Sasakian manifold; semi-symmetric nonmetric connection; τ-curvature tensor.*
2010 Mathematics Subject Classification: *53C15; 53C25.*

1. Introduction

Early in the 1980s, the idea of Ricci flow was developed by R. S. Hamilton [19]. The Ricci flow equation is an evolution equation for metric on a Riemannian manifold defined as follows:

$$\frac{\partial g}{\partial t} = -2S,$$

where g is a Riemannian metric, S is Ricci curvature tensor and t is time. The solitons for the Ricci flow are the solutions of the above equation, where the metric at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g,V,σ), where V is a vector field and σ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\sigma g = 0,$$

where $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. The vector field V is called potential vector field and if it is a gradient of a smooth function, then the Ricci soliton (g,V,σ) is called a gradient Ricci soliton. Ricci soliton was further studied by many researchers. For instance, we see [3, 4, 22, 30, 34, 40, 45] and their references.

Recent focus of study on Ricci solitons has been made examining different kinds of manifolds and geometric structures. Research on Ricci soliton in various geometric contexts and generalization of the idea have grown in importance. Therefore, there is continuous, intense research being done on the stability of Ricci solitons, the identification of unique solutions and their physical uses.

As a generalization of concept of Ricci flow, the Ricci-Bourguignon flow [9, 10, 11] is given by

$$\frac{\partial g}{\partial t} = -2\left[S - \beta r g\right],\ g(0) = g_0,$$

where β is a constant and r is scalar curvature tensor regarding g.

The Ricci-Bourguignon soliton (RB soliton for short) is a solution of the Ricci-Bourguignon flow given by

$$L_V g + 2S = 2(\lambda + \beta r)g, \tag{1.1}$$

where λ is the soliton constant.

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In 1990, the basic ideas of Ricci-Bourguignon solitons were developed. The connection between Ricci Bourguignon flows and solitons was thoroughly investigated by Catino *et al.* [11]. The Ricci-Bourguignon solitons on numerous geometric structures have been studied as subject of research, with an emphasis on the interactions between the metrics and curvatures that make up the manifold. In geometric analysis, these solitons have been crucial in helping to comprehend the structural characteristics of manifolds. Physical applications of Ricci-Bourguignon solitons are also significant [12]. Ricci-Bourguignon soliton has been used to investigate the connection between physical systems and the geometric structures of manifolds.
Considering a 1-form $\eta = df(X)$, the η-Ricci-Bourguignon soliton (η-RB soliton for short) on an n-dimensional Riemannian manifold (M,g) is defined as a quadruple (g,V,λ,μ) satisfying

$$(L_V g)(X,Y) + 2S(X,Y) = 2(\lambda + \beta r)g(X,Y) + 2\mu\eta(X)\eta(Y), \tag{1.2}$$

for all smooth vector fields X,Y on M, where λ and μ being the soliton constants, β is a non-zero constant. Here $\chi(M)$ is the set of all smooth vector fields on the manifold M. An η-RB soliton (g,V,λ,μ) is called expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$. The soliton is called gradient η-Ricci-Bourguignon soliton when the potential vector field V is a gradient of a differentiable function $f : M \to R$ such that

$$\nabla^2 f + 2S(X,Y) = 2(\lambda + \beta r)g(X,Y) + 2\mu\eta(X)\eta(Y),$$

where $\nabla^2 f$ stands for the Hessian of f. The η-RB soliton is said to be trivial when either the vector field V is trivial or the potential function f is constant. In this case the vector field become killing vector field. This soliton is further studied by Traore *et al.* [36, 37, 38].
In 1924, Friedmann and Schouten gave the notion of semi-symmetric connection on a differentiable manifold. A linear connection on a differentiable manifold M is said to be semi-symmetric if its torsion tensor T satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{1.3}$$

for all $X,Y \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M and π is a 1-form associated with the vector field P given by

$$\pi(X) = g(X,P).$$

A linear connection ∇ is said to be metric connection if

$$(\nabla_X g)(Y,Z) = 0,$$

otherwise it is nonmetric. In 1932, Hayden [18] introduced the semi-symmetric metric connection on a Riemannian manifold and later it was named as Hayden connection. A systematic study of semi-symmetric metric connection was initiated by Yano [41] in 1970. He proved that a Riemannian manifold with respect to the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. The study of semi-symmetric metric connection was further developed by Amur and Puzara [6], Binh [8], De [15], Ozgur and Sular [25], Singh and Pandey [33] and many others.
On the other hand, semi-symmetric nonmetric connetion whose torsion is given by (1.3) was introduced by Agashe and Chafle [5] in 1992. They showed that a Riemannian manifold is projectively flat if it's curvature tensor with respect to the SSNM-connection vanishes. This linear connection was further developed by many researchers such as Chaubey and Ojha [13], De and Kamilya [16], De, Han and Zhao [17], Prasad and Singh [26], Prasad and Verma [27] and many others. In [14], Chaubey and Yildiz defined a new type of SSNM-connection on Remannian manifolds. They investigated various curvature properties of Riemannian manifold with respect to the SSNM-connection and studied Ricci soliton on Riemannian manifold with respect to this connection. Recently, SSNM-connections are studied on statistical manifolds by Yıldırım [44]. Motivated by their studies we introduced here the SSNM-connection on para-Sasakian manifold to study some properties and η-Ricci Bourguignon soliton on this manifold.
In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds were introduced by Sato and Matsumoto [32]. Later, Adati and Matsumoto investigate some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold have been studied by many authors. For instance, we see [2, 20, 21, 23, 24, 31, 35] and their references.
In 2011, Tripathi and Gupta [39] introduced τ-curvature tensor on semi-Riemannian manifold. Recently, Bhunia *et al.* [7] studied application of τ-curvature tensor in spacetimes. We consider here some of the τ-curvature tensors on para-Sasakian manifold M and named their representative tensor as T_θ given by

$$T_\theta(X,Y)Z = l_0R(X,Y)Z + l_1S(Y,Z)X + l_2S(X,Z)Y + l_3g(Y,Z)QX + l_4g(Z,X)QY + l_5r[g(Y,Z)X - g(X,Z)Y], \tag{1.4}$$

for all $X,Y,Z \in \chi(M)$, where $l_0,l_1,...,l_5$ are some smooth functions on M and R,S,Q and r are the Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M, respectively.
In particular, the T_θ-curvature tensor is reduced to
(1) the *conformal curvature tensor* [41] if

$$l_0 = 1, l_1 = -l_2 = l_3 = -l_4 = -\frac{1}{n-2},\ l_5 = \frac{1}{(n-1)(n-2)}, \tag{1.5}$$

(2) the *conharmonic curvature tensor* [46] if

$$l_0 = 1, l_1 = -l_2 = l_3 = -l_4 = -\frac{1}{n-2},\ l_5 = 0, \tag{1.6}$$

(3) the *concircular curvature tensor* [42] if

$$l_0 = 1, l_1 = l_2 = l_3 = l_4 = o,\ l_5 = \frac{1}{n(n-1)}, \tag{1.7}$$

(4) the *projective curvature tensor* [43] if

$$l_0 = 1, l_1 = -l_2 = -\frac{1}{n-1},\ l_3 = l_4 = l_5 = 0, \tag{1.8}$$

(5) the *pseudo projective curvature tensor* [28] if

$$l_1 = -l_2,\ l_3 = l_4 = 0,\ l_5 = -\frac{1}{n}\left(\frac{l_0}{n-1} + l_1\right), \tag{1.9}$$

(6) the M-projective curvature tensor [29] if

$$l_0 = 1, l_1 = -l_2 = l_3 = -l_4 = -\frac{1}{2(n-1)},\ l_5 = 0. \tag{1.10}$$

Definition 1.1. *A para-Sasakian manifold M is said to be η-Einstein manifold if the Ricci tensor of type (0,2) is of the form:*

$$S(X,Y) = k_1 g(X,Y) + k_2 \eta(X)\eta(Y),$$

for all $X, Y \in \chi(M)$, where k_1, k_2 are scalars.

Definition 1.2. *A para-Sasakian manifold M is said to be generalized η-Einstein manifold if the Ricci tensor of type $(0,2)$ is of the form:*

$$S(X,Y) = k_1 g(X,Y) + k_2 \eta(X)\eta(Y) + k_3 \omega(X,Y),$$

for all $X, Y \in \chi(M)$, where k_1, k_2 and k_3 are scalars and ω is a 2-form.

This paper has been organized as follows:
After introduction, a short description of para-Sasakian manifold has been given in **Section-2**. In **Section-3**, we introduce SSNM-connection on para-Sasakian manifold and establish some properties of the manifold. In **Section-4**, we study η-Ricci-Bourguignon soliton on para-Sasakian manifold with respect to SSNM-connection. **Section-5** contains η-Ricci-Bourguignon soliton on T_θ-flat para-Sasakian manifold with respect to SSNM-connection. **Section-5** concerns with η-Ricci-Bourguignon soliton on T_θ-Ricci symmetric para-Sasakian manifold. Finally, **Section-6** contains an example of 3-dimensional para-Sasakian manifold to verify the results obtained.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with structure (ϕ, ξ, η), where η is a 1-form, ξ is the structure vector field, ϕ is a $(1,1)$-tensor field satisfying [32]

$$\phi^2(X) = X - \eta(X)\xi, \eta(\xi) = 1, \tag{2.1}$$

$$\phi(\xi) = 0, \eta \circ \phi = 0, \tag{2.2}$$

for all vector field X on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [47]

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ, ξ, η, g). From (2.3) one can deduce that

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.4}$$

$$g(X, \xi) = \eta(\xi). \tag{2.5}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [47] if

$$g(X, \phi Y) = d\eta(X,Y),$$

for all vector fields X, Y on M, where

$$d\eta(X,Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X,Y])\}.$$

The manifold M is called a para-Sasakian manifold if

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.6}$$

for any smooth vector fields X, Y on M.
In a para-Sasakian manifold the following relations also hold [47]

$$(\nabla_X \eta)Y = g(X, \phi Y), \nabla_X \xi = -\phi X, \tag{2.7}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.8}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.10}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{2.12}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.13}$$

$$S(\xi,\xi) = -(n-1), Q\xi = -(n-1)\xi, \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

for any smooth vector fields X, Y, Z on M.

3. Semi-symmetric nonmetric connection on para-Sasakian manifolds

In this section we get the relation between SSNM-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SSNM-connection. We also establish here the first Bianchi identity with respect to SSNM-connection on M.
Let $M\ (\phi,\xi,\eta,g)$ be an n-dimensional para-Sasakian manifold equipped with Levi-Civita connection ∇ corresponding to the Riemannian metric g. Let a linear connection $\overline{\nabla}$ on M be defined by

$$\overline{\nabla}_X Y=\nabla_X Y+\frac{1}{2}\left[\eta\left(Y\right)X-\eta\left(X\right)Y\right], \tag{3.1}$$

for all $X,Y\in\chi\left(M\right)$.
Using the fact that ∇ is a metric connection, we have from (3.1) that

$$\begin{aligned}\left(\overline{\nabla}_X g\right)(Y,Z) &= \frac{1}{2}\left[g\left(X,Y\right)\eta\left(Z\right)+g\left(X,Z\right)\eta\left(Y\right)\right]\\ &\quad -g\left(Y,Z\right)\eta\left(X\right),\end{aligned} \tag{3.2}$$

for all $X,Y,Z\in\chi\left(M\right)$. Therefore $\overline{\nabla}$ is a nonmetric connection on M. The torsion tensor of $\overline{\nabla}$ is given by

$$\overline{T}\left(X,Y\right)=\eta\left(Y\right)X-\eta\left(X\right)Y. \tag{3.3}$$

On para-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\left(\overline{\nabla}_X\eta\right)Y = -\frac{1}{2}g\left(\phi X,\phi Y\right), \tag{3.4}$$

$$\overline{\nabla}_X\xi = -\phi X+\frac{1}{2}\left[X-\eta\left(X\right)\xi\right], \tag{3.5}$$

for all $X,Y\in\chi\left(M\right)$.
Let $\overline{R}$ be the Riemannian curvature tensor with respect to SSNM-connection on a para-Sasakian manifold defined as

$$\overline{R}(X,Y)Z=\overline{\nabla}_X\overline{\nabla}_Y Z-\overline{\nabla}_Y\overline{\nabla}_X Z-\overline{\nabla}_{[X,Y]}Z. \tag{3.6}$$

In reference of (2.6), (2.7) and (3.1) we have

$$\begin{aligned}\overline{\nabla}_X\overline{\nabla}_Y Z &= \nabla_X\nabla_Y Z+\frac{1}{2}\left[g\left(X,\phi Z\right)Y+\eta\left(\nabla_X Z\right)Y+\eta\left(Z\right)\nabla_X Y\right]\\ &\quad -\frac{1}{2}\left[g\left(X,\phi Y\right)Z+\eta\left(\nabla_X Y\right)Z+\eta\left(Y\right)\nabla_X Z\right]\\ &\quad +\frac{1}{2}\left[\eta\left(\nabla_Y Z\right)X-\eta\left(X\right)\nabla_Y Z\right]\\ &\quad +\frac{1}{4}\left[\eta\left(X\right)\eta\left(Y\right)Z-\eta\left(X\right)\eta\left(Z\right)Y\right],\end{aligned} \tag{3.7}$$

$$\begin{aligned}\overline{\nabla}_{[X,Y]}Z &= \nabla_{[X,Y]}Z+\frac{1}{2}\left[\eta\left(Z\right)\nabla_X Y-\eta\left(Z\right)\nabla_Y X\right]\\ &\quad +\frac{1}{2}\left[\eta\left(\nabla_Y X\right)Z-\eta\left(\nabla_X Y\right)Z\right].\end{aligned} \tag{3.8}$$

Interchanging X and Y in (3.7) and using it along with (3.7) and (3.8) in (3.6) we get

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z+\frac{1}{2}\left[g\left(X,\phi Z\right)Y-g\left(Y,\phi Z\right)X-2g\left(X,\phi Y\right)Z\right]\\ &\quad +\frac{1}{4}\left[\eta\left(Y\right)X-\eta\left(X\right)Y\right]\eta\left(Z\right),\end{aligned} \tag{3.9}$$

for all $X,Y,Z\in\chi\left(M\right)$.
Taking inner product of (3.9) with a vector field U and contracting over X and U we get

$$\begin{aligned}\overline{S}(Y,Z) &= S(Y,Z)-\frac{1}{2}(n-3)g(Y,\phi Z)\\ &\quad +\frac{1}{4}(n-1)\eta\left(Y\right)\eta\left(Z\right),\end{aligned} \tag{3.10}$$

where $\overline{S}$ denotes Ricci curvature tensor with respect to $\overline{\nabla}$.

Lemma 3.1. *Let M be an n-dimensional para-Sasakian manifold admitting SSNM-connection, then*

$$\begin{aligned}
\eta\left(\overline{R}(X,Y)Z\right) &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X) - g(X,\phi Y)\eta(Z) \\
&\quad + \frac{1}{2}\left[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)\right], && (3.11)\\
\overline{R}(X,Y)\xi &= \frac{3}{4}\left[\eta(X)Y - \eta(Y)X\right] - g(X,\phi Y)\xi, && (3.12)\\
\overline{R}(\xi,Y)Z &= -g(Y,Z)\xi - \frac{1}{2}g(Y,\phi Z)\xi \\
&\quad + \frac{3}{4}\eta(Z)Y + \frac{1}{4}\eta(Y)\eta(Z)\xi, && (3.13)\\
\overline{R}(X,\xi)Z &= g(X,Z)\xi + \frac{1}{2}g(X,\phi Z)\xi \\
&\quad - \frac{3}{4}\eta(Z)X - \frac{1}{4}\eta(X)\eta(Z)\xi, && (3.14)\\
\overline{Q}X &= QX - \frac{1}{2}(n-3)\phi X + \frac{1}{4}(n-1)\eta(X)\xi, && (3.15)\\
\overline{S}(X,\xi) &= -\frac{3}{4}(n-1)\eta(X),\ \overline{Q}\xi = -\frac{3}{4}(n-1)\xi, && (3.16)\\
\overline{r} &= r + \frac{1}{4}(n-1) - \frac{1}{2}(n-3)\psi, && (3.17)
\end{aligned}$$

for all $X,Y,Z \in \chi(M)$, where $\psi = trace(\phi)$ and $\overline{R}$, $\overline{Q}$, $\overline{r}$ denote Riemannian curvature tensor, Ricci operator, scalar curvature with respect to $\overline{\nabla}$, respectively.

Remark 3.2. *Eigen value of Ricci operator with respect to SSNM-connection corresponding to the eigen vector is $-\frac{3}{4}(n-1)$.*

4. η-Ricci-Bourguignon soliton on para-Sasakian manifold with respect to SSNM-connection

Let (g,V,λ,μ) be an η-RB soliton with respect to SSNM-connection on an n-dimensional para-Sasakian manifold $M(\phi,\xi,\eta,g)$. Then the equation (1.2) gives

$$0 = \left(\overline{L}_V g\right)(X,Y) + 2\overline{S}(X,Y) - 2(\lambda + \beta\overline{r})g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{4.1}$$

for all $X,Y \in \chi(M)$, where $\overline{L}_V$ is the Lie derivative along the vector field V on M with respect to SSNM-connection.
By virtue of (3.1) we have

$$\begin{aligned}
\left(\overline{L}_V g\right)(X,Y) &= g(\overline{\nabla}_X V, Y) + g(X, \overline{\nabla}_Y V) \\
&= g(\nabla_X V, Y) + g(X, \nabla_Y V) + g(X,Y)\eta(V) \\
&\quad - \frac{1}{2}\left[g(X,V)\eta(Y) + g(Y,V)\eta(X)\right]. && (4.2)
\end{aligned}$$

Using (3.10), (3.17) and (4.2) in (4.1) we obtain

$$\begin{aligned}
0 &= (L_V g)(X,Y) + 2S(X,Y) - 2(\lambda + \beta r)g(X,Y) - 2\mu\eta(X)\eta(Y) \\
&\quad - \frac{1}{2}\left[g(X,V)\eta(Y) + g(Y,V)\eta(X)\right] + g(X,Y)\eta(V) - (n-3)g(X,\phi Y) \\
&\quad + \beta\left[\frac{1}{2}(n-1) + (n-3)\psi\right]g(X,Y) + \frac{1}{2}(n-1)\eta(X)\eta(Y). && (4.3)
\end{aligned}$$

If (g,V,λ,μ) is an η-RB soliton on M with respect to Levi-Civita connection, then (4.3) holds. Thus from (1.2) and (4.3) we can state the following:

Theorem 4.1. *An η-RB soliton on para-Sasakian manifold is invariant under SSNM-connection if and only if the relation*

$$\begin{aligned}
0 &= -\frac{1}{2}\left[g(X,V)\eta(Y) + g(Y,V)\eta(X)\right] \\
&\quad + g(X,Y)\eta(V) - (n-3)g(X,\phi Y) + \frac{1}{2}(n-1)\eta(X)\eta(Y) \\
&\quad + \beta\left[\frac{1}{2}(n-1) + (n-3)\psi\right]g(X,Y)
\end{aligned}$$

holds for arbitrary vector fields X,Y and V.

Let (g,ξ,λ,μ) be an η-RB soliton on an n-dimensional para-Sasakian manifold $M(\phi,\xi,\eta,g)$. Then equation (4.1) gives

$$\left(\overline{L}_\xi g\right)(X,Y) + 2\overline{S}(X,Y) = 2(\lambda + \beta\overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y), \tag{4.4}$$

for all $X,Y \in \chi(M)$.

Expanding $\overline{L}_{\xi}g$ and using (3.5) and (3.10) in (4.4) we get

$$\overline{S}(X,Y) = \left(\lambda + \beta\overline{r} - \frac{1}{2}\right) g(X,Y) + \left(\mu + \frac{1}{2}\right) \eta(X)\eta(Y). \tag{4.5}$$

Setting $Y = \xi$ in (4.5) we get

$$\overline{S}(X,\xi) = (\lambda + \mu + \beta\overline{r})\,\eta(X). \tag{4.6}$$

Using (3.10) and (3.17) in (4.5) we get

$$\begin{aligned} S(X,Y) &= \left[\lambda + \beta\left(r + \frac{1}{4}(n-1) - \frac{1}{2}(n-3)\psi\right) - \frac{1}{2}\right] g(X,Y) \\ &\quad + \frac{1}{4}\left[4\mu - (n-3)\right]\eta(X)\eta(Y) + \frac{1}{2}(n-3)g(X,\phi Y). \end{aligned}$$

Theorem 4.2. *If an n-dimensional para-Sasakian manifold $M(\phi,\xi,\eta,g)$ admits η-RB soliton (g,ξ,λ,μ) with respect to SSNM-connection then M becomes a generalized η-Einstein manifold.*

5. η-Ricci-Bourguignon soliton on T_θ-flat para-Sasakian manifold with respect to SSNM-connection

The T_θ-curvature tensor with respect to SSNM-connection is given by

$$\begin{aligned} \overline{T}_\theta(X,Y)Z &= l_0\overline{R}(X,Y)Z + l_1\overline{S}(Y,Z)X + l_2\overline{S}(X,Z)Y \\ &\quad + l_3 g(Y,Z)\overline{Q}X + l_4 g(Z,X)\overline{Q}Y \\ &\quad + l_5\overline{r}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{5.1}$$

for all $X,Y,Z \in \chi(M)$, where $l_0, l_1, ..., l_5$ are some smooth functions on M and $\overline{R},\overline{S},\overline{Q}$ and $\overline{r}$ are the Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to SSNM-connection, respectively.
Let the para-Sasakian manifold $M(\phi,\xi,\eta,g)$ be $\overline{T}_\theta$-flat, then equation (5.1) gives

$$\begin{aligned} 0 &= l_0\overline{R}(X,Y)Z + l_1\overline{S}(Y,Z)X + l_2\overline{S}(X,Z)Y \\ &\quad + l_3 g(Y,Z)\overline{Q}X + l_4 g(Z,X)\overline{Q}Y \\ &\quad + l_5\overline{r}\left[g(Y,Z)X - g(X,Z)Y\right]. \end{aligned} \tag{5.2}$$

Taking inner product of (5.2) with a vector field V, we have

$$\begin{aligned} 0 &= l_0 g(\overline{R}(X,Y)Z,V) + l_1\overline{S}(Y,Z)g(X,V) + l_2\overline{S}(X,Z)g(Y,V) \\ &\quad + l_3 g(Y,Z)\overline{S}(X,V) + l_4 g(Z,X)\overline{S}(Y,V) \\ &\quad + l_5\overline{r}\left[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\right]. \end{aligned} \tag{5.3}$$

Contracting (5.3) over X and V we get

$$\overline{S}(Y,Z) = -\overline{r}\left[\frac{l_3 + (n-1)l_5}{l_0 + nl_1 + l_2 + l_4}\right] g(Y,Z), \tag{5.4}$$

for all vector fields X,Y on M.

Theorem 5.1. *If an n-dimensional para-Sasakian manifold be $\overline{T}_\theta$-flat, then M becomes generalized η-Einstein.*

Setting $X = \xi$ in (5.4) we obtain

$$\overline{S}(Y,\xi) = -\overline{r}\left[\frac{l_3 + (n-1)l_5}{l_0 + nl_1 + l_2 + l_4}\right] \eta(Y). \tag{5.5}$$

In view of (3.17), (4.6) and (5.5) we get

$$\lambda + \mu = -\left[r + \frac{1}{4}(n-1) - \frac{1}{2}(n-3)\psi\right]\left[\beta + \frac{l_3 + (n-1)l_5}{l_0 + nl_1 + l_2 + l_4}\right]. \tag{5.6}$$

Theorem 5.2. *If an n-dimensional $\overline{T}_\theta$-flat para-Sasakian manifold contains an η-RB soliton (g,ξ,λ,μ) with respect to SSNM-connection, then the soliton constants are given by (5.6).*

6. T_θ-Ricci semi-symmetric para-Sasakian manifold admitting η-RB soliton with respect to SSNM-connection

Definition 6.1. *An n-dimensional para-Sasakian manifold $M(\phi,\xi,\eta,g)$ is said to be T_θ-Ricci semi-symmetric with respect to SSNM-connection if and only if*

$$\overline{T}_\theta.\overline{S}=0.$$

In reference to (3.13), (4.6) and (5.1) we get the following result:

$$\begin{aligned}\overline{T}_\theta(\xi,X)Y &= -l_0\left[g(X,Y)+\frac{1}{2}g(X,\phi Y)-\frac{1}{4}\eta(X)\eta(Y)\right]\xi \\ &+\frac{3l_0}{4}\eta(Y)X+l_1\overline{S}(X,Y)+al_2\eta(Y)X+al_3g(X,Y)\xi \\ &+al_4\eta(Y)X+l_5\overline{r}\left[g(X,Y)\xi-\eta(Y)X\right],\end{aligned} \tag{6.1}$$

where

$$a=\lambda+\mu+\beta\overline{r}. \tag{6.2}$$

Let us consider the product

$$(\overline{T}_\theta(\xi,X))\overline{S}(Y,Z)=\overline{S}(\overline{T}_\theta(\xi,X)Y,Z)+\overline{S}(Y,\overline{T}_\theta(\xi,X)Z), \tag{6.3}$$

for all $X,Y,Z\in\chi(M)$.
Using (3.13), (4.6) and (6.1) in (6.3) we get

$$\begin{aligned}(\overline{T}_\theta(\xi,X))\overline{S}(Y,Z) &= -al_0\left[g(X,Y)+\frac{1}{2}g(X,\phi Y)-\frac{1}{4}\eta(X)\eta(Y)\right]\eta(Z) \\ &+\frac{3l_0}{4}\overline{S}(X,Z)\eta(Y)+al_1\overline{S}(X,Y)\eta(Z)+al_2\overline{S}(X,Z)\eta(Y) \\ &+a^2l_3g(X,Y)\eta(Z)+al_4\overline{S}(X,Z)\eta(Y) \\ &+l_5\overline{r}\left[ag(X,Y)\eta(Z)-\overline{S}(X,Z)\eta(Y)\right] \\ &-al_0\left[g(X,Z)+\frac{1}{2}g(X,\phi Z)-\frac{1}{4}\eta(X)\eta(Z)\right]\eta(Y) \\ &+\frac{3l_0}{4}\overline{S}(X,Y)\eta(Z)+al_1\overline{S}(X,Z)\eta(Y)+al_2\overline{S}(X,Y)\eta(Z) \\ &+a^2l_3g(X,Z)\eta(Y)+al_4\overline{S}(X,Y)\eta(Z) \\ &+l_5\overline{r}\left[ag(X,Z)\eta(Y)-\overline{S}(X,Y)\eta(Z)\right].\end{aligned} \tag{6.4}$$

Setting $X=\xi$ in (6.4) we get

$$\overline{T}_\theta.\overline{S}=a\left[l_1+l_2+l_3+l_4\right]\eta(Y)\eta(Z). \tag{6.5}$$

Contracting (6.5) over Y and Z we get

$$\overline{T}_\theta.\overline{S}=a(l_1+l_2+l_3+l_4). \tag{6.6}$$

In reference to the equations (1.5)-(1.10) it follows that

$$\sum_{i=1}^{4}l_i=0. \tag{6.7}$$

In view of equations (6.6) and (6.7) it follows that

$$\overline{T}_\theta.\overline{S}=0.$$

Theorem 6.2. *A para-Sasakian manifold containing an η-RB soliton with respect to SSNM-connection is always T_θ-Ricci symmetric.*

7. Example of 3-dimensional para-Sasakian manifold

Let us consider 3-dimensional manifold

$M^3 = \left\{(x,y,z) \in \mathbb{R}^3\right\}$,

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. We choose the linearly independent vector fields

$E_1 = e^x \frac{\partial}{\partial y},\ E_2 = e^x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right),\ E_3 = -\frac{\partial}{\partial x}.$

Let g be the pseudo Riemannian metric defined by $g\left(E_i,E_j\right) = 0,\ if\ i \neq j\ for\ i,j = 1,2,3,$
and $g(E_1,E_1) = -1,\ g(E_2,E_2) = -1,\ g(E_3,E_3) = 1$
Let η be the 1-form defined by $\eta(X) = g(X,E_3)$ for any $X \in \chi\left(M^3\right)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi E_1 = E_1,\ \phi E_2 = E_2,\ \phi E_3 = 0. \tag{7.1}$$

$$trace(\phi) = \sum_{i=1}^{3} g(E_i,\phi E_i) = -2 \tag{7.2}$$

Let $X,\,Y,\,Z \in \chi\left(M^3\right)$ be given by

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3, \\ Y &= y_1E_1 + y_2E_2 + y_3E_3, \\ Z &= z_1E_1 + z_2E_2 + z_3E_3. \end{aligned}$$

Then, we have

$$\begin{aligned} g(X,Y) &= x_1y_1 + x_2y_2 + x_3y_3, \\ \eta(X) &= x_3, \\ g(\phi X,\phi Y) &= x_1y_1 + x_2y_2. \end{aligned}$$

Using the linearity of g and ϕ, $\eta(E_3) = 1,\ \phi^2X = X - \eta(X)E_3$ and $g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y)$ for all $X,\,Y \in \chi(M)$. We have

$$\begin{aligned} [E_1,E_2] &= 0, [E_1,E_3] = -E_1, [E_2,E_3] = E_2, \\ [E_2,E_1] &= 0, [E_3,E_1] = E_1, [E_3,E_2] = -E_2. \end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$$\begin{pmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above results we see that the structure (ϕ,ξ,η,g) satisfies

$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X,$

for all $X,\,Y \in \chi\left(M^3\right)$, where $\eta(\xi) = \eta(E_3) = 1$. Hence $M^3(\phi,\xi,\eta,g)$ is a para-Sasakian manifold.
The components of Riemannian curvature tensor of M^3 are given by

$$\begin{pmatrix} R(E_1,E_2)E_2 & R(E_1,E_3)E_3 & R(E_1,E_2)E_3 \\ R(E_2,E_1)E_1 & R(E_2,E_3)E_3 & R(E_2,E_3)E_1 \\ R(E_3,E_1)E_1 & R(E_3,E_2)E_2 & R(E_3,E_1)E_2 \end{pmatrix} = \begin{pmatrix} -E_1 & -E_1 & 0 \\ E_2 & E_2 & 0 \\ E_3 & E_3 & 0 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 are given by

$$S(E_1,E_1) = S(E_3,E_3) = 0,\ S(E_2,E_2) = 2. \tag{7.3}$$

Therefore the scalar curvature of M^3 is

$$r = \sum_{i=1}^{3} S(E_i,E_i) = 2. \tag{7.4}$$

Using (3.1) we have the following values of $\overline{\nabla}$:

$$\begin{pmatrix} \overline{\nabla}_{E_1}E_1 & \overline{\nabla}_{E_1}E_2 & \overline{\nabla}_{E_1}E_3 \\ \overline{\nabla}_{E_2}E_1 & \overline{\nabla}_{E_2}E_2 & \overline{\nabla}_{E_2}E_3 \\ \overline{\nabla}_{E_3}E_1 & \overline{\nabla}_{E_3}E_2 & \overline{\nabla}_{E_3}E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -\frac{1}{2}E_1 \\ 0 & E_3 & -\frac{1}{2}E_2 \\ \frac{1}{2}E_1 & \frac{1}{2}E_2 & 0 \end{pmatrix}.$$

By the help of (3.6) and above matrix we get the components of Riemannian curvature tensor of M^3 with respect to SSNM-connection as follows

$$\begin{pmatrix} \overline{R}(E_1,E_2)E_1 & \overline{R}(E_1,E_3)E_1 & \overline{R}(E_2,E_3)E_1 \\ \overline{R}(E_1,E_2)E_2 & \overline{R}(E_1,E_3)E_2 & \overline{R}(E_2,E_3)E_2 \\ \overline{R}(E_1,E_2)E_3 & \overline{R}(E_1,E_3)E_3 & \overline{R}(E_2,E_3)E_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}E_2 & -\frac{3}{2}E_3 & 0 \\ -\frac{1}{2}E_1 & 0 & -\frac{1}{2}E_3 \\ 0 & -\frac{1}{4}E_1 & \frac{1}{4}E_2 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 with respect to SSNM-connection are given by

$$\overline{S}(E_1,E_1) = \overline{S}(E_2,E_2) = 1, \ \overline{S}(E_3,E_3) = \frac{1}{2}.$$

Therefore the scalar curvature of M^3 with respect to SSNM-connection is

$$\bar{r} = \sum_{i=1}^{3} S(E_i,E_i) = \frac{5}{2}. \tag{7.5}$$

In view of (7.2), (7.4) and (7.5) we have

$$\begin{aligned} \bar{r} &= \frac{5}{2} \\ &= 2+\frac{1}{4}(3-1)-\frac{1}{2}(3-3).(-2) \\ &= r+\frac{1}{4}(n-1)-\frac{1}{2}(n-3)\psi, \end{aligned}$$

which verifies the relation (3.17). Similarly, we can verify all the results obtained.

8. Conclusion

This paper introduces a novel class of τ-curvature tensor and η-Ricci-Bourguignon soliton with respect to SSNM-connection on para-Sasakian manifolds. From Section-4 and Section-5 we can conclude that if a para-Sasakian manifold contains η-Ricci-Bourguignon soliton with SSNM-connection then it must be generalized η-Einstein. Also, T_θ-flat para-Sasakian manifold it becomes generalized η Einstein if it admits η-Ricci-Bourguignon with respect to SSBN-connection. Further, a para-Sasakian manifold containing η-Ricci-Bourguignon soliton with respect to SSNM-connection is always T_θ-Ricci symmetric.
These results can be used to study nature of η-Ricci-Bourguignon flow in a contact manifold as well as in other metric manifolds. Also the results can be useful for further studies on this topic and have their own advantages.

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