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RESEARCH ARTICLE

SUBCLASS OF BAZILEVIČ AND λ-PSEUDO-STARLIKE BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SAKAGUCHI TYPE FUNCTIONS AND ITS APPLICATION TO FIBONACCI-LIKE POLYNOMIAL

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Abstract

In this paper, we first introduce a new subclass of analytic and bi-univalent functions related to Sakaguchi type functions, λ-pseudo-starlike and, Bazilevič functions. We also use the generalized bivariate Fibonacci-like polynomial as a subordination function. In addition, we provide function examples to show that this class of functions is a non-empty set. We then establish bounds on certain coefficients of the Maclaurin series expansion of the functions belonging to this newly constructed subclass of bi-univalent functions. We also establish bounds on the Fekete-Szegö functional of the functions in the defined subclass. We note that the comprehensive function class in this work generalizes some previously studied function classes, and the results of this study re-establish certain results in the previously published papers.

Keywords

Analytic function,
Bazilevič functions,
Bi-univalent function,
Starlike functions,
Fibonacci-like polynomial

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathfrak{A}$ indicate the family of regular functions of the form

$$\mathfrak{f}(t) = t + a_2 t^2 + \cdots = t + \sum_{l=2}^{\infty} a_l t^l, \tag{1.1}$$

in the disk $\mathcal{U} = \{t \in \mathbb{C} : |t| < 1\}$. By $\mathcal{S}$, we show the subset of $\mathfrak{A}$ such that every function $\mathfrak{f} \in \mathcal{S}$ are univalent in $\mathcal{U}$. There are various subclasses of $\mathcal{S}$ such that they have nice geometric features.

If $\mathfrak{f}(\mathcal{U})$ is a starlike set with respect to the origin, then it is called $\mathfrak{f}$ is starlike in $\mathcal{U}$. The set of all starlike functions are denoted by $\mathcal{S}^{\star}$ and characterized as

$$\mathcal{S}^{\star} = \left\{ \mathfrak{f} : \mathfrak{f} \in \mathfrak{A} \text{ and } \Re\left(\frac{t\mathfrak{f}'(t)}{\mathfrak{f}(t)}\right) > 0,\ t \in \mathcal{U} \right\}. \tag{1.2}$$

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Koebe $\frac{1}{4}$-Theorem (see [1]) guarantees that if $\mathfrak{f} \in \mathcal{S}$, then there is an inverse function $\mathfrak{f}^{-1}$ satisfying

$$\mathfrak{f}^{-1}(\mathfrak{f}(t)) = t, (t \in \mathcal{U}) \text{ and } \mathfrak{f}(\mathfrak{f}^{-1}(s)) = s, \left(|s| < r_0(\mathfrak{f}), r_0(\mathfrak{f}) \geq \frac{1}{4}\right),$$

where

$$\mathfrak{f}^{-1}(s) = s - a_2 s^2 + (2a_2^2 - a_3)s^3 - (5a_2^3 - 5a_2 a_3 + a_4)s^4 + \cdots =: \mathfrak{g}(s). \tag{1.3}$$

If both functions $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are univalent in $\mathcal{U}$, then the function $\mathfrak{f} \in \mathfrak{A}$ is called bi-univalent in $\mathcal{U}$. The class of all bi-univalent functions in $\mathcal{U}$ is represented by Σ and characterized by

$$\Sigma = \{\mathfrak{f} \in \mathcal{S} : \mathfrak{f}^{-1} \in \mathcal{S},\ t \in \mathcal{U}\}. \tag{1.4}$$

The Bazilevič functions are closely related to the class $\mathcal{S}$ and Bazilevič [2] showed that these functions form a subclass of analytic and univalent function in $\mathcal{S}$. In [3], Singh studied a special case of Bazilevič functions which characterized by $\mathrm{Re}\left(\frac{z^{1-\gamma}\mathfrak{f}'(z)}{(\mathfrak{f}(z))^{1-\gamma}}\right) > 0,\ \gamma \in \mathbb{R}^+ \cup \{0\}$.

Similarly, λ-pseudo-starlike functions in $\mathcal{U}$ is defined by $\mathrm{Re}\left(\frac{z(\mathfrak{f}'(z))^{\lambda}}{\mathfrak{f}(z)}\right)$, for $\lambda \geq 1$, see [4]. For a recent study on λ-pseudo-starlike functions we refer to [5]. It is clear that the above mentioned function subclasses reduced to usual starlike functions class $\mathcal{S}^{\star}$ for $\gamma = 0$ and $\lambda = 1$, respectively. On the other hand, Sakaguchi introduced a new subclass of starlike functions with respect to symmetrical points and formed a new direction for studies in the geometric properties of analytic and univalent functions in 1959. This class of starlike functions is defined by [6]

$$\mathcal{S}_S^{\star} = \left\{\mathfrak{f} \in \mathfrak{A} : \mathrm{Re}\left(\frac{2t\mathfrak{f}'(t)}{\mathfrak{f}(t) - \mathfrak{f}(-t)}\right) > 0,\ t \in \mathcal{U}\right\}. \tag{1.5}$$

In the theory of univalent functions, the most remarkable problem was Bieberbach's conjecture and it was solved by Branges in 1985. According to Bieberbach conjecture nth coefficient of every function $\mathfrak{f} \in \mathcal{S}$ satisfy that $|a_n| \leq n$ for $n \geq 2$. Also, this problem was solved for convex and starlike functions such that if $\mathfrak{f} \in \mathcal{S}^*$ then $|a_n| \leq n$, while, if $\mathfrak{f} \in \mathcal{C}$ then $|a_n| \leq 1$, $n \geq 2$. But, any conjecture on nth coefficient of bi-univalent functions is not established yet. The first finding in this regard is that: if $\mathfrak{f} \in \Sigma$ then $|a_2| \leq 1.51$, see [7] . Also, Brannan and Clunie [8] conjectured that $|a_2| \leq \sqrt{2}$ for every $\mathfrak{f} \in \Sigma$. In the class Σ, a coefficient estimation on $|a_n|$ is still an open problem for $n \geq 3$. In 2010, Srivastava *et* al. (see [9]) published a paper on bi-univalent functions, which contains quite extensive information and examples of bi-univalent functions. This paper has been a pioneering study for recent papers and has attracted much more attention from researchers.

Another interesting problem in geometric function theory is known as the Fekete-Szegö problem. This problem is an upper bound search problem for a combination of coefficients of functions of class $\mathcal{S}$ and in [10] Fekete and Szegö proved the following sharp result for $f \in \mathcal{S}$:

$$\left|a_3-\tau a_2^2\right| \le \begin{cases} 4\tau-3, & \tau\ge 1 \\ 1+2e^{\left(\frac{-2\tau}{1-\tau}\right)}, & 0\le\tau<1 \\ 3-4\tau, & \tau<0. \end{cases}$$

The fundamental result $\left|a_3-a_2^2\right|\le 1$ is obtained when $\tau\to 1$. The combination of coefficients of the form $F_\tau(f)=a_3-\tau a_2^2$ on the functions $f\in\mathfrak{A}$ make a remarkable impact on univalent functions. Finding bounds for $|F_\tau(f)|$ is a maximization problem. In 2014, Zaprawa [11] published a very striking article on the solution of the Fekete-Szegö problem for bi-univalent functions. The author of [12] recently investigated the Fekete–Szegö problem for new classes of analytic functions defined in terms of Poisson and Borel distribution series.

Many researchers have defined new subclasses of bi-univalent functions based on previously published works such as [7-9, 11] and [13]. Also, the authors obtained non-sharp estimates for coefficients of functions of these subclasses. Of course, research on this topic is not limited to the above. Results on new subclasses of bi-univalent functions continue to be published without interruption. In recent years, subclasses of functions defined with the help of special number sequences and special polynomials have been frequently encountered. In particular, Fibonacci [14-17], Lucas [18], Chebyshev [19-29], Horadam [30-35], Gegenbauer [36], Lucas-Balancing [37] and Gregory [38] polynomials are the main polynomials used in the defined subclasses. In addition, various operators like Hohlov [39], q-derivative [40], fractional q-difference [41], differential [42] have been used to define such kind of function subclasses.

There are many techniques useful in research and one of them is known as the Subordination principle. This technique is defined as follows (see [43]):

If the functions $\mathfrak{f}$ and $\mathfrak{g}\in\mathfrak{A}$, then $\mathfrak{f}$ is said to be subordinate to $\mathfrak{g}$ if there exists a Schwarz function $\mathfrak{w}$ such that

$$\mathfrak{w}(0)=0, |\mathfrak{w}(t)|<1 \; and \; \mathfrak{f}(t)=\mathfrak{g}(\mathfrak{w}(t)) \quad (t\in\mathcal{U}).$$

This subordination is shown by

$$\mathfrak{f}\prec\mathfrak{g} \quad or \quad \mathfrak{f}(t)\prec\mathfrak{g}(t) \quad (t\in\mathcal{U}).$$

If $\mathfrak{g}$ is univalent in $\mathcal{U}$, then this subordination is equivalent to

$$\mathfrak{f}(0)=\mathfrak{g}(0), \quad \mathfrak{f}(\mathcal{U})\subset\mathfrak{g}(\mathcal{U}).$$

In this paper, we introduce a comprehensive class of bi-univalent functions using generalized bivariate Fibonacci-like polynomials. Then, we investigate upper bounds for the absolute values of the coefficients a_2 and a_3 in the Taylor series expansion of the members of the class of functions we define. We also solve the Fekete-Szegö problem for members of this class of functions. In addition, we give examples showing that the classes of functions we define are not empty sets and support the correctness of our results with these examples. Our results generalize many of the results for the function classes introduced earlier.

Suppose that p and q are positive integers and x and y are real numbers. Let generalized bivariate Fibonacci-like polynomials is denoted by $\mathfrak{F}_n$. Then, $\mathfrak{F}_n$ satisfy the recurrence relation (see [44]):

$$\mathfrak{F}_n(x,y) = px\mathfrak{F}_{n-1}(x,y) + qy\mathfrak{F}_{n-2}(x,y), \ \ n \geq 2, \tag{1.6}$$

where $\mathfrak{F}_0(x,y) = a,\ \mathfrak{F}_1(x,y) = b$ and $px,\ qy \neq 0, p^2x^2 + 4qy \neq 0$. The generating function of $\mathfrak{F}_n$ is (see [44])

$$\mathfrak{F}^{(x,y)}(s) = \sum_{n=0}^{\infty} \mathfrak{F}_n\,(x,y)s^n = \frac{a + (b - apx)s}{1 - pxs - qys^2}. \tag{1.7}$$

In equation (1.7), different choices of the parameters p, q, a, b and y yield the polynomial sequences listed in the Table 1 below:

(p,q)	(a,b)	(x,y)	$\mathfrak{F}_n(x,y)$
$(1,1)$	$(0,1)$	(x,y)	Bivariate Fibonacci, $F_n(x,y)$
$(1,1)$	$(0,1)$	$(x,1)$	Fibonacci, $F_n(x)$
$(2,1)$	$(0,1)$	$(x,1)$	Pell, $P_n(x)$
$(1,1)$	$(2,x)$	(x,y)	Bivariate Lucas, $L_n(x,y)$
$(2,1)$	$(1,2t)$	$(t,-1)$	Chebyshev of the second kind, $U_n(x)$
(p,q)	(a,bx)	$(x,1)$	Horadam, $H_{n+1}(x)$

Table 1. Special cases of $\mathfrak{F}_n(x,y)$

From now on we will write $\mathfrak{F}_n$ instead of $\mathfrak{F}_n(x,y)$ for convenience.

2. MAIN RESULTS

In this section, we first define a novel subclass of bi-univalent functions and then present our findings on members of this class of functions.

Definition 2.1. *Let suppose that* $0 \leq \delta \leq 1, \eta \geq 0, \lambda \geq 1, u \in \mathbb{C}$ and $|u| \leq 1$. *A function* $\mathfrak{f} \in \Sigma$ *is said to be in the set* ${}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u)$ *if the next subordinations hold true:*

$$(1-\delta)\frac{\big((1-u)t\big)^{1-\eta}\mathfrak{f}'(t)}{\big(\mathfrak{f}(t)-\mathfrak{f}(ut)\big)^{1-\eta}} + \delta\frac{(1-u)t\big(\mathfrak{f}'(t)\big)^{\lambda}}{\mathfrak{f}(t)-\mathfrak{f}(ut)} \prec \mathfrak{F}_n^{(x,y)}(t) + 1 - a \tag{2.1}$$

$$(1-\delta)\frac{\big((1-u)s\big)^{1-\eta}\mathfrak{g}'(s)}{\big(\mathfrak{g}(s)-\mathfrak{g}(us)\big)^{1-\eta}} + \delta\frac{(1-u)s\big(\mathfrak{g}'(s)\big)^{\lambda}}{\mathfrak{g}(s)-\mathfrak{g}(us)} \prec \mathfrak{F}_n^{(x,y)}(s) + 1 - a \tag{2.2}$$

where a *is real constant and* $\mathfrak{g} = \mathfrak{f}^{-1}$ *is given by* (1.3).

Remark 2.1. *The class* ${}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u)$ *generalizes some well-known subclasses of bi-univalent functions studied in previously published papers. Some of these works are listed below:*

1. *Setting* $\delta = u = \eta = 0$ *and* $\lambda = 1$, *we have*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{H}_{n,\Sigma,0}^{p;q;x;y}, $$

where $\mathcal{H}_{n,\Sigma,0}^{p;q;x;y}$ *is the subclass of bi-starlike functions investigated by Aktaş and Yılmaz in [16, Definition 1.4].*

2. *For* $\delta = u = 0$, *we get*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{H}_{n,\Sigma,\lambda,1}^{p;q;x;y}, $$

where $\mathcal{H}_{n,\Sigma,\lambda,1}^{p;q;x;y}$ *is the subclass of bi-Bazilevič functions investigated by Aktaş and Yılmaz in [16, Definition 1.6].*

3. *For* $\delta = u = 0$ *and* $\eta = 1$, *we get*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,1,u) =: \mathcal{H}_{n,\Sigma,1,1}^{p;q;x;y}, $$

where $\mathcal{H}_{n,\Sigma,\lambda,1}^{p;q;x;y}$ *is the subclass of bi-Bazilevič functions investigated by Aktaş and Yılmaz in [16, Definition 1.7].*

4. *Taking* $b \to bx$ *and* $y = 1$, *we have*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{D}(\delta,\eta,\lambda,u,x), $$

where $\mathcal{D}(\delta,\eta,\lambda,u,x)$ *is the subclass of bi-univalent functions investigated by Al-Shbeil et al. in [35].*

5. *Taking* $b \to bx$, $y = 1$, $\delta = u = 0$ *we have*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{N}_{\Sigma}(\eta,x), $$

where the class $\mathcal{N}_{\Sigma}(\eta,x)$ *studied by Wanas and Alb Lupas in [33].*

6. *Taking* $b \to bx$, $y = 1$, $\delta = u = \eta = 0$ *we have*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{W}_{\Sigma}(x), $$

where the class $\mathcal{W}_{\Sigma}(x)$ *studied by Srivastava et al. in [31].*

7. *Considering* $\delta = u = 0$, $y = \eta = 1$ *and* $b \to bx$ *we arrive at*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \Sigma'(x), $$

where $\Sigma'(x)$ *is the subclass of bi-univalent functions investigated by Alamoush in [30].*

8. *Setting* $\delta = u = 0$, $a = y = 1$, $b = p = 2$, $q = -1$ *and* $b \to bx$ *we obtain*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{B}(\beta,x), $$

where $\mathcal{B}(\beta,x)$ is the subclass of bi-Bazilevič functions investigated by Altınkaya and Yalçın in [22, Definition 1.2].

9. *Putting $\delta = u = 0,\ \eta = y = a = 1,\ b = p = 2,\ q = -1$ and $b \to bx$, we get*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{B}_{\Sigma}(x), $$

where $\mathcal{B}_{\Sigma}(x)$ is the subclass of bi-univalent functions introduced by Bulut et al. in [25, Remark 1-(iii)].

10. *Setting $u = 0,\ \delta = a = y = 1,\ b = p = 2,\ q = -1$ and $b \to bx$, we have*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{LB}_{\Sigma}(\lambda,x), $$

where $\mathcal{LB}_{\Sigma}(\lambda,x)$ is the subclass of bi-univalent functions introduced by Magesh and Bulut in [26].

11. *Setting $u = 0,\ \delta = a = y = \lambda = 1,\ b = p = 2,\ q = -1$ and $b \to bx$, we have*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{S}_{\Sigma}^{\star}(x), $$

where $\mathcal{S}_{\Sigma}^{\star}(x)$ is the subclass of bi-univalent functions introduced by Bulut et al. in [23, Remark 1-(iv)] and Magesh and Bulut in [26, see p. 205].

12. *Setting $\delta = \eta = u = 0,\ y = a = 1,\ b = p = 2,\ q = -1$ and $b \to bx$, we obtain*

$$ {}_{(p,q)}^{(a,b)}\mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u) =: \mathcal{S}_{\sigma}(x), $$

where $\mathcal{S}_{\sigma}(x)$ is the subclass of bi-univalent functions introduced by Altınkaya and Yalçın in [21, Definition 1.3].

Remark 2.2. *The following specific example shows that the defined subclasses here are not empty set.*

Example 2.1. *The function*

$$ h(t) = \sqrt{\log\frac{1+t}{1-t}} = t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots \in \Sigma $$

is in the class ${}_{(2,-1)}^{(1,2)}\mathfrak{B}_{\Sigma}^{(0.95,0.9)}(0,1,1,0)$. First of all, the function $h(t)$ has an inverse function of the form

$$ \rho(s) = \frac{e^{2s}-1}{e^{2s}+1} = s - \frac{s^3}{3} + \frac{2s^5}{15} - \cdots \in \mathfrak{A}. $$

On the other hand, if we set $\mathfrak{f}(t) = h(t) = \sqrt{\log\frac{1+t}{1-t}}$, $\mathfrak{g}(s) = \rho(s) = \frac{e^{2s}-1}{e^{2s}+1}$, $\lambda = a = \eta = 1,\ q = -1$, $\delta = u = 0,\ b = p = 2,\ b \to bx,\ x = 0.95$ and $y = 0.9$ in (2.1) and (2.2) we get that

$$m_1(t) = \frac{1}{1-t^2} \prec \frac{10}{9t^2 - 19t + 10} = m_2(t) \tag{2.3}$$

and

$$n_1(s) = \frac{4e^{2s}}{(e^{2s}+1)^2} \prec \frac{10}{9s^2 - 19s + 10} = n_2(s), \tag{2.4}$$

which are correct for all $t, s \in \mathcal{U}$. The correctness of the relations (2.3) and (2.4) may be shown using [45, Lemma 2.1, p. 36]. In fact, it is easily seen that the function $m_2(t) = \frac{10}{9t^2-19t+10}$ is analytic and univalent in $\mathcal{U}$. In addition, the functions $m_1(t)$, $n_1(s)$ and $m_2(t)$ satisfy hypothesis of [40, Lemma 2.1, p. 36]. More precisely, $m_1(0) = m_2(0) = n_1(0) = n_2(0) = 1$, $m_1(\mathcal{U}) \subset m_2(\mathcal{U})$ (see FIGURE 1.-(iv)) and $n_1(\mathcal{U}) \subset n_2(\mathcal{U})$ (see FIGURE 1.-(v)) for $\forall t, s \in \mathcal{U}$. Therefore, we see that the function $h(t) \in_{(2,-1)}^{(1,2)} \mathfrak{B}_{\Sigma}^{(0.95,0.9)}(0,1,\lambda,0)$.

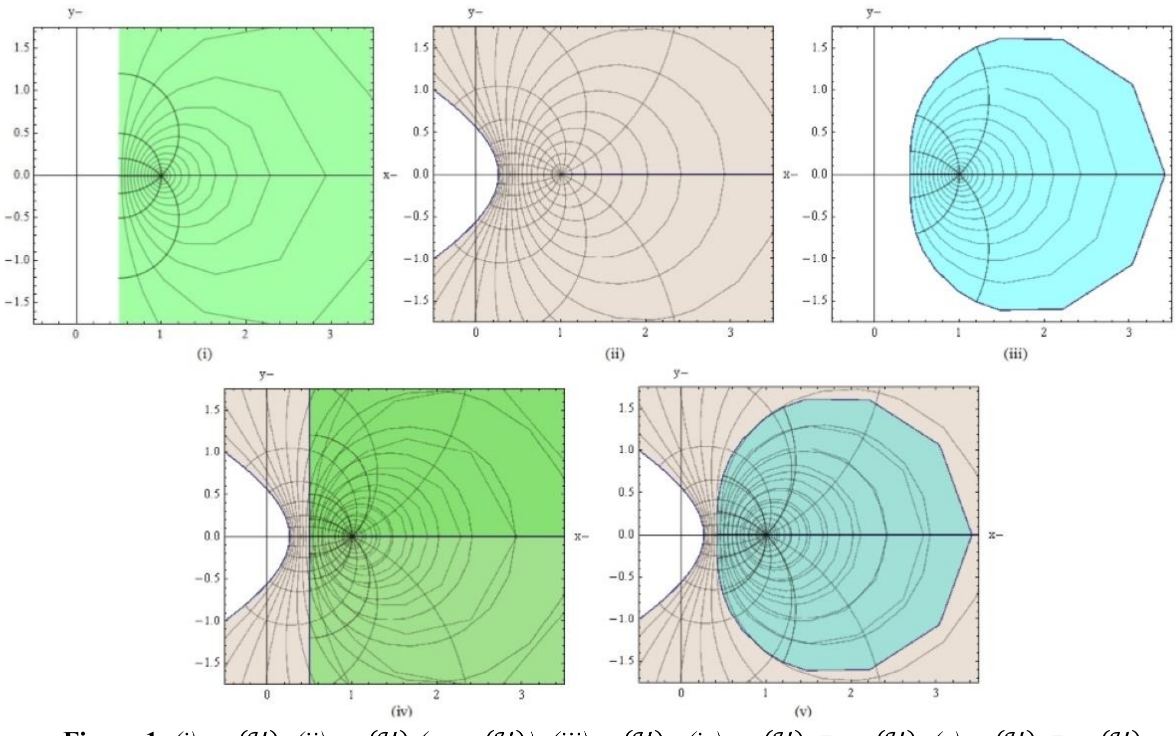

Figure 1. *(i) $m_1(\mathcal{U})$ (ii) $m_2(\mathcal{U})$ (or $n_2(\mathcal{U})$) (iii) $n_1(\mathcal{U})$ (iv) $m_1(\mathcal{U}) \subset m_2(\mathcal{U})$ (v) $n_1(\mathcal{U}) \subset n_2(\mathcal{U})$*

Theorem 2.1. *Suppose that $0 \leq \delta \leq 1, \eta \geq 0, \lambda \geq 1, u \in \mathbb{C}, |u| \leq 1$ and $r \in \mathbb{R}$. If the function $\mathfrak{f} \in_{(p,q)}^{(a,b)} \mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u)$, then*

$$|a_2| \leq \frac{|b|\sqrt{|b|}}{\sqrt{|b^2[\Omega(\delta,\eta,\lambda,u) + \Upsilon(\delta,\eta,\lambda,u)] - (bpx + aqy)\Lambda(\delta,\eta,\lambda,u)|}} \tag{2.5}$$

$$|a_3| \leq \frac{b^2}{|\Lambda(\delta,\eta,\lambda,u)|} + \frac{|b|}{|\Omega(\delta,\eta,\lambda,u)|} \tag{2.6}$$

where

$$\Omega(\delta,\eta,\lambda,u) = (1-\delta)[3-(1-\eta)(u^2+u+1)] + \delta(3\lambda - u^2 - u - 1),$$

$$\Upsilon(\delta,\eta,\lambda,u) = (1-\delta)(1-\eta)(u+1)\left[\frac{1}{2}(2-\eta)(u+1)-2\right] + \delta[(u+1)^2 - 2\lambda(u-\lambda+2)]$$

and

$$\Lambda(\delta,\eta,\lambda,u) = \{(1-\delta)[2-(1-\eta)(u+1)] + \delta(2\lambda - u - 1)\}^2.$$

Proof. Let $\mathfrak{f} \in_{(p,q)}^{(a,b)} \mathfrak{B}_{\Sigma}^{(x,y)}(\delta,\eta,\lambda,u)$ and $\mathfrak{g} = \mathfrak{f}^{-1}$. Then, there exist Schwarz functions $\rho, \kappa: \mathcal{U} \to \mathcal{U}$ given by

$$\rho(t) = \sum_{k=1}^{\infty} \rho_k t^k,\ (t \in \mathcal{U}) \quad \text{and} \quad \kappa(s) = \sum_{k=1}^{\infty} \kappa_k s^k,\ (s \in \mathcal{U})$$

such that the definition of Schwarz functions implies

$$|\rho_k| \leq 1 \tag{2.7}$$

and

$$|\kappa_k| \leq 1, \tag{2.8}$$

where $k \in \mathbb{N}$. Definition 2.1 and subordination principle yield that the next relations hold:

$$(1-\delta)\frac{\big((1-u)t\big)^{1-\eta}\mathfrak{f}'(t)}{\big(\mathfrak{f}(t)-\mathfrak{f}(ut)\big)^{1-\eta}} + \delta\frac{(1-u)t\big(\mathfrak{f}'(t)\big)^{\lambda}}{\mathfrak{f}(t)-\mathfrak{f}(ut)} = \mathfrak{F}_n^{(x,y)}\big(\rho(t)\big) + 1 - a \tag{2.9}$$

and

$$(1-\delta)\frac{\big((1-u)\mathfrak{g}\big)^{1-\eta}\mathfrak{g}'(s)}{\big(\mathfrak{g}(s)-\mathfrak{g}(us)\big)^{1-\eta}} + \delta\frac{(1-u)s\big(\mathfrak{g}'(s)\big)^{\lambda}}{\mathfrak{g}(s)-\mathfrak{g}(us)} = \mathfrak{F}_n^{(x,y)}\big(\kappa(s)\big) + 1 - a. \tag{2.10}$$

Taking into account the series expansion of the function $\rho(t)$ and comparing the coefficients in (2.9), we arrive at the following equations:

$$\Lambda(\delta,\eta,\lambda,u)a_2 = \mathfrak{F}_1\rho_1 \tag{2.11}$$

$$\Omega(\delta,\eta,\lambda,u)a_3+\Upsilon(\delta,\eta,\lambda,u)a_2^2=\mathfrak{F}_1\rho_2+\mathfrak{F}_2\rho_1^2. \tag{2.12}$$

Similarly, by the series expansion of $\kappa(s)$ and comparing the coefficients in (2.10), we have the next equations:

$$-\Lambda(\delta,\eta,\lambda,u)a_2=\mathfrak{F}_1\kappa_1 \tag{2.13}$$

$$\Omega(\delta,\eta,\lambda,u)(2a_2^2-a_3)+\Upsilon(\delta,\eta,\lambda,u)a_2^2=\mathfrak{F}_1\kappa_2+\mathfrak{F}_2\kappa_1^2. \tag{2.14}$$

Equalities (2.11) and (2.13) imply that

$$\rho_1=-\kappa_1 \tag{2.15}$$

and

$$\rho_1^2+\kappa_1^2=\frac{2\Lambda(\delta,\eta,\lambda,u)}{\mathfrak{F}_1^2}a_2^2. \tag{2.16}$$

Adding equations (2.12) and (2.14) and using equation (2.16) gives that

$$a_2^2=\frac{\mathfrak{F}_1^3(\rho_2+\kappa_2)}{2\left[\mathfrak{F}_1^2\big(\Omega(\delta,\eta,\lambda,u)+\Upsilon(\delta,\eta,\lambda,u)\big)-\mathfrak{F}_2\Lambda(\delta,\eta,\lambda,u)\right]}. \tag{2.17}$$

Using (2.7) and (2.8), and substituting $\mathfrak{F}_1=b$ and $\mathfrak{F}_2=bpx+aqy$ in (2.17), we conclude (2.5).

In order to obtain an upper bound on $|a_3|$, we firstly substract (2.14) from (2.12) and consider the fact that $\kappa_1^2-\rho_1^2=0$. Therefore, we get

$$a_3=a_2^2+\frac{\mathfrak{F}_1(\rho_2-\kappa_2)}{2\Omega(\delta,\eta,\lambda,u)}. \tag{2.18}$$

In (2.18), by substituting a_2^2, which we got from (2.16), we have

$$a_3=\frac{\mathfrak{F}_1^2(\rho_1^2+\kappa_1^2)}{2\Lambda(\delta,\eta,\lambda,u)}+\frac{\mathfrak{F}_1(\rho_2-\kappa_2)}{2\Omega(\delta,\eta,\lambda,u)}. \tag{2.19}$$

Using the inequalities (2.7) and (2.8) with the fact that $\mathfrak{F}_1=b$ in (2.19) a simple calculation gives (2.6).
□

Example 2.2. *The function* $h(t)=\sqrt{\log\frac{1+t}{1-t}}\in {}_{(2,-1)}^{(1,2)}\mathfrak{B}_{\Sigma}^{(0.95,0.9)}(0,1,1,0)$ *fits the requirements of Theorem 2.1. Replacing the values of the variables in the Theorem 2.1, we get that* $|a_2|=0\le\frac{19\sqrt{19}}{\sqrt{5410}}$ *and* $|a_3|=\frac{1}{3}\le\frac{1463}{600}$.

Theorem 2.2. *Suppose that* $0 \leq \delta \leq 1, \eta \geq 0, \lambda \geq 1, u \in \mathbb{C}, |u| \leq 1$ *and* $r \in \mathbb{R}$. *If the function* $\mathfrak{f} \in_{(p,q)}^{(a,b)} \mathfrak{B}_{\Sigma}^{(x,y)}(\delta, \eta, \lambda, u)$, *then*

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \dfrac{|b|}{\Omega(\delta,\eta,\lambda,u)}, & |1-\tau| \leq \dfrac{|\xi|}{b^2 \Omega(\delta,\eta,\lambda,u)} \\ \dfrac{|b^3||1-\tau|}{|\xi|}, & |1-\tau| \geq \dfrac{|\xi|}{b^2 \Omega(\delta,\eta,\lambda,u)} \end{cases}, \tag{2.20}$$

where $\tau \in \mathbb{R}$ *and* $\xi = b^2[\Omega(\delta,\eta,\lambda,u) + \Upsilon(\delta,\eta,\lambda,u)] - (bpx + aqy)\Lambda(\delta,\eta,\lambda,u)$.

Proof. In order to obtain a bound on Fekete-Szegő functional $\left|a_3 - \tau a_2^2\right|$ we will use the values of a_2^2 and a_3 which given in (2.17) and (2.18), respectively. For this purpose, we write the functional $a_3 - \tau a_2^2$ as:

$$\begin{aligned} a_3 - \tau a_2^2 &= a_3 - a_2^2 + (1-\tau) a_2^2 \\ &= \frac{\mathfrak{F}_1(\rho_2 - \kappa_2)}{2\Omega(\delta,\eta,\lambda,u)} + \frac{(1-\tau)\mathfrak{F}_1^3(\rho_2 + \kappa_2)}{2\left[\mathfrak{F}_1^2\big(\Omega(\delta,\eta,\lambda,u) + \Upsilon(\delta,\eta,\lambda,u)\big) - \mathfrak{F}_2 \Lambda(\delta,\eta,\lambda,u)\right]} \\ &= \mathfrak{F}_1 \left\{\left[\frac{1}{2\Omega(\delta,\eta,\lambda,u)} + \psi(\tau)\right]\rho_2 + \left[-\frac{1}{2\Omega(\delta,\eta,\lambda,u)} + \psi(\tau)\right]\kappa_2\right\}, \end{aligned} \tag{2.21}$$

where $\psi(\tau) = \frac{(1-\tau)\mathfrak{F}_1^2}{2\left[\mathfrak{F}_1^2\left(\Omega(\delta,\eta,\lambda,u) + \Upsilon(\delta,\eta,\lambda,u)\right) - \mathfrak{F}_2 \Lambda(\delta,\eta,\lambda,u)\right]}$. Considering the inequalities (2.7) and (2.8) with $\mathfrak{F}_1 = b$ and the properties of the absolute value we arrive at (2.20). □

Setting $\tau = 1$ in Theorem 2.2 we are led to the next corollary.

Corollary 2.1. *Suppose that* $0 \leq \delta \leq 1, \eta \geq 0, \lambda \geq 1, u \in \mathbb{C}, |u| \leq 1$ *and* $r \in \mathbb{R}$. *If* $\mathfrak{f} \in_{(p,q)}^{(a,b)} \mathfrak{B}_{\Sigma}^{(x,y)}(\delta, \eta, \lambda, u)$, *then*

$$\left|a_3 - a_2^2\right| \leq \frac{|b|}{\Omega(\delta,\eta,\lambda,u)}. \tag{2.22}$$

Remark 2.3. *Setting special values for the variables* $u, \delta, \eta, a, b, p, q$ *in Theorems 2.1 and 2.2 several known results may be deduced. These are listed below:*

1. *Taking* $\delta = u = \eta = 0$ *and* $\lambda = 1$ *in Theorems 2.1 and 2.2 we get [14, Remark 2.2-(i)] and [16, Remark 4.2-(ii)], respectively.*

2. *Taking* $\delta = u = 0$ *in Theorems 2.1 and 2.2 reduce to the [14, Remark 3.2-(i)] and [16, Remark 4.4-(ii)], respectively.*

3. *Taking* $\delta = u = 0$ *and* $\eta = 1$ *in Theorems 2.1 and 2.2 we arrive at [14, Remark 3.2-(iii)] and [16, Remark 4.4-(i)], respectively.*

4. *Taking* $b \to bx$ *and* $y = 1$ *in Theorems 2.1 and 2.2 we deduce Theorem 1 and Theorem 2 given by Al-Shbeil et al. in [35], respectively.*

5. *Taking* $u = \delta = 0,\ b \to bx$ *and* $y = 1$ *in Theorems 2.1 and 2.2 we get Corollary 2.1 and Corollary 2.2 given by Wanas and Lupas in [33], respectively.*

6. *Taking* $u = \delta = \eta = 0,\ b \to bx$ *and* $y = 1$ *in Theorems 2.1 and 2.2 we get Corollary 1 and Corollary 3 given by Srivastava et al. in [31], respectively.*

7. *Replacing* $\delta = u = 0,\ y = \eta = 1,\ b \to bx$ *and combining Theorems 2.1 and 2.2 we arrive at the Theorem 2.2 given by Alamoush in [30].*

8. *Replacing* $\delta = u = 0,\ y = a = 1,\ b = p = 2,\ q = -1,\ b \to bx$ *implies that Theorems 2.1 and 2.2 reduce to the Theorems 2.1 and 3.1 given by Altınkaya and Yalçın in [22].*

9. *Replacing* $\delta = u = 0,\ y = \eta = a = 1,\ b = p = 2,\ q = -1,\ b \to bx$ *and combining Theorems 2.1 and 2.2 we get Corollary 3 given by Bulut et al. in [25].*

10. *Replacing* $u = 0,\ \delta = a = y = 1,\ b = p = 2,\ q = -1$ *and* $b \to bx$*, combining Theorems 2.1 and 2.2 we get Theorem 1 given by Magesh and Bulut in [26].*

11. *Replacing* $u = 0,\ \delta = a = y = \lambda = 1,\ b = p = 2,\ q = -1$ *and* $b \to bx$*, combining Theorems 2.1 and 2.2 we get Corollary 4 given by Bulut et al. in [25].*

12. *Replacing* $\delta = \eta = u = 0,\ a = y = 1,\ b = p = 2,\ q = -1$ *and* $b \to bx$*, combining Theorems 2.1 and 2.2 we get Corollary 2.2 and Corollary 3.3 given by Altınkaya and Yalçın in [21].*

3. CONCLUSIONS

The main purpose of this study was to introduce and investigate a new subclass of bi-univalent functions. In order to define this function class Bazilevič and λ-pseudo-starlike functions were used. Certain bounds were obtained for coefficient estimates and Fekete-Szegő functional of this function class in terms of Fibonacci-like polynomial. It is important to note that this paper generalize some results of previously published papers.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Hasan Hüseyin Güleç: Conceptualization, Methodology, Software, Investigation, Writing-Original Draft, Writing-Review & Editing, **İbrahim Aktaş:** Conceptualization, Methodology, Software, Investigation, Writing-Original Draft, Writing-Review & Editing, Visualization, Supervision.

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