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*Research Article*

# Newton-type Inequalities for Fractional Integrals by Various Function Classes

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**Abstract**

The authors of the paper examine some Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals. Namely, we establish some Newton-type inequalities for bounded functions by fractional integrals. In addition, we construct some fractional Newton-type inequalities for Lipschitzian functions. Furthermore, we offer some Newton-type inequalities by fractional integrals of bounded variation. Finally, we provide our results by using special cases of theorems and obtained examples.

## 1. Introduction & Preliminaries

Inequality theory is a crucial subject in many branches of mathematics with numerous number of applications. Many mathematicians have established the Hermite-Hadamard, Simpson, and Newton-type inequalities and they are very interested in generalizing and extending it to the case of various classes of functions, including $s$-convex functions, quasi-convex functions, log-convex functions, etc. In recent years, fractional calculus has increased interest because of the its demonstrated applications in a range of the inequality theory on convex functions. It can be obtained the bounds of new formulas by using the Hermite-Hadamard-type, Simpson-type inequality, and Newton-type inequality. Simpson-type inequalities are derived from Simpson's rules and take the following form of inequalities:

i. Simpson's $1/3$ rule, or Simpson's quadrature formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. The Simpson's second formula, often known as the Simpson's 3/8 rule, or the Newton-Cotes quadrature formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

The most popular Newton-Cotes quadrature using a three-point Simpson-type inequality is as follows:

**Theorem 1.1.** *If $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on $(a,b)$, and let $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

According to the Simpson $3/8$ inequality, the Simpson $3/8$ rule is a classical closed type quadrature rule is as follows:

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**Theorem 1.2.** *Note that $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on $(a,b)$, and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, one has the inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{6480}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

**Definition 1.3** (See [1]). *Suppose that $I$ is an interval of real numbers. Then, a function $f:I\to\mathbb{R}$ is said to be convex, if*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

*is valid $\forall x,y\in I$ and $\forall t\in[0,1]$.*

The three-point Newton-Cotes quadrature rule is the basis for Simpson's second rule. Results for three-step quadratic kernel computations are commonly referred to as Newton-type results. It is known from the literature that these results are Newton-type inequalities. There have been several mathematicians who have been considered to Newton-type inequalities. For example, in paper [2], some Newton-type inequalities are proved for the case of functions whose second derivatives are convex. In paper [3], several Newton-type inequalities are constructed by post-quantum integrals. Noor et al. proved Newton-type inequalities connected with harmonic convex and $p$-harmonic convex functions in [4] and [5], respectively. Moreover, in paper [6], some Newton-type inequalities were considered for the case of quantum-differentiable convex functions. Furthermore, in paper [7], several error estimates of the Newton-type quadrature formula were presented by bounded variation and Lipschitzian mappings. For some recent results on Newton-type inequalities, see [8–10] and the references therein.

**Definition 1.4** (See [11, 12]). *Let us consider $f\in L_1[a,b]$, $a,b\in\mathbb{R}$ with $a<b$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ are given by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

*and*

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b,$$

*respectively. Here, $\Gamma$ denotes the Gamma function defined by*

$$\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$$

*Note that $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.*

By means of the well-known Riemann-liouville fractional integrals for differentiable convex functions, some Newton-type inequalities are given as follows:

**Theorem 1.5** (See [13]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping $(a,b)$ so that $f'\in L_1([a,b])$. Let us also consider that the function $|f'|$ is convex on $[a,b]$. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{(b-a)}{2}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

*where*

$$\begin{cases}\Omega_1(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{1}{8}\right)^{1+\frac{1}{\alpha}}+\frac{1}{(\alpha+1)3^{\alpha+1}}-\frac{1}{24},\\ \Omega_2(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+\frac{1+2^{\alpha+1}}{(\alpha+1)3^{\alpha+1}}-\frac{1}{2},\\ \Omega_3(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{7}{8}\right)^{1+\frac{1}{\alpha}}+\frac{2^{\alpha+1}+3^{\alpha+1}}{(\alpha+1)3^{\alpha+1}}-\frac{35}{24}.\end{cases}$$

The popularity of fractional calculus has increased in recent years because of its wide range of applications in various fields of science. Given the importance of fractional calculus, several operators for fractional integrals can be taken into account. For example, in paper [14], sundry Newton-type inequalities are acquired for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex. In addition, in paper [15], some Newton-type inequalities are proved using Riemann-Liouville fractional integrals for differentiable convex functions and several Riemann-Liouville fractional Newton-type inequalities are presented for functions of bounded variation. Please refer to the [16–22] articles for further information and topics that are not explained.

The structure of the paper is divided into four parts, starting with an overview of the introduction and preliminaries. The fundamental definitions of fractional calculus and other relevant research in this discipline are given above. In Section 2, we will demonstrate an integral equality that is essential to establish the main findings. The authors of the paper will be presented some Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals in Section 3. More precisely, in subsection 3.1, some Newton-type inequalities will be presented for differentiable convex functions by using Riemann-Liouville fractional integrals. Moreover, we will provide several graphical examples in order to demonstrate the accuracy of the newly established inequalities. In subsection 3.2, we will give several Newton-type for bounded functions by fractional integrals. In subsection 3.3, some fractional Newton-type inequalities will be established for Lipschitzian functions. Furthermore, some Newton-type inequalities will be proved by fractional integrals of bounded variation in subsection 3.4. Finally, we will discuss our opinions on Newton-type inequalities and their potential consequences for future research areas in Section 4.

## 2. Main Result

In this section, we establish an integral equality involving Riemann-Liouville fractional integrals.

**Lemma 2.1.** *If $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function $(a,b)$ such that $f'\in L_1[a,b]$, then the following equality holds:*

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]$$
$$=\frac{b-a}{4}\left[I_1+I_2\right].$$

*Here,*

$$\begin{cases} I_1=\int_0^{\frac{1}{3}}t^{\alpha}\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt,\\ I_2=\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{cases}$$

*Proof.* With the help of the integration by parts, we can quickly acquire

$$\begin{aligned}I_1&=\int_0^{\frac{1}{3}}t^{\alpha}\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt \qquad (2.1)\\
&=\frac{2}{b-a}t^{\alpha}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]\Bigg|_0^{\frac{1}{3}}\\
&\quad-\frac{2\alpha}{b-a}\int_0^{\frac{1}{3}}t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt\\
&=\frac{2}{b-a}\left(\frac{1}{3}\right)^{\alpha}\left[f\left(\frac{a+2b}{3}\right)+f\left(\frac{2a+b}{3}\right)\right]\\
&\quad-\frac{2\alpha}{b-a}\int_0^{\frac{1}{3}}t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

If we apply a similar process above, then we have

$$\begin{aligned}I_2&=\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt \qquad (2.2)\\
&=\frac{1}{2(b-a)}\left[f(a)+f(b)\right]-\frac{2}{b-a}\left(\left(\frac{1}{3}\right)^{\alpha}-\frac{3}{4}\right)\left[f\left(\frac{a+2b}{3}\right)+f\left(\frac{2a+b}{3}\right)\right]\\
&\quad-\frac{2\alpha}{b-a}\int_{\frac{1}{3}}^{1}t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

If we combine (2.1) and (2.2), then we readily get

$$\begin{aligned}I_1+I_2&=\frac{1}{2(b-a)}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] \qquad (2.3)\\
&\quad-\frac{2\alpha}{b-a}\int_0^{1}t^{\alpha-1}\left[f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

Let us consider the change of the variables $x=\frac{1+t}{2}b+\frac{1-t}{2}a$ and $y=\frac{1+t}{2}a+\frac{1-t}{2}b$ for $t\in[0,1]$. Then, the equality (2.3) can be rewritten as follows

$$I_1+I_2=\frac{1}{2(b-a)}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]. \qquad (2.4)$$

Multiplying both sides of (2.4) by $\frac{b-a}{4}$, we arrive the proof of Lemma 2.1. □

## 3. Inequalities for various function classes

In this section, we prove several Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals. To be more precise, some Newton-type inequalities established for differentiable convex functions by using Riemann-Liouville fractional integrals. In addition, we acquire several graphical examples in order to demonstrate the accuracy of the newly established inequalities. Moreover, we present some Newton-type inequalities for bounded functions by fractional integrals. Afterwards, several fractional Newton-type inequalities are obtained for Lipschitzian functions. Furthermore, some Newton-type inequalities are proved by fractional integrals of bounded variation.

### 3.1. Fractional Newton-type inequalities for convex functions

**Theorem 3.1.** *Assume that the assumptions of Lemma 2.1 hold and the function $|f'|$ is convex on the interval $[a,b]$. Then, one can prove fractional Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{3.1}$$

*Here,*

$$\Omega_1(\alpha)=\int_0^{\frac{1}{3}} t^{\alpha}dt=\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1},$$

*and*

$$\Omega_2(\alpha)=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{\alpha+1}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\\ \frac{1}{\alpha+1}\left[2\alpha\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+1}+1\right]-1, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}.\end{cases}$$

*Proof.* By taking into account the absolute value of Lemma 2.1, we can directly have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|\left[\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt\right\}.\end{aligned} \tag{3.2}$$

Since $|f'|$ is convex, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}t^{\alpha}\left[\left(\frac{1+t}{2}\right)|f'(b)|+\left(\frac{1-t}{2}\right)|f'(a)|+\left(\frac{1+t}{2}\right)|f'(a)|+\left(\frac{1-t}{2}\right)|f'(b)|\right]dt\right. \\ &\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left[\left(\frac{1+t}{2}\right)|f'(b)|+\left(\frac{1-t}{2}\right)|f'(a)|+\left(\frac{1+t}{2}\right)|f'(a)|+\left(\frac{1-t}{2}\right)|f'(b)|\right]dt\right\} \\ &= \frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}$$

This finishes the proof of Theorem 3.1. □

**Remark 3.2.** *If we choose $\alpha=1$ in Theorem 3.1, then we can obtain Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{25(b-a)}{576}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

*which is established by Sitthiwirattham et al. in paper [15, Remark 3].*

**Example 3.3.** *If a function $f:[a,b]=[0,4]\to\mathbb{R}$ is described by $f(x)=\frac{x^2}{2}$ with $\alpha\in(0,15]$, then the left-hand side of (3.1) reduces to*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \\ &=\left|\frac{1}{8}\left[f(0)+3f\left(\frac{4}{3}\right)+3f\left(\frac{8}{3}\right)+f(4)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{4^{\alpha}}\left[J_{0+}^{\alpha}f(2)+J_{4-}^{\alpha}f(2)\right]\right|=\frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|.\end{aligned} \tag{3.3}$$

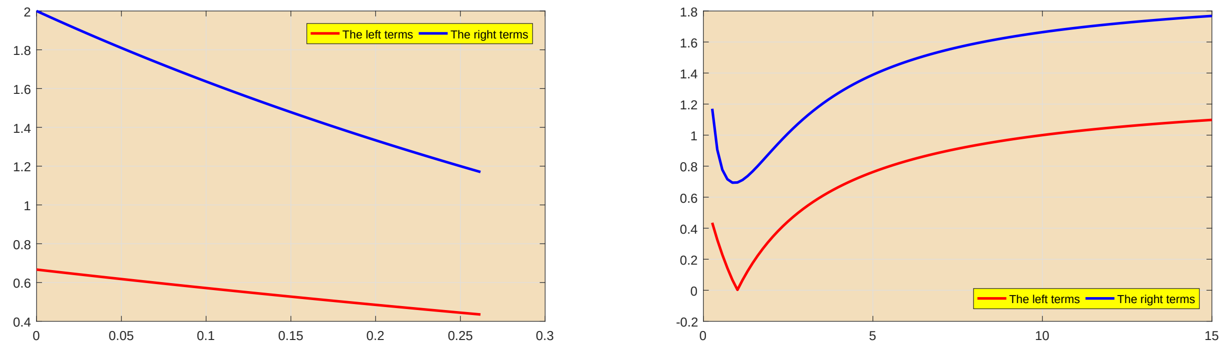
*The right hand-side of (3.1) coincides with*

$$\frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]=4\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)$$

$$=\begin{cases}\frac{2(1-\alpha)}{\alpha+1}, & 0<\alpha\le\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\ 4\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{2}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1\right], & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\le 15.\end{cases}$$

*Finally, we have*

$$\begin{cases}\frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\le\frac{2(1-\alpha)}{\alpha+1}, & 0<\alpha\le\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\ \frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\le 4\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{2}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1\right], & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\le 15.\end{cases}$$




**(a)** Graph based on the interval $0<\alpha\le\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}$.

**(b)** Graph based on the interval $\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\le 15$.

**Figure 3.1:** The left-hand side of (3.1) is consistently below the right-hand side of this inequality for all values of $\alpha\in(0,15]$ in Example 3.3.

**Theorem 3.4.** *Let us consider that the assumptions in Lemma 2.1 hold and the function* $|f'|^q$, $q>1$ *is convex on* $[a,b]$. *Then, the Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{3.4}$$

$$\le\frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{7|f'(a)|^q+5|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{5|f'(b)|^q+|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right\}$$

*is valid. Here,* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* By applying Hölder's inequality to (3.2), we obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\le\frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}|t^{\alpha}|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{3}}|t^{\alpha}|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Taking advantage of the convexity $|f'|^q$, we get

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{3}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{b-a}{4}\left\{\left(\frac{1}{(\alpha p+1)}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{7|f'(a)|^q+5|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{5|f'(b)|^q+|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right\},$$

which completes the proof of Theorem 3.4. □

**Corollary 3.5.** *If we assign $\alpha=1$ in Theorem 3.4, then we obtain*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\times\left[\left(\frac{5|f'(b)|^q+|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{7|f'(a)|^q+5|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right\}.$$

**Example 3.6.** *Let us consider a function $f:[a,b]=[0,4]\to\mathbb{R}$ given by $f(x)=\frac{x^2}{2}$. From Theorem 3.4 with $\alpha\in(0,15]$ and $p=q=2$, the left-hand side of (3.4) reduces to equality (3.3) and the right hand-side of (3.4) is equal to*

$$\left[\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}\right]^{\frac{1}{2}}\left(\frac{2\sqrt{7}+2\sqrt{5}}{3}\right)+\left[\frac{3}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\alpha+1}-\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}+\frac{1}{2\alpha+1}-\frac{3}{2(\alpha+1)}+\frac{3}{8}\right]^{\frac{1}{2}}\left(2+\sqrt{2}\right).$$

*Consequently, we have the inequality*

$$\frac{4}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\leq\left[\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}\right]^{\frac{1}{2}}\left(\frac{2\sqrt{7}+2\sqrt{5}}{3}\right)+\left[\frac{3}{2(\alpha+1)}\left(\frac{1}{3}\right)^{\alpha+1}\right.$$

$$\left.-\frac{1}{2\alpha+1}\left(\frac{1}{3}\right)^{2\alpha+1}+\frac{1}{2\alpha+1}-\frac{3}{2(\alpha+1)}+\frac{3}{8}\right]^{\frac{1}{2}}\left(2+\sqrt{2}\right).$$

*By using MATLAB software, as one can see in Example 3.6, it is easy to confirm that the left-hand side of (3.4) is always lower than the right-hand side of (3.4) in Figure 3.2 for all values of $\alpha\in(0,15]$.*

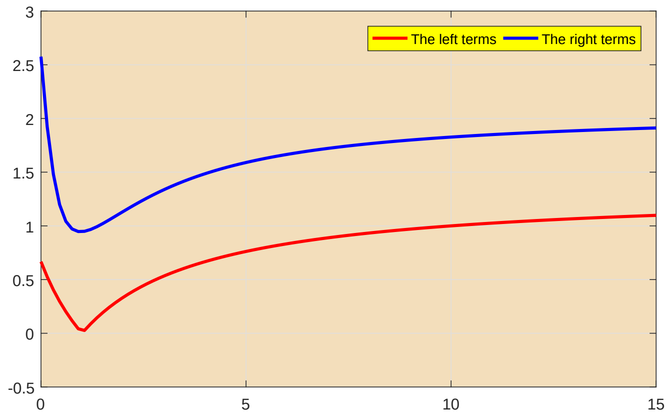



**Figure 3.2:** MATLAB has been evaluated and ploted the graph of both sides of (3.4) in Example 3.6.

**Theorem 3.7.** *Suppose that the assumptions of Lemma 2.1 hold and the function $|f'|^q$, $q \geq 1$ is convex on $[a,b]$. Then, one can obtain the Newton-type inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{3.5}$$
$$\leq \frac{b-a}{4}\left\{(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_3(\alpha)|f'(b)|^q+\Omega_4(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_3(\alpha)|f'(a)|^q+\Omega_4(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\right.$$
$$\left.+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_5(\alpha)|f'(b)|^q+\Omega_6(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_5(\alpha)|f'(a)|^q+\Omega_6(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\right\}.$$

*Here, $\Omega_1(\alpha)$ and $\Omega_2(\alpha)$ are specified in Theorem 3.1 and*

$$\Omega_3(\alpha)=\int_0^{\frac{1}{3}}\left(\frac{1+t}{2}\right)t^{\alpha}dt=\frac{4\alpha+7}{6(\alpha+1)(\alpha+2)}\left(\frac{1}{3}\right)^{\alpha+1},$$

$$\Omega_4(\alpha)=\int_0^{\frac{1}{3}}\left(\frac{1-t}{2}\right)t^{\alpha}dt=\frac{2\alpha+5}{6(\alpha+1)(\alpha+2)}\left(\frac{1}{3}\right)^{\alpha+1},$$

$$\Omega_5(\alpha)=\int_{\frac{1}{3}}^{1}\left(\frac{1+t}{2}\right)\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{2(\alpha+1)}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)+\frac{1}{2(\alpha+2)}\left(\left(\frac{1}{3}\right)^{\alpha+2}-1\right)-\frac{1}{12}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\\ \frac{1}{2(\alpha+1)}\left[2\alpha\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+1}+1\right] & \\ +\frac{1}{2(\alpha+2)}\left[\alpha\left(\frac{3}{4}\right)^{1+\frac{2}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+2}+1\right]-\frac{17}{24}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}.\end{cases}$$

$$\Omega_6(\alpha)=\int_{\frac{1}{3}}^{1}\left(\frac{1-t}{2}\right)\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{2(\alpha+1)}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2(\alpha+2)}\left(\left(\frac{1}{3}\right)^{\alpha+2}-1\right)-\frac{5}{12}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\\ \frac{1}{2(\alpha+1)}\left[2\alpha\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+1}+1\right] & \\ -\frac{1}{2(\alpha+2)}\left[\alpha\left(\frac{3}{4}\right)^{1+\frac{2}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+2}+1\right]-\frac{7}{24}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}.\end{cases}$$

*Proof.* When we first apply (3.2) to the power-mean inequality, one can obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}|t^{\alpha}|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{3}}|t^\alpha|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^\alpha|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

With the help of the convexity of $|f'|^q$, we get

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{a+b}{2}\right)+J_{b-}^\alpha f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{4}\Bigg\{\left(\int_0^{\frac{1}{3}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}t^\alpha\left[\left(\frac{1+t}{2}\right)|f'(b)|^q+\left(\frac{1-t}{2}\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{3}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}t^\alpha\left[\left(\frac{1+t}{2}\right)|f'(a)|^q+\left(\frac{1-t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left[\left(\frac{1+t}{2}\right)|f'(b)|^q+\left(\frac{1-t}{2}\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^\alpha-\frac{3}{4}\right|\left[\left(\frac{1+t}{2}\right)|f'(a)|^q+\left(\frac{1-t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}$$
$$=\frac{b-a}{4}\Bigg\{(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_3(\alpha)|f'(b)|^q+\Omega_4(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_3(\alpha)|f'(a)|^q+\Omega_4(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]$$
$$+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_5(\alpha)|f'(b)|^q+\Omega_6(\alpha)|f'(a)|^q\right]^{\frac{1}{q}}+\left[\Omega_5(\alpha)|f'(a)|^q+\Omega_6(\alpha)|f'(b)|^q\right]^{\frac{1}{q}}\right]\Bigg\}.$$

This ends the proof of Theorem 3.7. □

**Corollary 3.8.** *If we select $\alpha=1$ in Theorem 3.7, then we have the Newton-type inequality*

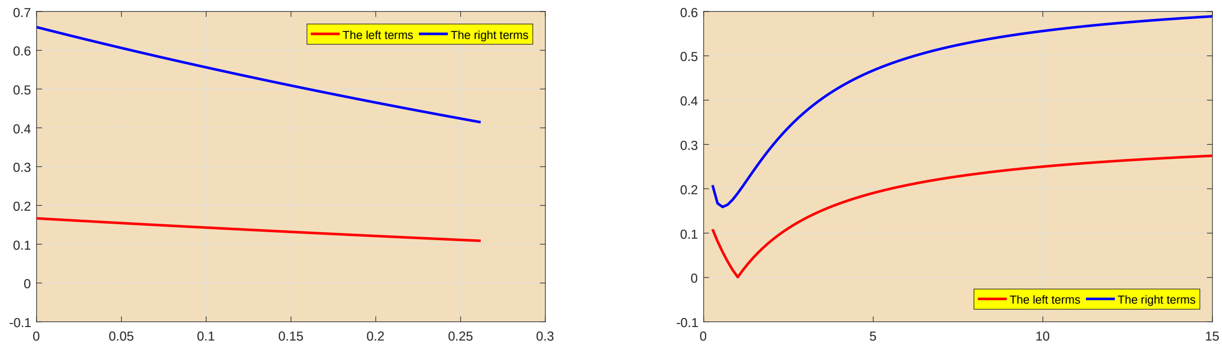
$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{b-a}{72}\Bigg\{\left[\left(\frac{11|f'(b)|^q+7|f'(a)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{11|f'(a)|^q+7|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right]$$
$$+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{973|f'(b)|^q+251|f'(a)|^q}{576}\right)^{\frac{1}{q}}+\left(\frac{973|f'(a)|^q+251|f'(b)|^q}{576}\right)^{\frac{1}{q}}\right]\Bigg\}.$$

**Example 3.9.** *A function $f:[a,b]=[0,4]\to\mathbb{R}$ is presented by $f(x)=\frac{x^2}{2}$. From Theorem 3.7 with $\alpha\in(0,15]$ and $q=2$, the left-hand side of (3.5) reduces to an equality (3.3) and the right hand-side of (3.5) is*

$$4\left\{(\Omega_1(\alpha))^{\frac{1}{2}}\left[[\Omega_3(\alpha)]^{\frac{1}{2}}+[\Omega_4(\alpha)]^{\frac{1}{2}}\right]+(\Omega_2(\alpha))^{\frac{1}{2}}\left[[\Omega_5(\alpha)]^{\frac{1}{2}}+[\Omega_6(\alpha)]^{\frac{1}{2}}\right]\right\}.$$

*Finally, we have the inequality*

$$\frac{1}{3}\left|\frac{1-\alpha}{\alpha+2}\right|\leq(\Omega_1(\alpha))^{\frac{1}{2}}\left[[\Omega_3(\alpha)]^{\frac{1}{2}}+[\Omega_4(\alpha)]^{\frac{1}{2}}\right]+(\Omega_2(\alpha))^{\frac{1}{2}}\left[[\Omega_5(\alpha)]^{\frac{1}{2}}+[\Omega_6(\alpha)]^{\frac{1}{2}}\right].$$




**(a)** Graph based on the interval $0<\alpha\leq \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}$.

**(b)** Graph based on the interval $\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15$.

**Figure 3.3:** As one can see in Example 3.9 that the left-hand side of (3.5) constantly stays below the right-hand side.

## 3.2. Fractional Newton-type inequalities for bounded functions

**Theorem 3.10.** *Consider that the conditions of Lemma 2.1 hold. If there exist $m,M\in\mathbb{R}$ such that $m\leq f'(t)\leq M$ for $t\in[a,b]$, then it follows*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{3.6}$$
$$\leq\frac{b-a}{4}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}(M-m).$$

*Proof.* By using the Lemma 2.1, we have

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right] \tag{3.7}$$
$$=\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}t^{\alpha}\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right]dt+\int_0^{\frac{1}{3}}t^{\alpha}\left[\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt\right.$$
$$\left.+\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right]dt+\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{3}{4}\right)\left[\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt\right\}.$$

When we use the absolute value of (3.7), we have

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right|dt+\int_0^{\frac{1}{3}}|t^{\alpha}|\left|\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right|dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right\}.$$

If we use $m\leq f'(t)\leq M$ for $t\in[a,b]$, then we get

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2} \tag{3.8}$$

and

$$\left|\frac{m+M}{2}-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\leq\frac{M-m}{2}. \tag{3.9}$$

With the help of (3.8) and (3.9), we obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{a+2b}{3}\right)+3f\left(\frac{2a+b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{4}(M-m)\left\{\int_0^{\frac{1}{3}} t^{\alpha}dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|dt\right\}=\frac{b-a}{4}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}(M-m).$$

□

**Corollary 3.11.** *If we choose $\alpha=1$ in Theorem 3.10, then one can obtain*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq \frac{25(b-a)}{576}(M-m).$$

**Corollary 3.12.** *Under assumptions of Theorem 3.10, if there exists $M\in\mathbb{R}^+$ such that $|f'(t)|\leq M$ for all $t\in[a,b]$, then it follows*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}M.$$

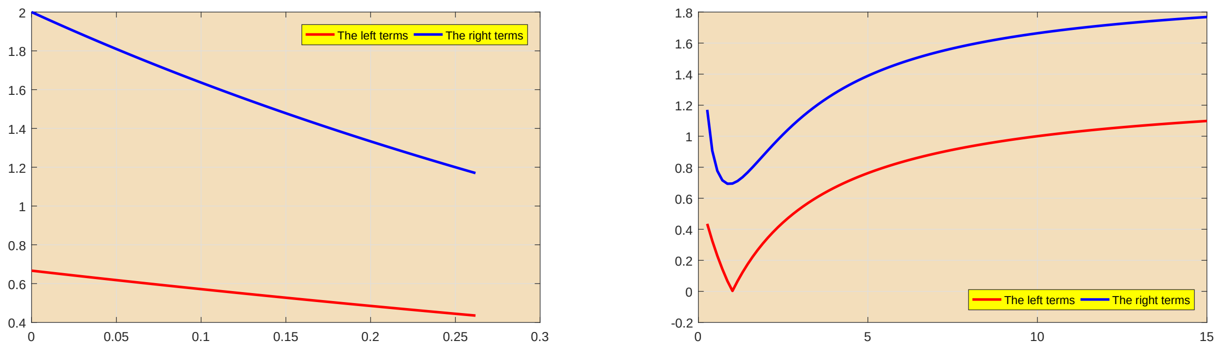
**Corollary 3.13.** *Let us consider $\alpha=1$ in Corollary 3.12. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq \frac{25(b-a)}{288}M.$$

**Example 3.14.** *A function $f:[a,b]=[0,4]\to\mathbb{R}$ is given by $f(x)=\frac{x^2}{2}$. From Theorem 3.10 with $\alpha\in(0,15]$ and $0\leq f'(t)\leq 4$, the left-hand side of (3.6) becomes to equality (3.3) and the right hand-side of (3.6) is*

$$4\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}$$

$$=\begin{cases}\frac{2(1-\alpha)}{\alpha+1}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\[2ex] 4\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{4}\right)^{1+\frac{1}{\alpha}}+\frac{2}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}+\frac{1}{\alpha+1}-1\right], & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15.\end{cases}$$




**(a)** Graph based on the interval $0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}$.

**(b)** Graph based on the interval $\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq 15$.

**Figure 3.4:** Example 3.14 illustrates how the left side of (3.6) consistently remains lower than the right side.

### 3.3. Fractional Newton-type inequalities for Lipschitzian functions

**Theorem 3.15.** *Suppose that the assumptions of Lemma 2.1 are valid. If $f'$ is a L-Lipschitzian function on $[a,b]$, then the following inequality*

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]$$
$$\leq \frac{(b-a)^2}{4}L\left\{\Omega_7(\alpha)+\Omega_8(\alpha)\right\}$$

*is valid. Here,*

$$\Omega_7(\alpha)=\int_0^{\frac{1}{3}} t^{\alpha+1}dt=\frac{1}{\alpha+2}\left(\frac{1}{3}\right)^{\alpha+2}$$

*and*

$$\Omega_8(\alpha)=\int_{\frac{1}{3}}^{1} t\left|t^{\alpha}-\frac{3}{4}\right|dt=\begin{cases}\frac{1}{\alpha+2}\left(1-\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{1}{3}, & 0<\alpha\leq\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)},\\[2ex] \frac{1}{\alpha+2}\left[\alpha\left(\frac{3}{4}\right)^{1+\frac{2}{\alpha}}+\left(\frac{1}{3}\right)^{\alpha+2}+1\right]-\frac{5}{12}, & \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha.\end{cases}$$

*Proof.* With the help of Lemma 2.1 and since $f'$ is $L$-Lipschitzian function, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.\\
&\quad\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right\}\\
&\leq\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}t^{\alpha}Lt(b-a)\,dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{3}{4}\right|Lt(b-a)\,dt\right\}\\
&=\frac{(b-a)^2}{4}L\left\{\Omega_7(\alpha)+\Omega_8(\alpha)\right\}.\end{aligned}$$

□

**Corollary 3.16.** *Consider $\alpha=1$ in Theorem 3.15. Then, the following Newton-type inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{425(b-a)^2}{20736}L.$$

### 3.4. Newton-type inequalities for functions of bounded variation

**Theorem 3.17.** *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then, we obtain*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{1}{2}\max\left\{\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4},\left(\frac{1}{3}\right)^{\alpha}\right\}\bigvee_a^b(f),\end{aligned}$$

*where $\bigvee_a^b(f)$ denotes the total variation of $f$ on $[a,b]$.*

*Proof.* Define the function $K_{\alpha}(x)$ by

$$K_{\alpha}(x)=\begin{cases}\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-x\right)^{\alpha}, & a\leq x<\frac{2a+b}{3},\\[1ex] \left(x-\frac{a+b}{2}\right)^{\alpha}, & \frac{2a+b}{3}\leq x<\frac{a+2b}{3},\\[1ex] \left(x-\frac{a+b}{2}\right)^{\alpha}-\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}, & \frac{a+2b}{3}\leq x\leq b.\end{cases}$$

By using the integrating by parts, we have

$$\int_a^b K_\alpha(x)df(x)$$
$$= \int_a^{\frac{2a+b}{3}} \left[\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right] df(x) + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x-\frac{a+b}{2}\right)^\alpha df(x)$$
$$+ \int_{\frac{a+2b}{3}}^{b} \left[\left(x-\frac{a+b}{2}\right)^\alpha - \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha\right] df(x)$$
$$= \left[\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right] f(x)\Big|_a^{\frac{2a+b}{3}} - \alpha \int_a^{\frac{2a+b}{3}} \left(\frac{a+b}{2}-x\right)^{\alpha-1} f(x)dx$$
$$+ \left(x-\frac{a+b}{2}\right)^\alpha f(x)\Big|_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} - \alpha \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x-\frac{a+b}{2}\right)^{\alpha-1} f(x)dx$$
$$+ \left[\left(x-\frac{a+b}{2}\right)^\alpha - \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha\right] f(x)\Big|_{\frac{a+2b}{3}}^{b} - \alpha \int_{\frac{a+2b}{3}}^{b} \left(x-\frac{a+b}{2}\right)^{\alpha-1} f(x)dx$$
$$= \frac{1}{4}\left(\frac{b-a}{2}\right)^\alpha f(a) + \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{2a+b}{3}\right) + \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{a+2b}{3}\right) + \frac{1}{4}\left(\frac{b-a}{2}\right)^\alpha f(b)$$
$$-\alpha \int_a^{\frac{2a+b}{3}} \left(\frac{a+b}{2}-x\right)^{\alpha-1} f(x)dx - \alpha \int_{\frac{2a+b}{3}}^{b} \left(x-\frac{a+b}{2}\right)^{\alpha-1} f(x)dx$$
$$= \frac{(b-a)^\alpha}{2^{\alpha-1}} \frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \Gamma(\alpha+1)\left[J_{a+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}\right)\right].$$

In other words, one can get

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{2^{\alpha-1}}{(b-a)^\alpha}\int_a^b K_\alpha(x)df(x).$$

It is known that if $g,f:[a,b]\to\mathbb{R}$ are such that $g$ is continuous on $[a,b]$ and $f$ is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right| \le \sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{3.10}$$

By using (3.10), it yields

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}\right)\right]\right|$$
$$= \frac{2^{\alpha-1}}{(b-a)^\alpha}\left|\int_a^b K_\alpha(x)df(x)\right|$$
$$\le \frac{2^{\alpha-1}}{(b-a)^\alpha}\left\{\left|\int_a^{\frac{2a+b}{3}} \left[\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right] df(x)\right| + \left|\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(x-\frac{a+b}{2}\right)^\alpha df(x)\right|\right.$$
$$\left. + \left|\int_{\frac{a+2b}{3}}^{b} \left[\left(x-\frac{a+b}{2}\right)^\alpha - \frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha\right] df(x)\right|\right\}$$
$$\le \frac{2^{\alpha-1}}{(b-a)^\alpha}\left\{\sup_{x\in\left[a,\frac{2a+b}{3}\right]}\left|\frac{3}{4}\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-x\right)^\alpha\right| \bigvee_a^{\frac{2a+b}{3}}(f) + \sup_{x\in\left[\frac{2a+b}{3},\frac{a+2b}{3}\right]}\left|\left(x-\frac{a+b}{2}\right)^\alpha\right| \bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)\right.$$

$$
\begin{aligned}
&+\sup_{x\in\left[\frac{a+2b}{3},b\right]}\left|\left(x-\frac{a+b}{2}\right)^{\alpha}-\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}\right|\bigvee_{\frac{a+2b}{3}}^{b}(f)\Bigg\}\\
&=\frac{2^{\alpha-1}}{(b-a)^{\alpha}}\Bigg\{\max\left[\left|\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{6}\right)^{\alpha}\right|,\frac{1}{4}\left(\frac{b-a}{2}\right)^{\alpha}\right]\bigvee_{a}^{\frac{2a+b}{3}}(f)+\left(\frac{b-a}{6}\right)^{\alpha}\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)\\
&+\max\left[\left|\frac{3}{4}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{6}\right)^{\alpha}\right|,\frac{1}{4}\left(\frac{b-a}{2}\right)^{\alpha}\right]\bigvee_{\frac{a+2b}{3}}^{b}(f)\Bigg\}\\
&=\frac{1}{2}\left\{\max\left[\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4}\right]\bigvee_{a}^{\frac{2a+b}{3}}(f)+\left(\frac{1}{3}\right)^{\alpha}\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)+\max\left[\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4}\right]\bigvee_{\frac{a+2b}{3}}^{b}(f)\right\}\\
&\leq\frac{1}{2}\max\left\{\left|\frac{3}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4},\left(\frac{1}{3}\right)^{\alpha}\right\}\bigvee_{a}^{b}(f).
\end{aligned}
$$

□

**Remark 3.18.** *Let us consider $\alpha=1$ in Theorem 3.17. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{5}{24}\bigvee_a^b(f).$$

*which is given by Alomari in [23].*

## 4. Conclusion

In this paper, some Newton-type inequalities are establish for various function classes involving Riemann-Liouville fractional integrals. First of all, we present an integral identity that is necessary in order to prove the main findings of the paper. Subsequently several Newton-type inequalities are investigated for differentiable convex functions by using the Riemann-Liouville fractional integrals. In addition to this, we give several examples using graphs in order to show that our main result is correct. Moreover we prove sundry Newton-type for bounded functions by fractional integrals. Furthermore, several fractional Newton-type inequalities are obtained for Lipschitzian functions. Finally, some Newton-type inequalities are acquired by fractional integrals of bounded variation.

The concepts and approaches for our findings about Newton-type inequalities using Riemann-Liouville fractional integrals could clear the way for additional studies in this area in subsequent publications. Improvements or generalizations of our results can be investigated by using different kinds of convex function classes or other types of fractional integral operators. Finally, one can acquire several Newton-type inequalities for various function classes with the help of the quantum calculus.

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