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Estimates on the Simpson Type Inequalities via Tempered Fractional Integrals

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Abstract

In this paper, we determine the upper and lower bounds for Simpson-type inequalities involving tempered fractional integrals, focusing on functions with bounded second derivatives. Additionally, by considering specific parameter values in the results, we recover previous studies, thus generalizing the findings of earlier research.

Keywords: *Bounded second derivatives; Differentiable function; Riemann-Liouville fractional integrals; Simpson-type inequalities; Tempered fractional integrals.*
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1. Introduction

Simpson's inequality is related to integral approximations in mathematics and it is often used in the context of Simpson's rule. This inequality provides an approximation to the true value of the integral of a function and includes error bounds to determine the accuracy of this approximation. Simpson's inequality can also be thought of as an error analysis of this rule and also provides an upper bound depending on the fourth derivative of the function. This inequality is stated as follows [15] :

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable function on (a,b) and let us consider $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

Recently, several researchers have highlighted Simpson-type inequalities for various kinds of functions, acknowledging their significant geometric relevance and wide range of applications. For example, Dragomir et al. [15] gave the Simpson type inequalities for function of bounded variation. Moreover, Sarikaya [37] and Alomari [2] analysed Simpson's inequality in the context of convex functions and related it to Hermite-Hadamard type inequalities. Also, other works have made important contributions to both numerical analysis and mathematical inequalities by adapting Simpson's inequality to different application areas [16, 28, 33, 30]. Studies on Simpson-type inequalities have also been conducted for functions, including those with twice differentiable, convex absolute values of first derivatives, and p-convex functions.[1, 34, 20, 22, 38, 39, 5]
Conversely, fractional calculus is a mathematical field that deals with derivatives and integrals of non-integer orders. It plays a crucial role in generalizing classical calculus, modeling complex systems, solving fractional differential equations, studying fractals, and has diverse applications in science and engineering. One of the oldest and fundamental building blocks of fractional calculus theory is the Riemann-Liouville fractional integral operator providing a powerful tool for the analysis of differential and integral inequalities as follows:

Definition 1.2. *[25, 18] Let f be the function with $f\in L_1[a,b]$. The Riemann-Liouville integrals of order $\alpha>0$ are:*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ \ x>a,$$

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and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\ \ x<b.$$

Here, Γ is the Gamma function and is described as follows:

$$\Gamma(\alpha)=\int_{0}^{\infty}e^{-u}u^{\alpha-1}du.$$

Taking $\alpha=1$ in the Riemann-Liouville integrals yields the classical integral.

Riemann-Liouville operator generalised the classical concept of integration, allowing a more flexible analysis structure. The reader is referred to Simpson type inequalities with fractional integrals [13, 14, 12, 23, 26, 35, 4, 29].
The advent of fractional derivatives and integrals like the Caputo-Fabrizio [10], Atangana-Baleanu [3] versions has significantly enhanced the importance and expanded the applications of this field. In order to create a wider range of applications, the tempered fractional integral operator was developed. The tempered operator introduces a modification to the Riemann-Liouville integral by adding an exponential weighting factor to the functions, thus providing both a new perspective on the theory and more precise results in practical applications. This transition increased the applicability of fractional integral inequalities to more complex systems and reinforced the importance of fractional calculus in modern research fields.
Now, we give the incomplete gamma function definiton which is used in the tempered fractional operator.

Definition 1.3. *[32] For the real numbers $\alpha>0$ and $x,\lambda\geq 0$, the λ-incomplete gamma function is defined as follows:*

$$\Upsilon_{\lambda}(\alpha,x)=\int_{0}^{x}t^{\alpha-1}e^{-\lambda t}dt.$$

If $\lambda=1$, incomplete gamma function occurs[11],

$$\Upsilon(\alpha,x)=\int_{0}^{x}t^{\alpha-1}e^{-t}dt.$$

Remark 1.4. *[32] For the real numbers $\alpha>0$; x, $\lambda\geq 0$ and $a<b$, there are the following equalities:*

i) $\Upsilon_{\lambda(b-a)}(\alpha,1)=\int_{0}^{1}t^{\alpha-1}e^{-\lambda(b-a)t}dt=\frac{\Upsilon_{\lambda}(\alpha,b-a)}{(b-a)^{\alpha}}$,

ii) $\int_{0}^{1}\Upsilon_{\lambda(b-a)}(\alpha,x)\,dx=\frac{\Upsilon_{\lambda}(\alpha,b-a)}{(b-a)^{\alpha}}-\frac{\Upsilon_{\lambda}(\alpha+1,b-a)}{(b-a)^{\alpha+1}}$.

Tempered fractional integral operators are as follows [27, 31]:

Definition 1.5. *Let $f\in L_1[a,b]$ and $\alpha>0,\lambda\geq 0$. The tempered fractional integral operators $\mathscr{J}_{a+}^{(\alpha,\lambda)}f(x)$ and $\mathscr{J}_{b-}^{(a,\lambda)}f(x)$ are presented by*

$$\mathscr{J}_{a+}^{(\alpha,\lambda)}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}e^{-\lambda(x-t)}f(t)dt,\ \ x\in[a,b],$$

and

$$\mathscr{J}_{b-}^{(\alpha,\lambda)}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}e^{-\lambda(t-x)}f(t)dt,\ \ \ x\in[a,b].$$

Tempered fractional calculus is an extension of the classical fractional calculus. If $\lambda=0$, then the tempered fractional integrals reduce to the Riemann-Liouville fractional integrals.
The foundational work by Buschman [8] introduced fractional integration with weak singular and exponential kernels, a step that eventually contributed to the creation of tempered fractional integration. In [32], Hermite-Hadamard-type inequalities for convex functions involving tempered fractional integrals were developed, expanding upon previous results, including those for Riemann integrals and Riemann-Liouville fractional integrals. In [9], Cai et al. demonstrated various fractional Simpson-type inequalities where the second derivatives, in absolute value, are convex. To learn more about tempered fractional integral inequalities, see and the references mentioned there [36, 21].
The purpose of this paper is to investigate Simpson-type inequalities for functions whose second derivatives are bounded, using tempered fractional integrals. After giving the necessary definitions, lower and upper bounds of Simpson type inequalities for the operators of tempered fractional integrals using functions whose second derivatives are bounded will be presented.

2. Essential outcomes of some Simpson-type inequalities applicable to bounded functions

In this section, we establish the following inequalities which give upper and lower bounds of Simpson type inequalities for tempered fractional integral operators. Here, fractional Simpson-type inequalities are defined for all $t\in[a,b]$; it is applied to functions satisfying the condition $m\leq f''(t)\leq M$ which is used instead of convexity.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ is a twice differentiable function, and $f\in L_1[a,b]$. If f'' is bounded, i.e., $m\leq f''(t)\leq M$ for all $t\in[a,b]$ with m, $M\in\mathbb{R}$ and $\alpha>0$, $\lambda\geq 0$, then the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[-M\frac{(b-a)^2}{4}\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)+[M+2m]\Upsilon_\lambda\left(\alpha+2,\frac{b-a}{2}\right)\right] \qquad (2.1)\\
\leq\ &\frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\
\leq\ &\frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[-m\frac{(b-a)^2}{4}\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)+[m+2M]\Upsilon_\lambda\left(\alpha+2,\frac{b-a}{2}\right)\right].
\end{aligned}$$

Proof. When we regularize the equation of the tempered fractional integral operators and then apply change of variables, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right] \qquad (2.2)\\
=\ &\frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}f(x)dx+\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha-1}e^{-\lambda\left(x-\frac{a+b}{2}\right)}f(x)dx\right]\\
=\ &\frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}f(x)dx+\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}f(a+b-x)dx\right]\\
=\ &\frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}[f(x)+f(a+b-x)]dx\right].
\end{aligned}$$

With the use of the equality (2.2), we obtain

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \qquad (2.3)\\
=\ &\frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}[f(x)+f(a+b-x)]dx\right]\\
&-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\
=\ &\frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}\left[3f(x)+3f(a+b-x)-\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right]dx\right].
\end{aligned}$$

It is obvious that we can represent these as:

$$f(x)-f(a)=\int_a^x f'(t)dt, \qquad (2.4)$$

$$f(a+b-x)-f(b)=-\int_{a+b-x}^{b}f'(t)dt=-\int_a^x f'(a+b-t)dt. \qquad (2.5)$$

Summing the equalities (2.4) and (2.5), we get

$$f(x)+f(a+b-x)-f(a)-f(b)=\int_a^x \left[f'(t)-f'(a+b-t)\right]dt. \tag{2.6}$$

On the other hand, based on the hypothesis, we know that $m \le f''(u) \le M$ for all $u \in [a,b]$. Integrating this inequality from t to $a+b-t$ gives us

$$m\int_t^{a+b-t} du \le \int_t^{a+b-t} f''(u)du \le M\int_t^{a+b-t} du$$

i.e

$$m(a+b-t) \le f'(a+b-t)-f'(t) \le M(a+b-t). \tag{2.7}$$

Integrating the inequality (2.7) with respect to t from a to x gives us

$$M(b-x)(a-x) \le \int_a^x \left[f'(t)-f'(a+b-t)\right]dt \le m(b-x)(a-x).$$

By applying the equality (2.6), we find that

$$M(b-x)(a-x) \le f(x)+f(a+b-x)-f(a)-f(b) \le m(b-x)(a-x). \tag{2.8}$$

In addition, it can be simply written as:

$$f(x)-f\left(\frac{a+b}{2}\right) = -\int_x^{\frac{a+b}{2}} f'(t)dt, \tag{2.9}$$

$$f(a+b-x)-f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-x} f'(t)dt = \int_x^{\frac{a+b}{2}} f'(a+b-t)dt. \tag{2.10}$$

Let's make the same arrangement for the equalities (2.9) and (2.10) as the equalities (2.4) and (2.5). When we sum the equalities (2.9) and (2.10), we have

$$f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) = \int_x^{\frac{a+b}{2}} \left[f'(a+b-t)-f'(t)\right]dt. \tag{2.11}$$

Integrating the inequality (2.7) with respect to t over $[x,\frac{a+b}{2}]$ and using the equality (2.11) leads to

$$2m\left(\frac{a+b}{2}-x\right)^2 \le 2\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \le 2M\left(\frac{a+b}{2}-x\right)^2. \tag{2.12}$$

Summing the equations (2.8) and (2.12) gives the following result:

$$\begin{aligned} &M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^2 \\ &\le 3f(x)+3f(a+b-x)-\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] \\ &\le m(b-x)(a-x)+2M\left(\frac{a+b}{2}-x\right)^2. \end{aligned} \tag{2.13}$$

Multiplying the inequality (2.13) by $\frac{\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}$ and then integrating the result obtained over the interval $\left[a,\frac{a+b}{2}\right]$, we get

$$\begin{aligned}
&\frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}\left[M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^2\right]dx\right\} \\
\leq\; & \frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}\left[3f(x)+3f(a+b-x)-\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right]dx\right\} \\
\leq\; & \frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}\left[m(b-x)(a-x)+2M\left(\frac{a+b}{2}-x\right)^2\right]dx\right\}.
\end{aligned} \tag{2.14}$$

The integral in the inequality (2.14) is equal to the following:

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{a+b}{2}-x\right)}\left[M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^2\right]dx \\
=\; & -M\frac{(b-a)^2}{4}\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)+[M+2m]\Upsilon_\lambda\left(\alpha+2,\frac{b-a}{2}\right).
\end{aligned} \tag{2.15}$$

Substituting the equality (2.15) in the inequality (2.3), we obtain

$$\begin{aligned}
&\frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[-M\frac{(b-a)^2}{4}\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)+[M+2m]\Upsilon_\lambda\left(\alpha+2,\frac{b-a}{2}\right)\right] \\
\leq\; & \frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
\leq\; & \frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[-m\frac{(b-a)^2}{4}\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)+[m+2M]\Upsilon_\lambda\left(\alpha+2,\frac{b-a}{2}\right)\right].
\end{aligned}$$

The proof is completed. □

Remark 2.2. *If we take $\lambda=0$ in the inequality (2.1), the following result is obtained:*

$$\begin{aligned}
&\frac{(b-a)^2}{12(\alpha+2)}[m\alpha-M] \\
\leq\; & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathscr{J}_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
\leq\; & \frac{(b-a)^2}{12(\alpha+2)}[M\alpha-m].
\end{aligned}$$

This inequality was derived by Budak et al. in the paper [6].

Remark 2.3. *When $\lambda=0$ and $\alpha=1$ are substituted in inequality (2.1), the following result follows:*

$$\frac{(b-a)^2}{36}[m-M]\leq\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\leq\frac{(b-a)^2}{36}[M-m], \tag{2.16}$$

which was proved by Budak et al. in [6].

Theorem 2.4. *Given the assumptions of Theorem 2.1, the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{[M+2m]\left[\Upsilon_{\lambda}\left(\alpha+2,\frac{b-a}{2}\right)-(b-a)\Upsilon_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)\right]+m\frac{(b-a)^2}{2}\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)\right\} \\
\leq\ &\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{I}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)+}f(b)+\mathscr{I}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
\leq\ &\frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{[m+2M]\left[\Upsilon_{\lambda}\left(\alpha+2,\frac{b-a}{2}\right)-(b-a)\Upsilon_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)\right]+M\,\frac{(b-a)^2}{2}\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)\right\}.
\end{aligned} \tag{2.17}$$

Proof. When we edit the equation of tempered integral operators and then change variables, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{I}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)+}f(b)+\mathscr{I}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)-}f(a)\right] \\
=\ &\frac{1}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_{\frac{a+b}{2}}^{b}(b-x)^{\alpha-1}e^{-\lambda(b-x)}f(x)dx+\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}f(x)dx\right\} \\
=\ &\frac{1}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}f(a+b-x)dx+\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}f(x)dx\right\} \\
=\ &\frac{1}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}[f(x)+f(a+b-x)]\,dx\right\}.
\end{aligned} \tag{2.18}$$

With the application of the equality (2.18), we obtain

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{I}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)+}f(b)+\mathscr{I}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
=\ &\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}[f(x)+f(a+b-x)]\,dx\right\} \\
&-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
=\ &\frac{\Gamma(\alpha)}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}\left[3f(x)+3f(a+b-x)-\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right]dx\right\}.
\end{aligned}$$

First, we multiply the inequality (2.13) by $\frac{(x-a)^{\alpha-1}e^{-\lambda(x-a)}}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$ and then integrate the resulting expression over the interval $\left[a,\frac{a+b}{2}\right]$, we have

$$\begin{aligned} & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}\left[M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^{2}\right]dx\right\} \\ \leq & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\ & \times\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}\left[3f(x)+3f(a+b-x)-\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right]dx\right\} \\ \leq & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}\left[m(b-x)(a-x)+2M\left(\frac{a+b}{2}-x\right)^{2}\right]dx\right\}. \end{aligned} \tag{2.19}$$

The integral in the inequality (2.19) is derived as follows:

$$\begin{aligned} & \int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}e^{-\lambda(x-a)}\left[M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^{2}\right]dx \\ = & [M+2m]\left[\Upsilon_{\lambda}\left(\alpha+2,\frac{b-a}{2}\right)-(b-a)\Upsilon_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)\right]+m\frac{(b-a)^{2}}{2}\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right). \end{aligned} \tag{2.20}$$

When we substitute the equality (2.20) into the inequality (2.19), we get

$$\begin{aligned} & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\ & \times\left\{[M+2m]\left[\Upsilon_{\lambda}\left(\alpha+2,\frac{b-a}{2}\right)-(b-a)\Upsilon_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)\right]+m\frac{(b-a)^{2}}{2}\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)\right\} \\ \leq & \frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)+}f(b)+\mathscr{J}^{(\alpha,\lambda)}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\ \leq & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\ & \times\left\{[m+2M]\left[\Upsilon_{\lambda}\left(\alpha+2,\frac{b-a}{2}\right)-(b-a)\Upsilon_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)\right]+M\frac{(b-a)^{2}}{2}\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)\right\}. \end{aligned}$$

Therefore the proof is completed. □

Remark 2.5. *Taking $\lambda=0$ in inequality (2.17) leads to the following result:*

$$\begin{aligned} & \frac{(b-a)^{2}}{24(\alpha+1)(\alpha+2)}[4m-\alpha(\alpha+3)M] \\ \leq & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\ \leq & \frac{(b-a)^{2}}{24(\alpha+1)(\alpha+2)}[4M-\alpha(\alpha+3)m]. \end{aligned}$$

Budak et al. established this inequality in [6].

Remark 2.6. *Taking $\lambda=0$ and $\alpha=1$ in inequality (2.17) gives the inequality (2.16).*

Theorem 2.7. *Given the assumptions of Theorem 2.1, the following double inequality holds:*

$$\begin{aligned}
&\frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{[M+2m]\left[\Upsilon_\lambda(\alpha+1,b-a)-(b-a)\Upsilon_\lambda(\alpha+1,b-a)\right]+m\left(\frac{b-a}{2}\right)^2\Upsilon_\lambda(\alpha,b-a)\right\} \\
\leq\;& \frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f(b)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
\leq\;& \frac{1}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{[m+2M]\left[\Upsilon_\lambda(\alpha+1,b-a)-(b-a)\Upsilon_\lambda(\alpha+1,b-a)\right]+M\left(\frac{b-a}{2}\right)^2\Upsilon_\lambda(\alpha,b-a)\right\}.
\end{aligned} \tag{2.21}$$

Proof. Through editing the equation of tempered integral operators and then changing the variables, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f(b)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f(a)\right] \\
=\;& \frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^b(b-x)^{\alpha-1}e^{-\lambda(b-x)}f(x)dx+\int_a^b(x-a)^{\alpha-1}e^{-\lambda(x-a)}f(x)dx\right\} \\
=\;& \frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^b\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]f(x)dx\right\} \\
=\;& \frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^b\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]f(a+b-x)dx\right\} \\
=\;& \frac{1}{4\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^b\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right][f(x)+f(a+b-x)]\,dx\right\} \\
=\;& \frac{1}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right][f(x)+f(a+b-x)]\,dx\right\}.
\end{aligned} \tag{2.22}$$

By applying the equality (2.22), we obtain

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f(b)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
=\;& \frac{\Gamma(\alpha)}{2\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\int_a^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right][f(x)+f(a+b-x)]\,dx\right\} \\
&-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
=\;& \frac{\Gamma(\alpha)}{6\Upsilon_\lambda\left(\alpha,\frac{b-a}{2}\right)} \\
&\times\left\{\int_a^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]\right. \\
&\qquad\left.\times\left[3f(x)+3f(a+b-x)-\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right]dx\right\}.
\end{aligned}$$

Multiplying the inequality (2.13) by $\frac{(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$ and then taking the integral over the interval $\left[a,\frac{a+b}{2}\right]$, we have

$$\begin{aligned}
& \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
& \times\left\{\int_{a}^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]\left[M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^{2}\right]dx\right\} \\
\leq\ & \frac{\Gamma(\alpha)}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
& \times\left\{\int_{a}^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]\right. \\
& \qquad\left.\times\left[3f(x)+3f(a+b-x)-\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right]dx\right\} \\
\leq\ & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
& \times\left\{\int_{a}^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]\left[m(b-x)(a-x)+2M\left(\frac{a+b}{2}-x\right)^{2}\right]dx\right\}.
\end{aligned} \tag{2.23}$$

The result of the integral in inequality(2.23) is obtained as follows:

$$\begin{aligned}
& \int_{a}^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}e^{-\lambda(x-a)}+(b-x)^{\alpha-1}e^{-\lambda(b-x)}\right]\left[M(b-x)(a-x)+2m\left(\frac{a+b}{2}-x\right)^{2}\right]dx \\
=\ & [M+2m]\left[\Upsilon_{\lambda}(\alpha+2,b-a)-(b-a)\Upsilon_{\lambda}(\alpha+1,b-a)\right]+m\frac{(b-a)^{2}}{2}\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right).
\end{aligned} \tag{2.24}$$

Substituting the equality (2.24) into the inequality (2.23) yields

$$\begin{aligned}
& \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
& \times\left\{[M+2m]\left[\Upsilon_{\lambda}(\alpha+2,b-a)-(b-a)\Upsilon_{\lambda}(\alpha+1,b-a)\right]+m\left(\frac{b-a}{2}\right)^{2}\Upsilon_{\lambda}(\alpha,b-a)\right\} \\
\leq\ & \frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathscr{J}_{a+}^{(\alpha,\lambda)}f(b)+\mathscr{J}_{b-}^{(\alpha,\lambda)}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
\leq\ & \frac{1}{6\Upsilon_{\lambda}\left(\alpha,\frac{b-a}{2}\right)} \\
& \times\left\{[m+2M]\left[\Upsilon_{\lambda}(\alpha+2,b-a)-(b-a)\Upsilon_{\lambda}(\alpha+1,b-a)\right]+M\left(\frac{b-a}{2}\right)^{2}\Upsilon_{\lambda}(\alpha,b-a)\right\}.
\end{aligned}$$

Thus, the desired result is obtained. □

Remark 2.8. *Setting $\lambda=0$ in inequality (2.21) gives the following result:*

$$\begin{aligned}
& \frac{(b-a)^{2}\alpha}{6(\alpha+1)(\alpha+2)}\left[m\frac{\alpha^{2}-\alpha+2}{2\alpha}-M\right] \\
\leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}_{\frac{a+b}{2}+}^{\alpha}f(b)+\mathscr{J}_{\frac{a+b}{2}-}^{\alpha}f(a)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
\leq\ & \frac{(b-a)^{2}\alpha}{6(\alpha+1)(\alpha+2)}\left[M\frac{\alpha^{2}-\alpha+2}{2\alpha}-m\right].
\end{aligned}$$

This inequality was derived by Budak et al. in [6].

Remark 2.9. *Setting $\lambda=0$ and $\alpha=1$ in inequality (2.21) gives the inequality (2.16).*

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