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$T-$Fuzzy Submodules of $R \times M$

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Abstaract − In this paper, we introduce the concept of T-fuzzy submodule of $R \times M$ and give new results on this subject. Next we study the concept of the extension of T-fuzzy submodule of $R \times M$ and prove some results on these. Also we investigate T-fuzzy submodule of $R \times M$ under homomorphisms or R-modules.

Keywords − *Theory of modules, extensions, homomorphism, fuzzy set theory, norms.*

1 Introduction

In algebra, ring theory is the study of rings algebraic structures in which addition and multiplication are defined and have similar properties to those operations defined for the integers. Ring theory studies the structure of rings, their representations, or, in different language, modules, special classes of rings (group rings, division rings, universal enveloping algebras), as well as an array of properties that proved to be of interest both within the theory itself and for its applications, such as homological properties and polynomial identities. In mathematics, a module is one of the fundamental algebraic structures used in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. Thus, a module, like a vector space, is an additive abelian group; a product is defined between elements of the ring and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication. Modules are very closely related to the representation theory of groups. They are also one of the central notions of commutative algebra and homological algebra, and are used widely in algebraic geometry and algebraic topology. In 1965, Zadeh [17] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. It provoked, at first (and as expected), a strong negative reaction from some influential scientists and mathematicians many of whom turned openly hostile. However, despite the controversy, the subject also attracted the attention of other mathematicians and in the following years, the field grew enormously, finding applications in areas as diverse as washing machines to handwriting recognition. In its trajectory of stupendous growth, it has also come to include the theory of fuzzy algebra and for the past five decades, several researchers have been working on concepts like fuzzy semigroup, fuzzy groups, fuzzy rings, fuzzy ideals, fuzzy semirings, fuzzy near-rings and so on. Solairaju and Nagarajan [5,6] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. The triangular norm, T-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [4], Alsina et al. [1] introduced the T-norm into fuzzy set theory and suggested that the T-norm be used for the intersection of fuzzy sets. Since then, many other researchers have presented various types of T-norms for particular purposes [3, 16]. Anthony and Sherwood [2] gave the definition of fuzzy subgroup based on t-norm. The author by using norms, investigated some properties of fuzzy submodules, fuzzy subrings, fuzzy ideals of subtraction semigroups, intuitionistic fuzzy subrings and ideals of a ring, fuzzy Lie algebra, fuzzy subgroups on direct product of groups, characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms, characterization of Q-fuzzy subrings (anti Q-fuzzy subrings) ([7, 8, 9, 10, 11, 12, 13, 14, 15]). In this work, by using a t-norm T, we introduce the notion of T-fuzzy submodule of $R \times M$, and investigate some of their properties. Also we use a t-norm to construct the concept of the extension of T-fuzzy submodule of $R \times M$ and prove some results on these. Finally we obtain some new results of T-fuzzy submodule of $R \times M$ with respect to t-norm T under homomorphisms of R-modules

2 Preliminary

Definition 2.1. A ring $< R, +, . >$ consists of a nonempty set R and two binary operations + and . that satisfy the axioms:
(1) $< R, +, . >$ is an abelian group;
(2) $(ab)c = a(bc)$ (associative multiplication) for all $a, b, c \in R$;
(3) $a(b + c) = ab + ac, (b + c)a = ba + ca$ (distributive laws) for all $a, b, c \in R$

Moreover, the ring R is a commutative ring if $ab = ba$ and ring with identity if R contains an element 1_R such that $1_R a = a 1_R = a$ for all $a \in R$.

Example 2.2. (1) The ring $\mathbb{Z}$ of integers is a commutative ring with identity. So are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_n$, $\mathbb{R}[x]$, etc.
(2) $3\mathbb{Z}$ is a commutative ring with no identity.
(3) The ring $\mathbb{Z}^{2\times 2}$ of 2×2 matrices with integer coe?cients is anoncommutative ring with identity.
(4) $(3\mathbb{Z})^{2\times 2}$ is a noncommutative ring with no identity.

Definition 2.3. Let R be a ring. A commutative group $(M, +)$ is called a left R-module or a left module over R with respect to a mapping

$$. : R \times M \to M$$

if for all $r, s \in R$ and $m, n \in M$,
(1) $r.(m+n) = r.m + r.n$,
(2) $r.(s.m) = (rs).m$,
(3) $(r+s).m = r.m + s.m$.

If R has an identity 1 and if $1.m = m$ for all $m \in M$, then M is called a unitary or unital left R-module.

A right R-module can be defined in a similar fashion.

Definition 2.4. Let X a non-empty sets. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$. Denote by $[0,1]^X$, the set of all fuzzy subset of X.

Definition 2.5. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),
for all $x, y, z \in [0,1]$.

We say that T be idempotent if $T(x,x) = x$ for all $x \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1,y_1) \geq T(x_2,y_2)$.

Example 2.6. (1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product T-norm $T_p(x,y) = xy$.
(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Lemma 2.7. Let T be a t-norm. Then

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

3 $T-$Fuzzy Submodules of $R \times M$

Definition 3.1. Let M be an R-module. A M-fuzzy subset μ of R is a function $\mu : R \times M \to [0,1]$. Denote by $[0,1]^{R\times M}$, the set of all M-fuzzy subset of R.

Definition 3.2. Let $S \subseteq R$ and $a \in [0,1]$. Define $a_{\{S\times M\}} \in [0,1]^{R\times M}$ as follows;

$$a_{\{S\times M\}}(r,m) = \begin{cases} a & \text{if } r \in S, m \in M \\ 0 & \text{if } r \in R - S, m \in M \end{cases}$$

Definition 3.3. Let $\mu \in [0,1]^{R\times M}$ and T be a t-norm. We say that μ is a T-fuzzy submodule of $R \times M$ if for all $r, s \in R$ and $x, y \in M$
(1) $\mu(r, 0_M) = 1$,
(2) $\mu(r, sx) \geq \mu(r, x)$,
(3) $\mu(r, x+y) \geq T(\mu(r,x), \mu(r,y))$.

We denote the set of all fuzzy submodules of $R \times M$ by $TF(R \times M)$.
Since $-1x = -x$, condition (2) implies that $\mu(r, -x) \geq \mu(r, x)$.

Example 3.4. Let $R = (\mathbb{Z}, +, .)$ be a ring of integer. If $M = \mathbb{Z}$, then M is an R-module. For all $x \in R$ we define a fuzzy subset μ of $\mathbb{Z} \times \mathbb{Z}$ as

$$\mu(r,x) = \begin{cases} 1 & \text{if } (r,x) \in \mathbb{Z} \times \{0_{\mathbb{Z}}\} \\ 0.90 & \text{if } (r,x) \in \mathbb{Z} \times (2\mathbb{Z} - \{0_{\mathbb{Z}}\}) \\ 0.80 & \text{if } (r,x) \in \mathbb{Z} \times (2\mathbb{Z} + 1) \end{cases}$$

Let $T(x,y) = T_p(x,y) = xy$ for all $x, y \in \mathbb{Z}$, then $\mu \in TF(\mathbb{Z} \times \mathbb{Z})$.

Definition 3.5. Let $\mu, \nu \in TF(R\times M)$ and $r \in R$ and $x \in M$. Define $\mu+\nu$, $\mu\cup\nu$, $\mu\cap\nu$, and $-\mu$ as follows:
$(\mu+\nu)(r,x) = \sup\{T(\mu(r,y), \nu(r,z)) \mid y, z \in M, y+z = x\}$,
$(\mu \cup \nu)(r,x) = \sup\{\mu(r,x), \nu(r,x)\}$,
$(\mu \cap \nu)(r,x) = T(\mu(r,x), \nu(r,x))$,
$(-\mu)(r,x) = \mu(r,-x)$.

Then $\mu+\nu, \mu\cup\nu, \mu\cap\nu$ are called the sum, union, intersection of μ and ν respectively, and $-\mu$ the negative of μ.

Let $\mu_i \in TF(R \times M)$. The least upper bound $\cup_{i\in I}\mu_i$ of the $x_i^{,}s$ is given by $(\cup_{i\in I}\mu_i)(r,x) = \sup\{\mu_i(r,x) \mid i \in I\}$ for all $i \in I, r \in R, x \in M$.

Definition 3.6. Let $\mu_i \in TF(R \times M), 1 \leq i \leq n$ and $n \in \mathbb{N}$. Since $+$ is associative and commutative, we can consider $\mu_1 + \mu_2 + ... + \mu_n$ and write it as $\Sigma_{i=1}^n \mu_i$.

If $\mu_i \in TF(R \times M)$ for each $i \in I$, then $\Sigma_{i\in I}\mu_i$ is defined by

$$(\Sigma_{i\in I}\mu_i)(r,x) = \sup\{T_{i\in I}(\mu_i(r,x_i)) \mid x_i \in M, i \in I, \Sigma x_i = x\}$$

such that $\Sigma x_i = \Sigma_{i\in I} x_i$ and there are at most finitely many $x_i^{,}s$ not equal to 0_M.

Definition 3.7. Let $r, s \in R, x \in M$ and $\mu \in TF(R \times M)$. Define $s\mu$ as follow: $s\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, sy = x\}$ which is called the product of s and μ.

Proposition 3.8. Let $r, s, t \in R$ and $\mu, \nu, \xi, \mu_i \in TF(R \times M), i \in I$. Then for all $x, y \in M$
(1) $1\mu = \mu, (-1)\mu = (-\mu)$.
(2) $s1_{\{R\times 0_M\}} = 1_{\{R\times 0_M\}}$.
(3) If $\mu \leq \nu$, then $s\mu \leq s\nu$.
(4) $(ts)\mu = t(s\mu)$.
(5) $s(\mu + \nu) = s\mu + s\nu$.
(6) $s(\cup_{i\in I}\mu_i) = \cup_{i\in I} s\mu_i$.
(7) $(s\mu)(r, sx) \geq \mu(r,x)$.
(8) $\xi(r, sx) \geq \mu(r,x)$ if and only if $s\mu \leq \xi$.
(9) $(s\mu + t\nu)(r, sx + ty) \geq T(\mu(r,x), \nu(r,y))$.
(10) $\xi(r, sx + ty) \geq T(\mu(r,x), \nu(r,y))$ if and only if $s\mu + t\nu \leq \xi$.

Proof. Let $r, s, t \in R$ and $x, y, z \in M$. Then

(1) $1\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, 1y = x\} = \mu(r,x)$. Also $(-1)\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, -1y = x\} = \mu(r,-x) = (-\mu)(r,x)$.

(2) It is clear.

(3) $s\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, sy = x\} \leq \sup\{\nu(r,y) \mid y \in M, sy = x\} = s\nu(r,x)$.

(4) $(ts)\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, (ts)y = x\} = \sup\{\mu(r,y) \mid y \in M, t(sy) = x\} = t(s\mu)(r,x)$.

(5) $(s\mu + s\nu)(r,x) = \sup\{T(s\mu(r,y), s\nu(r,z)) \mid y, z \in M, y + z = x\}$
$= \sup\{T(\sup\{\mu(r,y_1) \mid y_1 \in M, sy_1 = y\}, \sup\{\nu(r,z_1) \mid z_1 \in M, sz_1 = z\})\}$
$\mid y, z \in M, s(y_1 + z_1) = sy_1 + sz_1 = x = s(\mu + \nu)(r,x)\}$

(6) $s(\cup_{i\in I}\mu_i)(r,x) = \sup\{(\cup_{i\in I}\mu_i)(r,y) \mid y \in M, sy = x\}$
$= \sup\{\sup_{i\in I}\mu_i(r,y) \mid y \in M, sy = x\} = \sup_{i\in I}\{\sup \mu_i(r,y) \mid y \in M, sy = x\}$
$= \cup_{i\in I} s\mu_i(r,x)$.

(7) $s\mu(r, sx) = \sup\{\mu(r,y) \mid y \in M, sy = sx\} \geq \mu(r,x)$.

(8) Let $\xi(r,sx) \geq \mu(r,x)$. Then $s\mu(r,x) = \sup\{\mu(r,y) \mid y \in M, sy = x\}$
$\leq \sup\{\xi(r,sy) \mid y \in M, sy = x\} = \xi(r,x)$.

(9) By Definition 3.5 and (part 7) we obtain that
$(s\mu + t\nu)(r, sx + ty) = \sup\{T((s\mu)(r,sx), (t\nu)(r,ty))\} \geq \sup\{T\mu(r,x), \nu(r,y))\}$
$\geq T(\mu(r,x), \nu(r,y))$.

(10) Let $\xi(r, sx + ty) \geq T(\mu(r,x), \nu(r,y))$. Then $(s\mu + t\nu)(r,z)$
$= \sup\{T(s\mu(r,z_1), t\nu(r,z_2) \mid z_1, z_2 \in M, z_1 + z_2 = z\}$

$= \sup\{T(\sup\{\mu(r,x) \mid x \in M, sx = z_1\}, \sup\{\nu(r,y) \mid y \in M, ty = z_2\}) \mid z_1, z_2 \in M, sx + ty = z\}$
$= \sup\{T(\mu(r,x), \nu(r,y)) \mid x, y \in M, sx + ty = z\} \leq \sup\{\xi(r, sx + ty) \mid x, y \in M, sx + ty = z\} = \xi(r,z)$.

Conversely, suppose that $s\mu + t\nu \leq \xi$. Then $\xi(r, sx+ty) \geq (s\mu + t\nu)(r, sx+ty) \geq T(s\mu(r,sx), t\nu(r,ty)) \geq T(\mu(r,x), \nu(r,y))$ $(by(7))$. □

Corollary 3.9. Let $r, s, t \in R$ and $\mu \in TF(R \times M)$. Then
(1) $s\mu \leq \mu$ if and only if $\mu(r, sx) \geq \mu(r,x)$,
(2) $s\mu + t\mu \leq \mu$ if and only if $\mu(r, sx+ty) \geq T(\mu(r,x), \mu(r,y))$.

Proof. (1) In Proposision 3.8 part(8), put $\mu = \xi$.

(2) In Proposision 3.8 part(10), put $\mu = \nu = \xi$. □

Corollary 3.10. Let $s \in R$ and $\mu \in [0,1]^{R \times M}$. Then $\mu \in TF(R \times M)$ if and only if μ satisfies the following conditions:
(1) $1_{\{R \times 0_M\}} \leq \mu$,
(2) $s\mu \leq \mu$,
(3) $\mu + \mu \leq \mu$.

Proof. Let $\mu \in TF(R \times M)$. Then

(1) $\mu(r, 0_M) = 1 \geq 1 = 1_{\{R \times 0_M\}(r,0_M)}$ and so $1_{\{R \times 0_M\}} \leq \mu$.

(2) For all $r, s \in R$ and $x \in M$ we have that $\mu(r,sx) \geq \mu(r,x)$, and by Corollary 3.9 (part 1) we get $s\mu \leq \mu$.

(3) Let $r, s, t \in R$ and $x, y \in M$. Then from $\mu(r, x+y) \geq T(\mu(r,x), \mu(r,y))$ and Corollary 3.9 (part 2 with $s = 1 = t$) we obtain that $\mu + \mu \leq \mu$.

Conversely, we prove that $\mu \in TF(R \times M)$.

From condition (1) we have $\mu(r, 0_M) \geq 1_{\{R \times 0_M\}(r,0_M)}$ and so $\mu(r, 0_M) = 1$.

By condition (2) and Corollary 3.9 (part 1) we get $\mu(r, sx) \geq \mu(r,x)$.

Also as condition (3) and Corollary 3.9 (part 2) we have $\mu(r, x+y) \geq T(\mu(r,x), \mu(r,y))$. Therefore $\mu \in TF(R \times M)$. □

Proposition 3.11. Let $r, s, t \in R$ and $x, y \in M$. If $\mu \in [0,1]^{R \times M}$, then $\mu \in TF(R \times M)$ if and only if μ satisfies condition (1) from Definition 3.3 and the following condition:
(4) $\mu(r, sx + ty) \geq T(\mu(r,x), \mu(r,y))$.

Proof. Suppose $\mu \in TF(R \times M)$. By Definition 3.3, μ satisfies condition (1). Since μ also satisfies conditions (2) and (3), it follows that $\mu(r, sx+ty) \geq T(\mu(r,sx), \mu(r,ty)) \geq$

$T(\mu(r,x),\mu(r,y))$.

Conversely, assume that μ satisfies conditions (1) and (4). Then $\mu(r,sx) = \mu(r,sx+s0_M) \geq T(\mu(r,x),\mu(r,0_M)) = T(\mu(r,x),1) = \mu(r,x)$.

Also $\mu(r,x+y) = \mu(r,1x+1y) \geq T(\mu(r,x),\mu(r,y))$. Hence μ satisfies conditions (2) and (3) and so $\mu \in TF(R\times M)$. □

Corollary 3.12. Let $r,s \in R$ and $\mu \in [0,1]^{R\times M}$. Then $\mu \in TF(R\times M)$ if and only if μ satisfies the following conditions:
(1) $1_{\{R\times 0_M\}} \leq \mu$,
(2) $r\mu + s\mu \leq \mu$.

Proof. Let $\mu \in TF(R\times M)$. Then from Corollary 3.10 we get that $1_{\{R\times 0_M\}} \leq \mu$ and $r\mu+s\mu \leq \mu$. Conversely, we show that $\mu \in TF(R\times M)$. As $1_{\{R\times 0_M\}} \leq \mu$ so $\mu(r,0_M) = 1$. By $r\mu+s\mu \leq \mu$ and Proposision 3.8(part 10) and Proposion 3.11 we obtain that $\mu \in TF(R\times M)$. □

Proposition 3.13. Let $\mu,\nu \in TF(R\times M)$. Then $\mu\cap\nu \in TF(R\times M)$.

Proof. Let $r,s \in R$ and $x,y \in M$. If $\mu,\nu \in TF(R\times M)$, then

(1) $(\mu\cap\nu)(r,0_M) = T(\mu(r,0_M),\nu(r,0_M)) = T(1,1) = 1$.

(2) $(\mu\cap\nu)(r,sx) = T(\mu(r,sx),\nu(r,sx)) \geq T(\mu(r,x),\nu(r,x)) = (\mu\cap\nu)(r,x)$.

(3) $(\mu\cap\nu)(r,x+y) = T(\mu(r,x+y),\nu(r,x+y)) \geq T(T(\mu(r,x),\mu(r,y)),T(\nu(r,x),\nu(r,y))) = T(T(\mu(r,x),\nu(r,x)),T(\mu(r,y),\nu(r,y)))(by\ Lemma\ 2.7) = T((\mu\cap\nu)(r,x),(\mu\cap\nu)(r,y))$.

Thus $\mu\cap\nu \in TF(R\times M)$. □

Corollary 3.14. If $\{\mu_i \mid i=1,2,...\} \subseteq TF(R\times M)$, then $\cap_i\mu_i \in TF(R\times M)$.

Proposition 3.15. Let $\mu,\nu \in TF(R\times M)$ abd T be idempotent. Then $\mu+\nu \in TF(R\times M)$.

Proof. Let $\mu,\nu \in TF(R\times M)$.

(1) Let $r \in R, x \in M$. Then
$(\mu+\nu)(r,x) = \sup\{T(\mu(r,x_1),\nu(r,x_2)) \mid x_1,x_2 \in M, x_1+x_2 = x\}$
$\geq \sup\{T(1_{\{R\times 0_M\}}(r,x_1),1_{\{R\times 0_M\}}(r,x_2)) \mid x_1,x_2 \in M, x_1+x_2 = x\}$
$= \sup\{T(1,1) \mid x_1,x_2 \in M, x_1+x_2 = x\} = 1 = 1_{\{R\times 0_M\}}(r,x)$.

(2) Let $s \in R$. Then $s(\mu+\nu) = s\mu+s\nu \subseteq \mu+\nu$.

(3) $(\mu+\nu)+(\mu+\nu) = (\mu+\mu)+(\nu+\nu) \subseteq (\mu+\nu)$.

Hence from Corollary 3.10 we have $\mu+\nu \in TF(R\times M)$. □

Corollary 3.16. If $\{\mu_i \mid i=1,2,...\} \subseteq TF(R\times M)$, then $\Sigma_i\mu_i \in TF(R\times M)$.

Definition 3.17. Let $\mu \in [0,1]^{R\times M}$ and $s \in R$. For all $(r,y) \in R\times M$ the fuzzy subset $< s,\mu >\in [0,1]^{R\times M}$ defined by $< s,\mu > (r,y) = \mu(r,sy)$ is called the extension of μ by s.
Also we define $Supp\mu = \{(r,x) \in R \times M \mid \mu(r,x) > 0\}$.

Proposition 3.18. Let $\mu \in TF(R \times M)$ and $s \in R$. Then $< s,\mu >\in TF(R \times M)$.

Proof. Let $r,s,t \in R$ and $x,y \in M$. If $\mu \in TF(R \times M)$, then
(1) $< s,\mu > (r,0_M) = \mu(r,0_M) = 1$.
(2) $< s,\mu > (r,tx) = \mu(r,stx) = \mu(r,tsx) \geq \mu(r,sx) =< s,\mu > (r,x)$.
(3) $< s,\mu > (r,x+y) = \mu(r,s(x+y)) = \mu(r,sx+sy) \geq T(\mu(r,sx),\mu(r,sy)) = T(< s,\mu > (r,x), < s,\mu > (r,y))$.
Hence $< s,\mu >\in TF(R \times M)$. □

Corollary 3.19. If $s \in R$ and $\{\mu_i \mid i = 1,2,...\} \subseteq TF(R \times M)$, then $< s,\cap_i\mu_i >\in TF(R \times M)$.

Proposition 3.20. Let $\mu \in TF(R \times M)$ and $s \in R$. Then we have the following:
(1) $\mu \subseteq< s,\mu >$,
(2) $< s^n,\mu >\subseteq< s^{n+1},\mu >$ for every $n \in N$,
(3) If $x \in M$ and $\mu(r,x) > 0$, then $Supp < s,\mu >= R \times M$.

Proof. (1) If $(r,x) \in R \times M$, then $< s,\mu > (r,x) = \mu(r,sx) \geq \mu(r,x)$.

(2) From every $n \in N$ and $(r,x) \in R \times M$ we have that $< s^{n+1},\mu > (r,x) = \mu(r,s^{n+1}x) = \mu(r,ss^nx) \geq \mu(r,s^nx) =< s^n,\mu > (r,x)$.

(3) By Definition 3.17, $Supp < s,\mu >\subseteq R \times M$. Now if $(r,x) \in R \times M$, then $< s,\mu > (r,x) = \mu(r,sx) \geq \mu(r,x) > 0$ and so $Supp < s,\mu >= R \times M$. □

4 Homomorphisms Over T-Fuzzy Submodules of $R \times M$

Definition 4.1. Let f be a mapping from R-module M into R-module N. Let $\mu \in TF(R\times M)$ and $\nu \in TF(R\times N)$. Define $f(\mu) \in [0,1]^{R\times N}$ and $f^{-1}(\nu) \in [0,1]^{R\times M}$ as $\forall y \in N$, $\forall r \in R$, $f(\mu)(r,y) = \sup\{\mu(r,x) \mid x \in M, f(x) = y\}$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(r,y) = 0$ if $f^{-1}(y) = \emptyset$. Also $\forall x \in M$, $\forall r \in R$ $f^{-1}(\nu)(r,x) = \nu(r,f(x))$.

Proposition 4.2. Let f be a mapping from R-module M into R-module N. Let $\mu,\mu_1,\mu_2 \in TF(R \times M)$ and $\nu,\nu_1,\nu_2 \in TF(R \times N)$.
(1) Let $\mu_1 \leq \mu_2$. Then $f(\mu_1) \leq f(\mu_2)$.
(2) Let $\nu_1 \leq \nu_2$. Then $f^{-1}(\nu_1) \leq f^{-2}(\nu_2)$.
(3) $\mu \leq f^{-1}(f(\mu))$. Inparticular, if f is an injection, then $\mu = f^{-1}(f(\mu))$.
(4) $\nu \geq f(f^{-1}(\nu))$. Inparticular, if f is a surjection,, then $\nu = f(f^{-1}(\nu))$.
(5) $f(\mu) \leq \nu$ if and only if $\mu \leq f^{-1}(\nu)$.

Proof. Clearly, assertions (1) and (2) hold.

(3) $f^{-1}(f(\mu))(r,x) = f(\mu)(r,f(x)) = \sup\{\mu(r,z) \mid z \in M, f(z) = f(x)\} \geq \mu(r,x)$.
If f is an injection, then $f^{-1}(f(\mu))(r,x) = \sup\{\mu(r,z) \mid z \in M, f(z) = f(x)\} = \mu(r,x)$.

(4) $f(f^{-1}(\nu))(r,y) = \sup\{f^{-1}(\nu)(r,x) \mid x \in M, f(x) = y\} = \sup\{\nu(r,f(x)) \mid x \in M, f(x) = y\} = \{\nu(r,y) \mid y \in f(M)\} \leq \nu(y)$.

Assertion (5) is an immediate consequence of the four preceding assertions. □

Proposition 4.3. Suppose that f be an epiomorphism from R-module M into R-module N. Let $r,s,t \in R$ and $\mu,\nu \in TF(R \times M)$. Then
(1) $f(\mu+\nu) = f(\mu) + f(\nu)$,
(2) $f(s\mu) = sf(\mu)$,
(3) $f(s\mu + t\nu) = sf(\mu) + tf(\nu)$.

Proof. (1) If $y_1, y_2 \in N$, then we have $x_1, x_2 \in M$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now $f(\mu+\nu)(r,y) = \sup\{(\mu+\nu)(r,x) \mid x \in M, f(x) = y\}$
$= \sup\{\sup\{T(\mu(r,x_1),\nu(r,x_2)) \mid x_1,x_2 \in M, x_1+x_2 = x\} \mid y = f(x) = f(x_1) + f(x_2) = y_1 + y_2\}$
$= \sup\{T(\sup\{\mu(r,x_1) \mid x_1 \in M, f(x_1) = y_1\}, \sup\{\mu(r,x_2) \mid x_2 \in M, f(x_2) = y_2\}) \mid y = y_1 + y_2\} = (f(\mu) + f(\nu))(r,y)$.

(2) $f(s\mu)(r,y) = \sup\{(s\mu)(r,x_1) \mid x_1 \in M, f(x_1) = y\}$
$= \sup\{\sup\{\mu(r,x_2) \mid x_2 \in M, x_1 = sx_2\} \mid x_1 \in M, f(x_1) = y\}$
$= \sup\{\sup\{\mu(r,x_2) \mid x_2 \in M, x_1 = sx_2\} \mid x_1 \in M, sf(x_2) = y\} = sf(\mu)(r,y)$.

(3) This assertion follows from (1) and (2). □

Proposition 4.4. Let $\mu \in TF(R \times M)$ and N be an R-module. Suppose that f is an isomorphism of M onto N. Then $f(\mu) \in TF(R \times N)$.

Proof. (1) $f(\mu)(r,0_N) = \sup\{\mu(r,x) \mid f(x) = 0_N\} = \sup\{\mu(r,x) \mid x \in kerf = 0\} = \sup\{\mu(r,0_M)\} = 1$.

(2) $f(\mu)(r,sy) = \sup\{\mu(r,z) \mid f(z) = sy = sf(x) = f(sx)\} = \sup\{\mu(r,sx) \mid f(x) = y\} \geq \sup\{\mu(r,x) \mid f(x) = y\} = f(\mu)(r,y)$.

(3) $f(\mu)(r,y_1+y_2) = \sup\{\mu(r,z) \mid f(z) = y_1+y_2 = f(x_1)+f(x_2) = f(x_1+x_2)\}$
$= \sup\{\mu(r,x_1+x_2) \mid y_1 = f(x_1), y_2 = f(x_2)\}$
$\geq \sup\{T(\mu(r,x_1),\mu(r,x_2)) \mid y_1 = f(x_1), y_2 = f(x_2)\}$
$\geq T(\sup\{\mu(r,x_1) \mid f(x_1) = y_1\}, \sup\{\mu(r,x_2) \mid f(x_2) = y_2\})$
$= T(f(\mu)(r,y_1), f(\mu)(r,y_2))$. □

Proposition 4.5. Let $\nu \in TF(R \times N)$ and M be an R-module. Suppose that f is a homomorphism of M onto N. Then $f^{-1}(\nu) \in TF(R \times M)$.

Proof. Let $r,s \in R$ and $x_1, x_2 \in M$. Then

(1) $f^{-1}(\nu)(r, 0_M) = \nu(r, f(0_M)) = \nu(r, 0_N) = 1.$

(2) $f^{-1}(\nu)(r, sx) = \nu(r, f(sx)) = \nu(r, sf(x)) \geq \nu(r, f(x)) = f^{-1}(\nu)(r, x).$

(3) $f^{-1}(\nu)(r, x_1 + x_2) = \nu(r, f(x_1 + x_2)) = \nu(r, f(x_1) + f(x_2))$
$\geq T(\nu(r, f(x_1)), \nu(r, f(x_2))) = T(f^{-1}(\nu)(r, x_1), f^{-1}(\nu)(r, x_2)).$ □

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