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Combinatorial Results on Some Nilpotent Subsemigroups of a Semigroup of Order-Decreasing Full Transformations

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Abstract— Let $\mathcal{D}_n$ be the semigroup of all order-decreasing full transformations on $X_n = \{1, 2, \ldots, n\}$ under its natural order, and let $N(\mathcal{D}_n)$ be the subsemigroup of all nilpotent elements of $\mathcal{D}_n$, where $n \in \mathbb{Z}^+$, the set of all positive integers. In this paper, for $1 \leq r \leq n-1$, we determine the cardinality and rank of nilpotent subsemigroup $N(\mathcal{D}_{n,r}) = \{\alpha \in N(\mathcal{D}_n) : |\mathrm{im}(\alpha)| \leq r\}$ of $N(\mathcal{D}_n)$. We then find the cardinalities of $\mathcal{D}_n^{2,2}$ and $N(\mathcal{D}_n)^{p,p}$. Furthermore, we present an alternative combinatorial approach to determine the cardinality and rank of $\mathcal{D}_n(\xi) = \{\alpha \in \mathcal{D}_n : \alpha^k = \xi, \text{ for some } k \in \mathbb{Z}^+\}$, for all idempotent $\xi \in \mathcal{D}_n$ within the scope of this study. Here, for all $\alpha \in \mathcal{D}_n$, $\mathrm{im}^c(\alpha) = \{t \in \mathrm{im}(\alpha) : |t\alpha^{-1}| \geq 2\}$. Besides, for all $2 \leq p \leq r \leq n$ and $\mathcal{C} \in \{N(\mathcal{D}_n), \mathcal{D}_n\}$, $\mathcal{C}^p = \left\{\alpha \in \mathcal{C} : t \in \mathrm{im}^c(\alpha) \text{ and } |t\alpha^{-1}| = p\right\}$ and $\mathcal{C}^{p,r} = \left\{\alpha \in \mathcal{C}^p : \left|\bigcup_{t \in \mathrm{im}^c(\alpha)} t\alpha^{-1}\right| = r\right\}$.

Keywords — *Order-decreasing, transformations, collapse, nilpotent, rank*

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1. Introduction

For an arbitrary set $X_n = \{1, 2, \ldots, n\}$, ordered standard way, such that $n \in \mathbb{Z}^+$, the set of all positive integers, the notation $\mathcal{T}_n$ denote the full transformation semigroup on X_n, i.e., all mappings from X to X, under the operation of composition. We compose the functions from left to right. A transformation $\alpha \in \mathcal{T}_n$ is called order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X_n$, and decreasing if $x\alpha \leq x$ ($x\alpha \geq x$), for all $x \in X_n$. Let

$$\mathcal{D}_n = \{\alpha \in T_n : \forall x \in X_n,\, x\alpha \leq x\}$$

be the semigroup of all order-decreasing full transformations. For any transformation $\alpha \in \mathcal{T}_n$, the collapse, the image, and the fix of α are defined as follows, respectively:

$$c(\alpha) = \bigcup_{t \in \mathrm{im}(\alpha)} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}$$

$$\mathrm{im}(\alpha) = \{x\alpha : x \in X_n\}$$

and

$$\mathrm{fix}(\alpha) = \{x \in X_n : x\alpha = x\}$$

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For any semigroup S, an element e in S is said to be an idempotent if $e^2 = e$. It is known that $\alpha \in E(\mathcal{T}_n)$ if and only if $\mathrm{fix}(\alpha) = \mathrm{im}(\alpha)$. Let S be a semigroup with zero element 0. An element a in S is said to be a nilpotent element if there exists a $k \in \mathbb{Z}^+$ such that $a^k = 0$. A subsemigroup $T \subseteq S$ is called nilpotent if there exists an $m \in \mathbb{Z}^+$ such that $T^m = \{0\}$ or T is a nilpotent as a semigroup with different 0 which means that there exists an idempotent $e \in T$ differs from 0 such that $T^k = \{e\}$, for some $k \in \mathbb{Z}^+$. It is proven in [1] that a finite semigroup S with 0 is nilpotent if and only if the unique idempotent of S is 0. The set of all nilpotent subsemigroups of S is partially ordered with respect to inclusions, and each maximal elements of this set are called maximal nilpotent subsemigroup of S. Throughout this study, let $E(S)$ and $N(S)$ denote the set of all idempotent and nilpotent elements of S, respectively. It should be noted that $N(S)$ may not be a subsemigroup of S.

A subset W of a semigroup S is a generating set of S if every element of S is expressible as a product of the elements of W. Further, $\langle W \rangle$ denotes the subsemigroup generated by a non-empty subset W of S. If S is finitely generated, then its rank and idempotent rank are defined as follows, respectively:

$$\mathrm{rank}(S) = \min\{|W| : W \subseteq S \text{ and } \langle W \rangle = S\}$$

and

$$\mathrm{idrank}(S) = \min\{|W| : W \subseteq E(S) \text{ and } \langle W \rangle = S\}$$

The problem of determining the cardinality and aforementioned ranks of a certain finite transformation semigroups is closely related to combinatorics, is classical, and has been extensively explored. The authors [2] considered the ideal (and hence subsemigroup) $\mathcal{K}_{n,r} = \{\alpha \in \mathcal{T}_n : \mathrm{im}(\alpha) \leq r\}$ and demonstrated that $\mathrm{rank}(\mathcal{T}_{n,r}) = \mathrm{idrank}(\mathcal{T}_{n,r}) = S(n,r)$. Afterward, Ruškuc [3] gave an alternative proof for the rank of $\mathcal{K}_{n,r}$. These results provided that to be applicable in other semigroups as well as $\mathcal{D}_n$. This was followed by a number of articles on $\mathcal{D}_n$ in terms of algebraic, combinatorial, and (idempotent) rank properties [4–9]. In particular,

$$\mathrm{rank}(\mathcal{D}_n) = \mathrm{idrank}(\mathcal{D}_n) = \frac{n(n-1)}{2}$$

$$\mathrm{rank}(N(\mathcal{D}_n)) = (n-2)\,!\,(n-2)$$

$$|\mathcal{D}_n| = n!$$

$$|N(\mathcal{D}_n)| = (n-1)!$$

and

$$|E(\mathcal{D}_n)| = \sum_{r=0}^{n} S(n,r)$$

For further research on transformations semigroups within the scope of this study, see [10–12], and for more information about semigroup theory, see [1, 13].

The rest of the paper is organized as follows: Section 2 provides some combinatorial results on a nilpotent subsemigroup of $\mathcal{D}_n$ and certain invariants related to the collapse. Section 3 demonstrates the efficacy of enumerative techniques by providing alternative proofs for the cardinality and rank of maximal nilpotent subsemigroups of $\mathcal{D}_n$. The last section concludes the paper.

2. Cardinality and Rank Properties

Given a finite semigroup S with zero 0, S is nilpotent if and only if $S^m = \{0\}$, for some positive integer m. Let S be a finite nilpotent semigroup with $|S| \geq 2$. In [15], it was shown that $S \setminus S^2$ is the minimum generating set of S, and thus

$$\text{rank}(S) = |S| - |S^2|$$

Therefore, throughout this paper, we consider non-trivial nilpotent semigroups by determining their ranks.

For $1 \leq r \leq n-1$, let α and β be two elements in

$$N(\mathcal{D}_{n,r}) = \{\alpha \in N(\mathcal{D}_n) : |\text{im}(\alpha)| \leq r\}$$

Given that $\text{im}(\alpha\beta) \subseteq \text{im}(\beta)$ implies $|\text{im}(\alpha\beta)| \leq r$, then $N(\mathcal{D}_{n,r})$ is a nilpotent subsemigroup of both $\mathcal{D}_n$ and $N(\mathcal{D}_n)$ with the zero element 0_n. It can be observed that $N(\mathcal{D}_{n,1}) = \{0_n\}$ and $N(\mathcal{D}_{n,n-1}) = N(\mathcal{D}_n)$. We aim to discover a formula for the cardinality of $N(\mathcal{D}_{n,r})$, for $2 \leq r \leq n-1$, and then utilize this formula to determine the rank of $N(\mathcal{D}_{n,r})$.

Lemma 2.1. For $2 \leq r \leq n-1$,

$$|N(\mathcal{D}_{n,r})| = \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(k-i)^{n-1}$$

Proof. For $\alpha \in \mathcal{D}_{n-1}$, let $\widehat{\alpha} : X_n \to X_n$ be defined by $1\widehat{\alpha} = 1\alpha = 1$ and $i\widehat{\alpha} = (i-1)\alpha$, for all $2 \leq i \leq n$. It can be observed that $\widehat{\alpha} \in N(\mathcal{D}_n)$. For all $\alpha \in \mathcal{D}_{n-1}$, since the function $\varphi : \mathcal{D}_{n-1} \to N(\mathcal{D}_n)$ defined by $(\alpha)\varphi = \widehat{\alpha}$ is a bijection, then $|\mathcal{D}_{n-1}| = |N(\mathcal{D}_n)|$. For $1 \leq k \leq r \leq n-1$, consider the sets

$$\mathcal{D}_n(k) = \{\alpha \in \mathcal{D}_n : |\text{im}(\alpha)| = k\} \quad \text{and} \quad N(\mathcal{D}_n(k)) = \{\alpha \in N(\mathcal{D}_n) : |\text{im}(\alpha)| = k\}$$

It is known from [6] that $|\mathcal{D}_n(k)| = \sum_{i=0}^{k-1}(-1)^i\binom{n+1}{i}(k-i)^n$. It can be observed from the aforementioned bijection that $\alpha \in \mathcal{D}_{n-1}(k)$ if and only if $\widehat{\alpha} \in N(\mathcal{D}_n(k))$. Hence,

$$|N(\mathcal{D}_n(k))| = |\mathcal{D}_{n-1}(k)| = \sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(k-i)^{n-1}$$

Since $N(\mathcal{D}_{n,r})$ is the union of disjoint sets $N(\mathcal{D}_n(k))$, for $1 \leq k \leq r$, then

$$|N(\mathcal{D}_{n,r})| = \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(k-i)^{n-1}$$

□

Afterward, we present one of the key results of this study.

Theorem 2.2. For $2 \leq r \leq n-2$,

$$\text{rank}(N(\mathcal{D}_{n,r})) = \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(k-i)^{n-1} - \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n-1}{i}(k-i)^{n-2}$$

Proof. Since $N(\mathcal{D}_{n,r})$ is a nilpotent semigroup, it follows that

$$\text{rank}(N(\mathcal{D}_{n,r})) = |N(\mathcal{D}_{n,r}) \setminus N(\mathcal{D}_{n,r})^2| = |N(\mathcal{D}_{n,r})| - |N(\mathcal{D}_{n,r})^2|$$

The cardinality of $N(\mathcal{D}_{n,r})^2$ can be calculated by constructing the well-defined bijection $f : N(\mathcal{D}_{n-1,r}) \to N(\mathcal{D}_{n,r})^2$ in the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 \\ 1 & 1 & c_2 & c_3 & \cdots & c_{n-1} \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 1 & 1 & c_2 & c_3 & \cdots & c_{n-1} \end{pmatrix}$$

where $1 \leq c_i \leq i$, for $2 \leq i \leq n-1$. As a result, it follows from Lemma 2.1 that

$$|N(\mathcal{D}_{n,r})^2| = |N(\mathcal{D}_{n-1,r})| = \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n-1}{i}(k-i)^{n-2}$$

and thus

$$\text{rank}(N(\mathcal{D}_{n,r})) = \sum_{k=1}^{r-1}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(k-i)^{n-1} - \sum_{k=1}^{r}\sum_{i=0}^{k-1}(-1)^i\binom{n-1}{i}(k-i)^{n-2}$$

□

Example 2.3. Let $N(\mathcal{D}_{6,4}) = \{\alpha \in N(\mathcal{D}_6) : |\text{im}(\alpha)| \leq 4\}$. Then,

$$|N(\mathcal{D}_6)(1)| = 1$$

$$|N(\mathcal{D}_6)(2)| = \sum_{i=0}^{1}(-1)^i\binom{6}{i}(2-i)^5 = 19$$

$$N(\mathcal{D}_6)(3)| = \sum_{i=0}^{2}(-1)^i\binom{6}{i}(3-i)^5 = 66$$

$$|N(\mathcal{D}_6)(4)| = \sum_{i=0}^{3}(-1)^i\binom{6}{i}(4-i)^5 = 26$$

and

$$|N(\mathcal{D}_{6,4}| = \sum_{k=1}^{4}\sum_{i=0}^{k-1}(-1)^i\binom{6}{i}(k-i)^5 = 112$$

Furthermore,

$$|N(\mathcal{D}_{6,4})^2| = |N(\mathcal{D}_{5,4})| = |N(\mathcal{D}_5)| = \sum_{k=1}^{4}\sum_{i=0}^{k-1}(-1)^i\binom{5}{i}(k-i)^4 = 24$$

Therefore,

$$\text{rank}(N(\mathcal{D}_{6,4})) = |N(\mathcal{D}_{6,4}| - |N(\mathcal{D}_{6,4})^2| = 112 - 24 = 88$$

Lemma 2.4. For $n \geq 2$,

$$|\mathcal{D}_n^{2,2}| = 2^{n-1} - 1$$

Proof. For a given $\alpha \in \mathcal{D}_n^{2,2}$, there exists an $i \in \text{im}(\alpha)$ such that $|i\alpha^{-1}| = 2$ and $\min\{i\alpha^{-1}\} = i$. Thus,

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & \{i, i+1\} & \{i+2\} & \cdots & \{n\} \\ 1 & 2 & \cdots & i-1 & i & (i+2)\alpha & \cdots & n\alpha \end{pmatrix}$$

where $1 \leq i \leq n-1$. It can be observed that there is only one transformation in $\mathcal{D}_n^{2,2}$, for $i = n-1$. Hence, suppose that $i \neq n-1$. Since each block $\{j\}$ is a singleton for $i+2 \leq j \leq n$, we deduce that there are exactly two possibilities for $j\alpha$. Consequently, the number of transformations in $\mathcal{D}_n^{2,2}$ for a fixed i is given by 2^{n-i-1}. Since $1 \leq i \leq n-2$, then

$$\left|\mathcal{D}_n^{2,2}\right| = 1 + \sum_{i=1}^{n-2} 2^{n-i-1} = 2^{n-1} - 1$$

□

Lemma 2.5. For $n \geq 3$ and $2 \leq p \leq n-1$, let $\left|1\alpha^{-1}\right| = p$ and $X_n \setminus 1\alpha^{-1} = \{b_1, \ldots, b_{n-p}\}$. Then,

$$|N(\mathcal{D}_n)^{p,p}| = (b_1 - 2)(b_2 - 3)(b_3 - 3)\ldots(b_{n-k} - 3)$$

PROOF. Given $\alpha \in N(\mathcal{D}_n)^{p,p}$ the conditions $1\alpha = 2\alpha = 1$ implies that $|1\alpha^{-1}| = p$. If $X_n \setminus 1\alpha^{-1} = \{b_1, \ldots, b_{n-p}\}$, then

$$\alpha = \begin{pmatrix} A & \{b_1\} & \{b_2\} & \cdots & \{b_{n-p}\} \\ 1 & b_1\alpha & b_2\alpha & \cdots & b_{n-p}\alpha \end{pmatrix}$$

Recall that the conditions $1\alpha = 2\alpha = 1$ implies that $b_j \geq 3$, for each $1 \leq j \leq n-p$. Since $\alpha \in N(\mathcal{D}_n)$ and $\{b_j\}$ is a singletion, we deduce that there are exactly $b_1 - 2$ possibilities, for $b_1\alpha$, and $b_i - 3$ possibilities, for $b_i\alpha$, where $2 \leq i \leq n-p$. Hence,

$$|N(\mathcal{D}_n)^{p,p}| = (b_1 - 2)(b_2 - 3)(b_3 - 3)\ldots(b_{n-k} - 3)$$

□

Example 2.6. Consider the sets

$$\mathcal{D}_4^{2,2} = \left\{ \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} \\ 1 & 3 & 4 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \{1\} & \{2,3\} & \{4\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2,3\} & \{4\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2\} & \{3,4\} \\ 1 & 2 & 3 \end{pmatrix} \right\}$$

and

$$N(\mathcal{D}_5)^{3,3} = \left\{ \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} \{1,2,3\} & \{4\} & \{5\} \\ 1 & 3 & 4 \end{pmatrix} \right\}$$

Thus, $\left|\mathcal{D}_4^{2,2}\right| = 2^{4-1} - 1 = 7$ and $\left|N(\mathcal{D}_5)^{3,3}\right| = (4-2)(5-3) = 4$, where $1\alpha^{-1} = \{1,2,3\}$ and $X_n \setminus 1\alpha^{-1} = \{4,5\} = \{b_1, b_2\}$.

3. Maximal Nilpotent Subsemigroups of $\mathcal{D}_n$

This section demonstrates the efficacy of enumerative techniques by providing alternative proofs for the cardinality and rank of maximal nilpotent subsemigroups of $\mathcal{D}_n$. If $\alpha \in \mathcal{D}_n$, then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

to indicate that $\mathrm{im}(\alpha) = \{1 = a_1, a_2, \ldots, a_r\}$ and $a_i\alpha^{-1} = A_i$, for each $1 \leq i \leq r$, and A_1, A_2, …, A_r also called blocks of α are all non-empty. It is known that every transformation $\alpha \in \mathcal{D}_n$ is idempotent if and only if $a_i = \min A_i$, for each $1 \leq i \leq r$. Every transformation $\alpha \in \mathcal{D}_n$ is nilpotent if and only if $\mathrm{fix}(\alpha) = \{1\}$. In this paper, we denote the zero and the identity elements of $\mathcal{D}_n$ by 0_n and 1_n, respectively.

For each $\xi \in E(\mathcal{D}_n)$, let $\mathcal{D}_n(\xi) = \{\alpha \in \mathcal{D}_n : \alpha^k = \xi,\ \text{for some } k \in \mathbb{Z}^+\}$ be the maximum nilpotent subsemigroup of $\mathcal{D}_n$ with the zero element ξ. For any $\xi \in E(\mathcal{D}_n)$, the cardinality and rank of $\mathcal{D}_n(\xi)$ were computed in [14].

In this section, we demonstrate the efficacy of enumerative techniques by providing alternative proofs for the cardinality and rank of $\mathcal{D}_n(\xi)$. These proofs, derived through distinct block enumerations of ξ, highlight the potency of these techniques in tackling complex combinatorial challenges.

Theorem 3.1. For any $\xi \in E(\mathcal{D}_n)$, let $\mathrm{fix}(\xi) = \{1 = a_1 < a_2 < \cdots < a_r\}$ and A_i be blocks of ξ with $|A_i| = k_i$, for each $1 \leq i \leq r$. Then,

$$|\mathcal{D}_n(\xi)| = \prod_{i=1}^{r} (k_i - 1)!$$

PROOF. Let $A_i = \{a_i + s_{i_1}, a_i + s_{i_2}, \ldots, a_i + s_{i_{k_i}}\}$ with $s_{i_1} = 0$. For $\alpha \in \mathcal{D}_n(\xi)$, if $(a_i + s_{i_j})\alpha = a_i + s_{i_{m_j}}$, for $1 \le j \le k_i$, then $m_1 = 1$ and $m_j \in \{1, 2, \ldots, j-1\}$, for each $2 \le j \le k_i$. Consider

$$\alpha_i = \begin{pmatrix} 1 & 2 & \cdots & k_i \\ m_1 & m_2 & \cdots & m_{k_i} \end{pmatrix}$$

It can be observed that $\alpha_i \in N(\mathcal{D}_{k_i})$, for each $1 \le i \le r$. Therefore, the function

$$f : \mathcal{D}_n(\xi) \to N(\mathcal{D}_{k_1}) \times N(\mathcal{D}_{k_2}) \times \cdots \times N(\mathcal{D}_{k_r})$$

defined by $\alpha f = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a well-defined bijection. Thus,

$$|\mathcal{D}_n(\xi)| = \prod_{i=1}^{r} (k_i - 1)!$$

□

Theorem 3.2. For any $\xi \in E(\mathcal{D}_n)$, let $\operatorname{fix}(\xi) = \{1 = a_1 < a_2 < \cdots < a_r\}$ and A_i be blocks of ξ with $|A_i| = k_i$, for each $1 \le i \le r$. Then,

$$\operatorname{rank}(\mathcal{D}_n(\xi)) = \prod_{i=1}^{r} (k_i - 1)! - \prod_{i=1}^{r} (k_i - 2)!$$

PROOF. Let $A_i = \{a_i + s_{i_1}, a_i + s_{i_2}, \ldots, a_i + s_{i_{k_i}}\}$ with $s_{i_1} = 0$. For $\alpha \in \mathcal{D}_n(\xi)^2$, let $(a_i + s_{ij})\alpha = a_i + s_{i_{m_j}}$, for $1 \le j \le k_i$. Then, $m_1 = m_2 = 1$ and $m_j \in \{1, \ldots, j-2\}$, for each $3 \le j \le k_i$. Consider

$$\alpha_i = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & k_i \in \{1, 2\} \\ \begin{pmatrix} 1 & 2 & \cdots & k_i - 1 \\ m_1 & m_2 & \cdots & m_{k_i - 1} \end{pmatrix}, & k_i \ge 3 \end{cases}$$

Assuming that $N(\mathcal{D}_{k_i-1}) = N(\mathcal{D}_1)$, for $k_i = 1$, then $\alpha_i \in N(\mathcal{D}_{k_i-1})$, for each $1 \le i \le r$. Consequently, the function

$$f : \mathcal{D}_n(\xi)^2 \to N(\mathcal{D}_{k_1-1}) \times N(\mathcal{D}_{k_2-1}) \times \cdots \times N(\mathcal{D}_{k_r-1})$$

defined by $\alpha f = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a well-defined bijection. By taking into account the assumption that $N(\mathcal{D}_{k_i-1}) = N(\mathcal{D}_1)$ when $k_i = 1$, that is, $|N(\mathcal{D}_{k_i-1})| = (k_i - 2)! = 1$ for $k_i = 1$, the result follows from the fact that $\mathcal{D}_n(\xi) \setminus \mathcal{D}_n(\xi)^2$ is the minimum generating set of $\mathcal{D}_n(\xi)$. Therefore,

$$\operatorname{rank}(\mathcal{D}_n(\xi)) = \left|\mathcal{D}_n(\xi) \setminus \mathcal{D}_n(\xi)^2\right| = \left|\mathcal{D}_n(\xi)\right| - \left|\mathcal{D}_n(\xi)^2\right| = \prod_{i=1}^{r} (k_i - 1)! - \prod_{i=1}^{r} (k_i - 2)!$$

□

4. Conclusion

In this study, we determined the cardinality and rank of the nilpotent subsemigroup $N(\mathcal{D}_{n,r})$ of $N(\mathcal{D}_n)$ and, consequently, of $\mathcal{D}_n$. We also found the cardinalities of $\mathcal{D}_n^{2,2}$ and $N(\mathcal{D}_n)^{p,p}$. Furthermore, we provided an alternative combinatorial approach to determine the cardinality and rank of $\mathcal{D}_n(\xi)$ for each $\xi \in \mathcal{D}_n$. Future studies may delve deeper into the structural properties of nilpotent subsemigroups within other transformation semigroups, expanding the scope beyond $\mathcal{D}_n$. Moreover, exploring the interrelations between combinatorial approaches and other algebraic methods could yield new insights and simplify complex calculations.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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