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On the Francois Polynomials and Hybrinomials

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Abstract

This paper introduces the Francois polynomials and Francois hybrinomials, which are generalized forms of the Francois numbers and Francois hybrid numbers, respectively. We explore their recurrence relations, summation formulas, generating functions, and Binet-type formulas. Moreover, we derive the Catalan, Cassini, and d'Ocagne identities for these polynomials and hybrinomials.

Keywords: *Binet formula; Catalan identity; d'Ocagne identity; Francois hybrinomials; Francois polynomials; Generating function.*
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1. Introduction

Integer number sequences are essential in mathematics and applied sciences, with applications involving number theory, combinatorics, algebra, analysis, and computational mathematics. They also play a key role in algorithm development. In particular, well-known number sequences such as the Fibonacci, Lucas, Pell, and Leonardo sequences serve as fundamental tools in the study of mathematical structures. The Fibonacci and Lucas number sequences are known by the following recurrence relations ($n \geq 2$)

$$F_n = F_{n-1} + F_{n-2}, \quad \text{with} \quad F_0 = 0, \quad F_1 = 1, \tag{1.1}$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad \text{with} \quad L_0 = 2, \quad L_1 = 1, \tag{1.2}$$

respectively [19]. The recurrence relations (1.1) and (1.2) hold the characteristic equation

$$t^2 - t - 1 = 0, \tag{1.3}$$

whose roots are $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. As evident from the similarity of their recurrence relations (1.1) and (1.2), the Fibonacci and Lucas number sequences share similar properties. Another sequence with a comparable structure is the Leonardo number sequence, which is defined as follows ($n \geq 2$)

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \tag{1.4}$$

with initial conditions $Le_0 = Le_1 = 1$ [10]. It is possible to come across many generalizations of these number sequences in the literature [3, 6, 11, 13, 21, 27, 33, 34, 35, 41]. Some of these are Fibonacci, Lucas and Leonardo polynomials. The Fibonacci and Lucas polynomials are introduced by Catalan [8] and Bicknell [5] as follows

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad \text{with} \quad F_0(x) = 0, \quad F_1(x) = 1$$

and

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \quad \text{with} \quad L_0(x) = 2, \quad L_1(x) = x,$$

respectively, where x denotes any variable quantity and $n \geq 2$. For more details on the properties and identities of Fibonacci and Lucas polynomials, the reader is referred to [14, 15, 20]. The Leonardo Pisano polynomials are defined by the rule ($n \geq 3$) [22]

$$Le_n(x) = 2xLe_{n-1}(x) - Le_{n-3}(x) \quad \text{with} \quad Le_0(x) = Le_1(x) = 1 \quad \text{and} \quad Le_2(x) = x + 2.$$

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Dişkaya et *al*. [12] introduced the Francois number sequence, a variant of the Leonardo number sequence. The Francois number sequence is defined by the recurrence relation ($n \geq 2$)

$$\mathscr{F}_n = \mathscr{F}_{n-1} + \mathscr{F}_{n-2} + 1 \quad \text{with} \quad \mathscr{F}_0 = 2, \quad \mathscr{F}_1 = 1. \tag{1.5}$$

The recurrence relation (1.5) can be rewritten as ($n \geq 3$)

$$\mathscr{F}_n = 2\mathscr{F}_{n-1} - \mathscr{F}_{n-3},$$

which has the same initial conditions and satisfies the characteristic equation [12]

$$t^3 - 2t^2 + 1 = 0. \tag{1.6}$$

The roots of equation (1.6) are ϕ, ψ and 1, where ϕ and ψ are equal to the roots of equation (1.3). Alp [2] studied a generalized form of the Francois number sequence, exploring various properties, including its generating function, Binet-type formula, and summation formulas. Moreover, some relations between Fibonacci, Lucas, and generalized Francois numbers were examined by the author.

The complex, hyperbolic, and dual number systems are well known for their geometric and physical applications. For instance, these numbers define the geometry of the Euclidean, Minkowski, and Galilean planes, respectively. Their algebraic properties and geometric interpretations have been compared in Rooney's paper [32]. In addition, they play a role in describing Euclidean, Lorentzian, and Galilean rotations. Furthermore, complex, hyperbolic, and dual numbers can be represented as Clifford algebras associated with elliptic, hyperbolic, and parabolic bilinear forms, respectively. A combination of these numbers, known as the hybrid number system, was introduced by Özdemir [30] with the set

$$\mathbb{K} = \left\{ \zeta_0 + i\zeta_1 + \varepsilon\zeta_2 + h\zeta_3 : \zeta_0, \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i \right\}.$$

The elements ζ of this set are called hybrid numbers [30]. Some of the basic definitions given by Özdemir [30] regarding hybrid numbers are as follows: The character of the hybrid number $\zeta = \zeta_0 + i\zeta_1 + \varepsilon\zeta_2 + h\zeta_3$ is evaluated by

$$C(\zeta) = \zeta\overline{\zeta} = \zeta_0^2 + (\zeta_1 - \zeta_2)^2 - \zeta_2^2 - \zeta_3^2,$$

where $\overline{\zeta} = \zeta_0 - i\zeta_1 - \varepsilon\zeta_2 - h\zeta_3$ is the conjugate of a hybrid number ζ. The norm of the hybrid number ζ is defined by the real number

$$\|\zeta\| = \sqrt{|C(\zeta)|}.$$

The matrix equivalents of the hybrid number ζ is

$$M(\zeta) = \begin{bmatrix} \zeta_0 + \zeta_2 & \zeta_1 - \zeta_2 + \zeta_3 \\ \zeta_2 - \zeta_1 + \zeta_3 & \zeta_0 - \zeta_2 \end{bmatrix}.$$

The matrix forms of the hybrid units $1, i, \varepsilon$ and h are

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Recently, researchers have shown interest in the hybrid number system. In particular, choosing the components of hybrid numbers from well-known number sequences has yielded interesting results [4, 7, 9, 16, 24, 26, 28, 29, 36, 38, 40]. Szynal-Liana [36] introduced the Horadam hybrid numbers and analysed their characteristics, including the Binet-type formula and generating function. The Fibonacci and Lucas hybrid numbers are defined by the rules

$$FH_n = F_n + iF_{n+1} + \varepsilon F_{n+2} + hF_{n+3}$$

and

$$LH_n = L_n + iL_{n+1} + \varepsilon L_{n+2} + hL_{n+3},$$

where F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively [36]. Kızılateş [16], Tan and Ait-Amrane [40] studied on some generalizations of the Fibonacci hybrid numbers. The hybrid Leonardo numbers are introduced by Alp and Koçer [4] by

$$LeH_n = Le_n + iLe_{n+1} + \varepsilon Le_{n+2} + hLe_{n+3},$$

where Le_n denotes the nth Leonardo number. The authors gave some of their algebraic properties including the recurrence relation, generating function, summation formula, Binet-type formula, Catalan and Cassini identities. Szynal-Liana and Włoch [38] defined the Pell and Pell-Lucas hybrid numbers and described some of their algebraic properties. Catarino [9] and Bród et *al*. [7] studied on some generalizations of the Pell hybrid numbers. The components of hybrid numbers need not be limited to numbers only; they are chosen from known polynomials or even functions [1, 17, 18, 22, 23, 25, 31, 37, 39]. Szynal-Liana and Włoch [37] generalized Fibonacci and Lucas hybrid numbers by introducing the Fibonacci and Lucas hybrinomials. These hybrinomials are

$$FH_n(x) = F_n(x) + iF_{n+1}(x) + \varepsilon F_{n+2}(x) + hF_{n+3}(x) \tag{1.7}$$

and

$$LH_n(x) = L_n(x) + iL_{n+1}(x) + \varepsilon L_{n+2}(x) + hL_{n+3}(x), \tag{1.8}$$

where $F_n(x)$ and $L_n(x)$ are the nth Fibonacci and Lucas polynomials, respectively [37]. The authors gave the recurrence relation, Binet-type formula, Catalan, Cassini, d'Ocagne identities for the Fibonacci and Lucas hybrinomials. The papers [39] and [18] presented some generalizations of the Fibonacci hybrinomials. Kızılateş [17] defined the Horadam hybrinomials and examined some of their special cases and algebraic properties including the recurrence relation, generating function, Binet-type formula and summation formulas. The author also obtained some identities and applications for the Horadam hybrinomials. The Pell hybrinomials are introduced by Liana et *al*. [23]. The authors investigated some of their basic properties. Kürüz et *al*. [22] defined the Leonardo Pisano hybrinomials

$$LeH_n(x) = Le_n(x) + iLe_{n+1}(x) + \varepsilon Le_{n+2}(x) + hLe_{n+3}(x),$$

where $Le_n(x)$ denotes the nth Leonardo Pisano polynomial. They described some of their fundamental algebraic properties and identities. Following the approach of the aforementioned papers, we introduce the Francois polynomials and Francois hybrinomials as generalized forms of Francois numbers and Francois hybrid numbers, respectively. We examine their recurrence relations, summation formulas, generating functions, and Binet-type formulas. Furthermore, we establish basic identities for the Francois polynomials and hybrinomials. The paper expands on the algebraic aspects of the Francois hybrid numbers by incorporating a polynomial framework, providing further perspectives for mathematical analysis and application.

2. Main Results

Definition 2.1. *The Francois polynomials are defined by the recurrence relation for $n \geq 3$*

$$\mathscr{F}_n(x) = 2x\mathscr{F}_{n-1}(x) - \mathscr{F}_{n-3}(x),$$

with initials $\mathscr{F}_0(x) = 2$, $\mathscr{F}_1(x) = 1$ and $\mathscr{F}_2(x) = x+3$.

The first few Francois polynomials are

$$\begin{aligned}
&\mathscr{F}_3(x) = 2x^2 + 6x - 2,\\
&\mathscr{F}_4(x) = 4x^3 + 12x^2 - 4x - 1,\\
&\mathscr{F}_5(x) = 8x^4 + 24x^3 - 8x^2 - 3x - 3,\\
&\mathscr{F}_6(x) = 16x^5 + 48x^4 - 16x^3 - 8x^2 - 12x + 2.
\end{aligned}$$

Note that for $x = 1$, the Francois polynomials yield the Francois numbers, the first few terms of which for $n \geq 2$ are $4, 6, 11, 18, 30, \ldots$.

Definition 2.2. *The nth Francois hybrinomial is defined by the rule*

$$\mathscr{F}H_n(x) = \mathscr{F}_n(x) + i\mathscr{F}_{n+1}(x) + \varepsilon\mathscr{F}_{n+2}(x) + h\mathscr{F}_{n+3}(x),$$

where $\mathscr{F}_n(x)$ denotes the nth Francois polynomials.

The first few Francois hybrinomial are

$$\begin{aligned}
&\mathscr{F}H_0(x) = 2 + i + \varepsilon(x+3) + h\left(2x^2 + 6x - 2\right),\\
&\mathscr{F}H_1(x) = 1 + i(x+3) + \varepsilon\left(2x^2 + 6x - 2\right) + h\left(4x^3 + 12x^2 - 4x - 1\right),\\
&\mathscr{F}H_2(x) = (x+3) + i\left(2x^2 + 6x - 2\right) + \varepsilon\left(4x^3 + 12x^2 - 4x - 1\right) + h\left(8x^4 + 24x^3 - 8x^2 - 3x - 3\right),\\
&\mathscr{F}H_3(x) = \left(2x^2 + 6x - 2\right) + i\left(4x^3 + 12x^2 - 4x - 1\right) + \varepsilon\left(8x^4 + 24x^3 - 8x^2 - 3x - 3\right) + h\left(16x^5 + 48x^4 - 16x^3 - 8x^2 - 12x + 2\right).
\end{aligned}$$

Note that the Francois hybrinomials generalise the Francois hybrid numbers, as they yield the Francois hybrid numbers for $x = 1$. That is, the Francois hybrid numbers are defined by

$$\mathscr{F}H_n = \mathscr{F}_n + i\mathscr{F}_{n+1} + \varepsilon\mathscr{F}_{n+2} + h\mathscr{F}_{n+3},$$

where $\mathscr{F}_n$ is the nth Francois number.

Definition 2.3. *The character of the Francois hybrinomials is*

$$C(\mathscr{F}H_n(x)) = \mathscr{F}_n^2(x) + \mathscr{F}_{n+1}^2(x) - 2\mathscr{F}_{n+1}(x)\mathscr{F}_{n+2}(x) - \mathscr{F}_{n+3}^2(x).$$

Theorem 2.4. *The norm of the nth Francois hybrinomial is*

$$\|\mathscr{F}H_n(x)\| = \sqrt{\left|\mathscr{F}_{n+1}^2(x) - 2\mathscr{F}_{n+1}(x)\mathscr{F}_{n+2}(x) - 4x^2\mathscr{F}_{n+2}^2(x) + 4x\mathscr{F}_{n+2}(x)\mathscr{F}_n(x)\right|}.$$

Proof. Considering Definition 2.3, we have

$$\begin{aligned}
\|\mathscr{F}H_n(x)\|^2 &= \left|\mathscr{F}_n^2(x) + \mathscr{F}_{n+1}^2(x) - 2\mathscr{F}_{n+1}(x)\mathscr{F}_{n+2}(x) - \mathscr{F}_{n+3}^2(x)\right|\\
&= \left|\mathscr{F}_n^2(x) + \mathscr{F}_{n+1}^2(x) - 2\mathscr{F}_{n+1}(x)\mathscr{F}_{n+2}(x) - \left(2x\mathscr{F}_{n+2}(x) - \mathscr{F}_n(x)\right)^2\right|\\
&= \left|\mathscr{F}_{n+1}^2(x) - 2\mathscr{F}_{n+1}(x)\mathscr{F}_{n+2}(x) - 4x^2\mathscr{F}_{n+2}^2(x) + 4x\mathscr{F}_{n+2}(x)\mathscr{F}_n(x)\right|.
\end{aligned}$$

□

Theorem 2.5. *The matrix form of the nth Francois hybrinomial is*

$$M(\mathscr{F}H_n(x)) = \begin{bmatrix} \mathscr{F}_n(x) + \mathscr{F}_{n+2}(x) & \mathscr{F}_{n+1}(x) - \mathscr{F}_{n+2}(x) + \mathscr{F}_{n+3}(x) \\ \mathscr{F}_{n+2}(x) - \mathscr{F}_{n+1}(x) + \mathscr{F}_{n+3}(x) & \mathscr{F}_n(x) - \mathscr{F}_{n+2}(x) \end{bmatrix}.$$

Proof. Using Definition 2.2 and the equivalents of the hybrid units, we have

$$M\left(\mathscr{F}H_n(x)\right)=\mathscr{F}_n(x)\begin{bmatrix}1&0\\0&1\end{bmatrix}+\mathscr{F}_{n+1}(x)\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\mathscr{F}_{n+2}(x)\begin{bmatrix}1&-1\\1&-1\end{bmatrix}+\mathscr{F}_{n+3}(x)\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

which gives the desired result. □

Theorem 2.6. *The Francois hybrinomials have the following recurrence relation*

$$\mathscr{F}H_n(x)=2x\mathscr{F}H_{n-1}(x)-\mathscr{F}H_{n-3}(x).$$

Proof.

$$\begin{aligned}2x\mathscr{F}H_{n-1}(x)-\mathscr{F}H_{n-3}(x) &= 2x\left(\mathscr{F}_{n-1}(x)+i\mathscr{F}_n(x)+\varepsilon\mathscr{F}_{n+1}(x)+h\mathscr{F}_{n+2}(x)\right)\\ &\qquad-\left(\mathscr{F}_{n-3}(x)+i\mathscr{F}_{n-2}(x)+\varepsilon\mathscr{F}_{n-1}(x)+h\mathscr{F}_n(x)\right)\\ &=\left(2x\mathscr{F}_{n-1}(x)-\mathscr{F}_{n-3}(x)\right)+i\left(2x\mathscr{F}_n(x)-\mathscr{F}_{n-2}(x)\right)\\ &\qquad+\varepsilon\left(2x\mathscr{F}_{n+1}(x)-\mathscr{F}_{n-1}(x)\right)+h\left(2x\mathscr{F}_{n+2}(x)-\mathscr{F}_n(x)\right)\\ &=\mathscr{F}_n(x)+i\mathscr{F}_{n+1}(x)+\varepsilon\mathscr{F}_{n+2}(x)+h\mathscr{F}_{n+3}(x)\\ &=\mathscr{F}H_n(x).\end{aligned}$$

□

Theorem 2.7. *The generating function of the Francois polynomials is*

$$g(t)=\sum_{n=0}^{\infty}\mathscr{F}_n(x)t^n=\frac{2+(1-4x)t+(3-x)t^2}{1-2xt+t^3}.$$

Proof.

$$\begin{aligned}g(t)=\sum_{n=0}^{\infty}\mathscr{F}_n(x)t^n &=\mathscr{F}_0(x)+\mathscr{F}_1(x)t+\mathscr{F}_2(x)t^2+\sum_{n=3}^{\infty}\mathscr{F}_n(x)t^n\\ &=2+t+(x+3)t^2+\sum_{n=3}^{\infty}\left(2x\mathscr{F}_{n-1}(x)-\mathscr{F}_{n-3}(x)\right)t^n\\ &=2+t+(x+3)t^2+2xt\sum_{n=3}^{\infty}\mathscr{F}_{n-1}(x)t^{n-1}-t^3\sum_{n=3}^{\infty}\mathscr{F}_{n-3}(x)t^{n-3}\\ &=2+t+(x+3)t^2+2xt\left(\sum_{n=1}^{\infty}\mathscr{F}_{n-1}(x)t^{n-1}-\mathscr{F}_0(x)-\mathscr{F}_1(x)t\right)-t^3\sum_{n=3}^{\infty}\mathscr{F}_{n-3}(x)t^{n-3}\\ &=2+t+(x+3)t^2+2xt\left(g(t)-2-t\right)-t^3g(t).\end{aligned}$$

Hence,

$$g(t)\left(1-2xt+t^3\right)=2+(1-4x)t+(3-x)t^2$$

gives the desired result. □

Note that for $x=1$ the generating function given in Theorem 2.7 turns into

$$g(t)=\sum_{n=0}^{\infty}\mathscr{F}_nt^n=\frac{2-3t+2t^2}{1-2t+t^3},$$

which is the generating function of the Francois numbers [2].

Corollary 2.8. *The generating function of the Francois hybrinomials is*

$$G(t)=\frac{\mathscr{F}H_0(x)+\left(\mathscr{F}H_1(x)-2x\mathscr{F}H_0(x)\right)t+\left(\mathscr{F}H_2(x)-2x\mathscr{F}H_1(x)\right)t^2}{1-2xt+t^3}.$$

Theorem 2.9. *The Binet-type formula for the Francois polynomials is*

$$\mathscr{F}_n(x)=A_1\phi^n+A_2\psi^n+A_3\xi^n,$$

where ϕ, ψ and ξ are the roots of the characteristic equation $t^3-2xt^2+1=0$ and A_1,A_2,A_3 are

$$A_1=\frac{2\phi^2+\phi(1-4x)+3-x}{(\phi-\psi)(\phi-\xi)},\quad A_2=\frac{2\psi^2+\psi(1-4x)+3-x}{(\psi-\phi)(\psi-\xi)},\quad A_3=\frac{2\xi^2+\xi(1-4x)+3-x}{(\xi-\phi)(\xi-\psi)}.$$

Proof. By decomposing the generating function of Francois polynomials into partial fractions, we obtain

$$\begin{aligned} g(t) &= \frac{2+(1-4x)t+(3-x)t^2}{1-2xt+t^3} \\ &= \frac{A_1}{1-\phi t}+\frac{A_2}{1-\psi t}+\frac{A_3}{1-\xi t} \\ &= \frac{A_1(1-\psi t)(1-\xi t)+A_2(1-\phi t)(1-\xi t)+A_3(1-\phi t)(1-\psi t)}{(1-\phi t)(1-\psi t)(1-\xi t)}. \end{aligned}$$

Thus, we have

$$2+(1-4x)t+(3-x)t^2 = A_1(1-\psi t)(1-\xi t)+A_2(1-\phi t)(1-\xi t)+A_3(1-\phi t)(1-\psi t).$$

With some computations, A_1, A_2 and A_3 are obtained as desired. Additionally,

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} \mathscr{F}_n(x)t^n \\ &= \frac{A_1}{1-\phi t}+\frac{A_2}{1-\psi t}+\frac{A_3}{1-\xi t} \\ &= A_1\sum_{n=0}^{\infty}\phi^n t^n + A_2\sum_{n=0}^{\infty}\psi^n t^n + A_3\sum_{n=0}^{\infty}\xi^n t^n \\ &= \sum_{n=0}^{\infty}(A_1\phi^n + A_2\psi^n + A_3\xi^n)t^n. \end{aligned}$$

Hence,

$$\mathscr{F}_n(x) = A_1\phi^n + A_2\psi^n + A_3\xi^n.$$

□

Theorem 2.10. *The Binet-type formula for the Francois hybrinomials is*

$$\mathscr{F}H_n(x) = A_1\underline{\phi}\phi^n + A_2\underline{\psi}\psi^n + A_3\underline{\xi}\xi^n,$$

where $\underline{\phi} = 1+i\phi+\varepsilon\phi^2+h\phi^3$, $\underline{\psi} = 1+i\psi+\varepsilon\psi^2+h\psi^3$, $\underline{\xi} = 1+i\xi+\varepsilon\xi^2+h\xi^3$ *and* A_1, A_2, A_3 *are as in Theorem 2.9.*

Proof. Considering Definition 2.2 and Theorem 2.9, we have

$$\begin{aligned} \mathscr{F}H_n(x) &= \mathscr{F}_n(x)+i\mathscr{F}_{n+1}(x)+\varepsilon\mathscr{F}_{n+2}(x)+h\mathscr{F}_{n+3}(x) \\ &= (A_1\phi^n+A_2\psi^n+A_3\xi^n)+i\left(A_1\phi^{n+1}+A_2\psi^{n+1}+A_3\xi^{n+1}\right) \\ &\qquad +\varepsilon\left(A_1\phi^{n+2}+A_2\psi^{n+2}+A_3\xi^{n+2}\right)+h\left(A_1\phi^{n+3}+A_2\psi^{n+3}+A_3\xi^{n+3}\right) \\ &= A_1\left(1+i\phi+\varepsilon\phi^2+h\phi^3\right)\phi^n + A_2\left(1+i\psi+\varepsilon\psi^2+h\psi^3\right)\psi^n + A_3\left(1+i\xi+\varepsilon\xi^2+h\xi^3\right)\xi^n \\ &= A_1\underline{\phi}\phi^n + A_2\underline{\psi}\psi^n + A_3\underline{\xi}\xi^n. \end{aligned}$$

□

Theorem 2.11. *For* $n \geq 1$, *the Francois polynomials and hybrinomials satisfy the following summation formulas, respectively*

i) $\displaystyle\sum_{i=0}^{n}\mathscr{F}_i(x) = \frac{-5x+6+\mathscr{F}_{n-1}(x)+\mathscr{F}_n(x)-\mathscr{F}_{n+1}(x)}{2(1-x)}$,

ii) $\displaystyle\sum_{i=0}^{n}\mathscr{F}H_i(x) = \frac{(1-2x)(\mathscr{F}H_0(x)+\mathscr{F}H_1(x))+\mathscr{F}H_2(x)+\mathscr{F}H_{n-1}(x)+\mathscr{F}H_n(x)-\mathscr{F}H_{n+1}(x)}{2(1-x)}$.

Proof. i) Using Theorem 2.6, the following equalities hold

$$\begin{aligned} \mathscr{F}_3(x) &= 2x\mathscr{F}_2(x)-\mathscr{F}_0(x), \\ \mathscr{F}_4(x) &= 2x\mathscr{F}_3(x)-\mathscr{F}_1(x), \\ &\vdots \\ \mathscr{F}_n(x) &= 2x\mathscr{F}_{n-1}(x)-\mathscr{F}_{n-3}(x). \end{aligned}$$

Summing the above equalities, we have

$$\sum_{i=3}^{n}\mathscr{F}_i(x) = 2x\sum_{i=2}^{n-1}\mathscr{F}_i(x) - \sum_{i=0}^{n-3}\mathscr{F}_i(x).$$

Then,

$$\begin{aligned} \sum_{i=0}^{n}\mathscr{F}_i(x)-\mathscr{F}_0(x)-\mathscr{F}_1(x)-\mathscr{F}_2(x) &= 2x\left(\sum_{i=0}^{n}\mathscr{F}_i(x)-\mathscr{F}_0(x)-\mathscr{F}_1(x)-\mathscr{F}_n(x)\right) \\ &\quad -\left(\sum_{i=0}^{n}\mathscr{F}_i(x)-\mathscr{F}_n(x)-\mathscr{F}_{n-1}(x)-\mathscr{F}_{n-2}(x)\right). \end{aligned}$$

We obtain

$$\begin{aligned}(2-2x)\sum_{i=0}^{n}\mathscr{F}_i(x) &= \mathscr{F}_0(x)+\mathscr{F}_1(x)+\mathscr{F}_2(x)-2x\left(\mathscr{F}_0(x)+\mathscr{F}_1(x)+\mathscr{F}_n(x)\right)+\mathscr{F}_n(x)+\mathscr{F}_{n-1}(x)+\mathscr{F}_{n-2}(x)\\ &= -5x+6+\mathscr{F}_{n-1}(x)+\mathscr{F}_n(x)-\mathscr{F}_{n+1}(x),\end{aligned}$$

which gives the desired result.

ii) The proof follows a similar approach to that of (i).

□

Now, we provide an example to illustrate Theorem 2.11.

Example 2.12. *The sum of the first four terms of the Francois polynomials is*

$$\mathscr{F}_0(x)+\mathscr{F}_1(x)+\mathscr{F}_2(x)+\mathscr{F}_3(x)=(2)+(1)+(x+3)+\left(2x^2+6x-2\right)=2x^2+7x+4.$$

The same calculation using Theorem 2.11 is as follows

$$\begin{aligned}\sum_{i=0}^{3}\mathscr{F}_i(x) &= \frac{-5x+6+\mathscr{F}_2(x)+\mathscr{F}_3(x)-\mathscr{F}_4(x)}{2(1-x)}\\ &= \frac{-5x+6+(x+3)+\left(2x^2+6x-2\right)-\left(4x^3+12x^2-4x-1\right)}{2(1-x)}\\ &= \frac{-4x^3-10x^2+6x+8}{2(1-x)}\\ &= 2x^2+7x+4.\end{aligned}$$

Next, we present several basic identities for the newly defined polynomials and hybrinomials, derived from their Binet-type formulas.

Theorem 2.13. *The Catalan identities for the Francois polynomials and hybrinomials are*

i)

$$\begin{aligned}\mathscr{F}_{n+r}(x)\mathscr{F}_{n-r}(x)-\mathscr{F}_n^2(x) &= A_1A_2\phi^{n-r}\psi^{n-r}(\phi^r-\psi^r)^2\\ &+A_1A_3\phi^{n-r}\xi^{n-r}(\phi^r-\xi^r)^2\\ &+A_2A_3\psi^{n-r}\xi^{n-r}(\psi^r-\xi^r)^2,\end{aligned}$$

ii)

$$\begin{aligned}\mathscr{F}H_{n+r}(x)\mathscr{F}H_{n-r}(x)-\mathscr{F}H_n^2(x) &= A_1A_2\phi^{n-r}\psi^{n-r}(\phi^r-\psi^r)\left(\underline{\phi}\,\underline{\psi}\phi^r-\underline{\psi}\,\underline{\phi}\psi^r\right)\\ &+A_1A_3\phi^{n-r}\xi^{n-r}(\phi^r-\xi^r)\left(\underline{\phi}\,\underline{\xi}\phi^r-\underline{\xi}\,\underline{\phi}\xi^r\right)\\ &+A_2A_3\psi^{n-r}\xi^{n-r}(\psi^r-\xi^r)\left(\underline{\psi}\,\underline{\xi}\psi^r-\underline{\xi}\,\underline{\psi}\xi^r\right).\end{aligned}$$

Proof. i) Considering Theorem 2.9, we have

$$\begin{aligned}\mathscr{F}_{n+r}(x)\mathscr{F}_{n-r}(x)-\mathscr{F}_n^2(x) &= \left(A_1\phi^{n+r}+A_2\psi^{n+r}+A_3\xi^{n+r}\right)\left(A_1\phi^{n-r}+A_2\psi^{n-r}+A_3\xi^{n-r}\right)-\left(A_1\phi^n+A_2\psi^n+A_3\xi^n\right)^2\\ &= A_1^2\phi^{2n}+A_1A_2\phi^{n+r}\psi^{n-r}+A_1A_3\phi^{n+r}\xi^{n-r}+A_2A_1\psi^{n+r}\phi^{n-r}+A_2^2\psi^{2n}+A_2A_3\psi^{n+r}\xi^{n-r}\\ &\quad+A_3A_1\xi^{n+r}\phi^{n-r}+A_3A_2\xi^{n+r}\psi^{n-r}+A_3^2\xi^{2n}-A_1^2\phi^{2n}-A_2^2\psi^{2n}-A_3^2\xi^{2n}\\ &\quad-2A_1A_2\phi^n\psi^n-2A_1A_3\phi^n\xi^n-2A_2A_3\psi^n\xi^n\\ &= A_1A_2\left(\phi^{n+r}\psi^{n-r}+\psi^{n+r}\phi^{n-r}-2\phi^n\psi^n\right)+A_1A_3\left(\phi^{n+r}\xi^{n-r}+\xi^{n+r}\phi^{n-r}-2\phi^n\xi^n\right)\\ &\quad+A_2A_3\left(\psi^{n+r}\xi^{n-r}+\xi^{n+r}\psi^{n-r}-2\psi^n\xi^n\right)\\ &= A_1A_2\phi^{n-r}\psi^{n-r}(\phi^r-\psi^r)^2+A_1A_3\phi^{n-r}\xi^{n-r}(\phi^r-\xi^r)^2+A_2A_3\psi^{n-r}\xi^{n-r}(\psi^r-\xi^r)^2.\end{aligned}$$

ii) The proof follows a similar approach to that of (i).

□

Corollary 2.14. *The Cassini identities for the Francois polynomials and hybrinomials are*

i)

$$\begin{aligned}\mathscr{F}_{n+1}(x)\mathscr{F}_{n-1}(x)-\mathscr{F}_n^2(x) &= A_1A_2\phi^{n-1}\psi^{n-1}(\phi-\psi)^2\\ &+A_1A_3\phi^{n-1}\xi^{n-1}(\phi-\xi)^2\\ &+A_2A_3\psi^{n-1}\xi^{n-1}(\psi-\xi)^2,\end{aligned}$$

ii)

$$\begin{aligned}\mathscr{F}H_{n+1}(x)\mathscr{F}H_{n-1}(x)-\mathscr{F}H_n^2(x) &= A_1A_2\phi^{n-1}\psi^{n-1}(\phi-\psi)\left(\underline{\phi}\,\underline{\psi}\phi-\underline{\psi}\,\underline{\phi}\psi\right)\\ &+A_1A_3\phi^{n-1}\xi^{n-1}(\phi-\xi)\left(\underline{\phi}\,\underline{\xi}\phi-\underline{\xi}\,\underline{\phi}\xi\right)\\ &+A_2A_3\psi^{n-1}\xi^{n-1}(\psi-\xi)\left(\underline{\psi}\,\underline{\xi}\psi-\underline{\xi}\,\underline{\psi}\xi\right).\end{aligned}$$

Theorem 2.15. *The d'Ocagne identities for the Francois polynomials and hybrinomials are* *i)*

$$\begin{aligned}\mathscr{F}_m(x)\mathscr{F}_{n+1}(x)-\mathscr{F}_{m+1}(x)\mathscr{F}_n(x) &= A_1A_2(\phi-\psi)(\phi^n\psi^m-\psi^n\phi^m)\\ &+A_1A_3(\phi-\xi)(\phi^n\xi^m-\xi^n\phi^m)\\ &+A_2A_3(\psi-\xi)(\psi^n\xi^m-\xi^n\psi^m),\end{aligned}$$

ii)

$$\begin{aligned}\mathscr{F}H_m(x)\mathscr{F}H_{n+1}(x)-\mathscr{F}H_{m+1}(x)\mathscr{F}H_n(x) &= A_1A_2(\phi-\psi)\left(\underline{\phi\psi}\phi^n\psi^m-\underline{\psi\phi}\psi^n\phi^m\right)\\ &+A_1A_3(\phi-\xi)\left(\underline{\phi\xi}\phi^n\xi^m-\underline{\xi\phi}\xi^n\phi^m\right)\\ &+A_2A_3(\psi-\xi)\left(\underline{\psi\xi}\psi^n\xi^m-\underline{\xi\psi}\xi^n\psi^m\right)\end{aligned}$$

Proof. According to the theorem 2.9, we obtain

i)

$$\begin{aligned}&(A_1\phi^m+A_2\psi^m+A_3\xi^m)\left(A_1\phi^{n+1}+A_2\psi^{n+1}+A_3\xi^{n+1}\right)-\left(A_1\phi^{m+1}+A_2\psi^{m+1}+A_3\xi^{m+1}\right)(A_1\phi^m+A_2\psi^m+A_3\xi^m)\\ &=A_1^2\phi^{m+n+1}+A_1A_2\phi^m\psi^{n+1}+A_1A_3\phi^m\xi^{n+1}+A_2A_1\psi^m\phi^{n+1}+A_2^2\psi^{m+n+1}+A_2A_3\psi^m\xi^{n+1}\\ &\quad+A_3A_1\xi^m\phi^{n+1}+A_3A_2\xi^m\psi^{n+1}+A_3^2\xi^{m+n+1}-A_1^2\phi^{m+n+1}-A_1A_2\phi^{m+1}\psi^n-A_1A_3\phi^{m+1}\xi^n\\ &\quad-A_2A_1\psi^{m+1}\phi^n-A_2^2\psi^{m+n+1}-A_2A_3\psi^{m+1}\xi^n-A_3A_2\xi^{m+1}\psi^n-A_3A_2\xi^{m+1}\psi^n-A_3^2\xi^{m+n+1}\\ &=A_1A_2\left(\phi^m\psi^{n+1}+\psi^m\phi^{n+1}-\phi^{m+1}\psi^n-\psi^{m+1}\phi^n\right)+A_1A_3\left(\phi^m\xi^{n+1}+\xi^m\phi^{n+1}-\phi^{m+1}\xi^n-\xi^{m+1}\phi^n\right)\\ &\quad+A_2A_3\left(\psi^m\xi^{n+1}+\xi^m\psi^{n+1}-\psi^{m+1}\xi^n-\xi^{m+1}\psi^n\right)\\ &=A_1A_2(\phi^m\psi^n(\psi-\phi)+\psi^m\phi^n(\phi-\psi))+A_1A_3(\phi^m\xi^n(\xi-\phi)+\xi^m\phi^n(\phi-\xi))\\ &\quad+A_2A_3(\psi^m\xi^n(\xi-\psi)+\xi^m\psi^n(\psi-\xi))\\ &=A_1A_2(\phi-\psi)(\phi^n\psi^m-\psi^n\phi^m)+A_1A_3(\phi-\xi)(\phi^n\xi^m-\xi^n\phi^m)+A_2A_3(\psi-\xi)(\psi^n\xi^m-\xi^n\psi^m).\end{aligned}$$

ii) The proof follows a similar approach to that of (i).

□

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