

Biharmonic Hypersurfaces in Projective Spaces Revisited

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ABSTRACT

We study biharmonic homogeneous real hypersurfaces in complex projective space and quaternion projective space. We provide a classification of biharmonic homogeneous real hypersurfaces in quaternion projective space. We also classify pseudo-harmonic, subelliptic biharmonic, and Levi-harmonic homogeneous Hopf hypersurfaces in complex space forms.

Keywords: Biharmonic maps, complex projective space, quaternion projective space, Riemannian symmetric spaces, hypersurfaces, CR-manifolds; pseudo-harmonic map, subelliptic biharmonic map, Levi-harmonic map.

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1. Introduction

Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds. For a smooth map $f : M \rightarrow \widetilde{M}$, its *Dirichlet energy* $E(f; \Omega)$ over a relatively compact domain $\Omega \subset M$ is defined by

$$E(f; \Omega) = \int_{\Omega} \frac{1}{2} |df|^2 dv_g.$$

Then f is said to be *harmonic* if it is a critical point of the Dirichlet energy with respect to compactly supported variations [32]. The Euler-Lagrange equation of this variational problem is

$$\tau(f) = \text{tr}_g(\nabla df) = 0.$$

Here $\tau(f)$ is the tension field of f (see Section 2.2).

The theory of harmonic maps is a central topic in geometric analysis with numerous applications in differential geometry. Examples include Siu's strong rigidity theorem on Kähler structures with strongly negative curvature [86], Corlette's superrigidity over archimedean fields [26], Mok-Siu-Yeung's geometric superrigidity [65].

If a smooth map $f : M \rightarrow \widetilde{M}$ is an isometric immersion, i.e., $f^*\widetilde{g} = g$, then f is harmonic if and only if f is a minimal immersion. Thus, minimal submanifolds are specific example of harmonic maps.

However, some mapping spaces do not contain harmonic maps. For example, Eells and Wood [34] proved that the space $\text{Map}_1(\mathbb{T}^2, \mathbb{S}^2)$ of all smooth maps of mapping degree 1 from a 2-torus $\mathbb{T}^2$ into a unit 2-sphere $\mathbb{S}^2$ does not contain harmonic maps. To find alternative representatives in each homotopy class, another variational problem was proposed by Eells and Sampson [32]. They suggested the following functional (*bienergy*):

$$E_2(f; \Omega) = \int_{\Omega} \frac{1}{2} |\tau(f)|^2 dv_g.$$

Jiang [54] deduced the Euler-Lagrange equation

$$\tau_2(f) = 0$$

for the bienergy (see (2.4)). A smooth map f is said to be *biharmonic* if it satisfies $\tau_2(f) = 0$. Over the last two decades, there has been growing interest in the theory of biharmonic maps and biharmonic submanifolds. For a general theory and fundamental results on biharmonic maps, see the monograph [75] by Chen and Ou. In particular, we refer to [100] for a recent study on biharmonic maps from 2-tori into the unit 2-sphere.

In this article, we revisit the biharmonic hypersurface geometry of complex as well as quaternion projective spaces. Additionally, we study other types of harmonicity and biharmonicity of real hypersurfaces in complex projective space. Specifically, we study pseudo-harmonicity (Section 8), subelliptic biharmonicity (Section 8) and Levi-harmonicity (Section 9) of homogeneous (Hopf) hypersurfaces of $\mathbb{C}P^n$.

This article is organized as follows: In Section 2, we recall fundamental facts on vector bundle calculus, harmonic maps and biharmonic maps as well as hypersurface geometry. Section 3 is devoted to CR-manifolds (Cauchy-Riemann manifolds) and contact metric manifolds. We recall fundamental theory of real hypersurfaces in complex space forms in Sections 4–6.

We study biharmonic real hypersurfaces of complex projective space $\mathbb{C}P^n$ in Section 7. We give a (correct) classification of biharmonic homogeneous hypersurfaces in $\mathbb{C}P^n$ (Theorem 7.9).

Section 8 turns our attention to harmonic maps and biharmonic map in CR-geometry. The notion of harmonicity for smooth maps between Riemannian manifolds was adapted to smooth maps from strongly pseudo-convex CR-manifolds into Riemannian manifolds as the “pseudo-harmonicity” in [4]. Next, the biharmonicity is adapted for those maps as “subelliptic biharmonicity” in [30]. In Section 8, we study pseudo-harmonicity and subelliptic biharmonicity of homogeneous real hypersurfaces in $\mathbb{C}P^n$.

The notion of Levi-harmonicity for smooth maps from almost contact metric manifolds into Riemannian manifolds was introduced by Dragomir and Perrone [31]. In Section 9, we study Levi-harmonicity of homogeneous Hopf hypersurfaces in the complex projective space $\mathbb{C}P^n$ and the complex hyperbolic space $\mathbb{C}H^n$.

In the final section, we return to the original (Riemannian geometric) harmonicity and biharmonicity of isometric immersions. We provide the (correct) classification of biharmonic homogeneous real hypersurfaces in the quaternion projective space $\mathbb{H}P^n$.

Throughout this article, we denote by $\Gamma(E)$ the space of all smooth sections of a vector bundle E .

2. Preliminaries

2.1. Vector bundles

Let E be a vector bundle over a manifold M . Take a pair (h^E, ∇^E) consisting of a fiber metric h^E and a connection ∇^E of E satisfying the condition $\nabla^E h^E = 0$, i.e.,

$$(\nabla_X^E h^E)(V, W) = X \cdot h^E(V, W) - h^E(\nabla_X^E V, W) - h^E(V, \nabla_X^E W) = 0$$

for all $X \in \Gamma(TM)$ and $V, W \in \Gamma(E)$. Such a pair is called a *Riemannian structure* on E and (E, h^E, ∇^E) is called a *Riemannian vector bundle*. The curvature form R^{∇^E} of E is defined by

$$R^{\nabla^E}(X, Y)V = \nabla_X^E \nabla_Y^E V - \nabla_Y^E \nabla_X^E V - \nabla_{[X, Y]}^E V, \quad X, Y \in \Gamma(TM), \quad V \in \Gamma(E).$$

Let us denote by $A^r(E) = \Gamma(\wedge^r T^*M \otimes E)$, the space of all E -valued smooth r -forms. The *exterior-covariant differential* $d^{\nabla^E} : A^r(E) \rightarrow A^{r+1}(E)$ is defined by

$$\begin{aligned} (d^{\nabla^E} \omega)(X_1, X_2, \dots, X_{r+1}) &= \sum_{i=1}^{r+1} (-1)^{i+1} \nabla_{X_i}^E \omega(X_1, X_2, \dots, \hat{X}_i, \dots, X_{r+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, X_2, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{r+1}). \end{aligned}$$

Let us assume that M is an oriented Riemannian m -manifold with Riemannian metric g . Then the *codifferential* $\delta^{\nabla^E} : A^r(E) \rightarrow A^{r-1}(E)$ is defined by

$$(\delta^{\nabla^E} \omega)(X_1, X_2, \dots, X_{r-1}) = - \sum_{i=1}^m (\nabla_{e_i}^E \omega)(e_i, X_1, X_2, \dots, X_{r-1}),$$

where $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal frame field of (M, g) . The *rough Laplacian* $\overline{\Delta} = \overline{\Delta}^E$ and *Hodge-de Rham Laplacian* Δ^E are defined by

$$\overline{\Delta} = (\nabla^E)^* \nabla^E = - \sum_{i=1}^m \left(\nabla_{e_i}^E \nabla_{e_i}^E - \nabla_{\nabla_{e_i}^E e_i}^E \right), \quad \Delta^E = d^{\nabla^E} \delta^{\nabla^E} + \delta^{\nabla^E} d^{\nabla^E}, \quad (2.1)$$

respectively. These two operators are related by *Weitzenböck formula*:

$$\Delta^{\nabla^E} = \overline{\Delta} + S^{\nabla^E}, \quad (2.2)$$

where

$$(S^{\nabla^E} \omega)(X_1, X_2, \dots, X_r) = \sum_{j,k} (-1)^{k+1} (R^{\nabla^E}(e_j, X_k) \omega)(e_j, X_1, X_2, \dots, \widehat{X_k}, \dots, X_r).$$

In particular, for an E -valued 1-form ω , we have

$$(\Delta^{\nabla^E} \omega)(X) = (\overline{\Delta} \omega)(X) + (S^{\nabla^E} \omega)(X), \quad (S^{\nabla^E} \omega)(X) = \sum_{i=1}^m (R^{\nabla^E}(e_i, X) \omega)(e_i). \quad (2.3)$$

2.2. Harmonic maps

Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and N , respectively. The Riemannian curvatures of M and N are denoted by R and $\tilde{R}$, respectively. For a smooth map $f : M \rightarrow \widetilde{M}$, the Levi-Civita connection $\tilde{\nabla}$ induces a connection ∇^f on the pull-back bundle $f^*T\widetilde{M}$ over M described explicitly by

$$\nabla_X^f (V \circ f) = (\tilde{\nabla}_{df(X)} V) \circ f$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(f^*T\widetilde{M})$. For any sections X and $Y \in \Gamma(TM)$,

$$\nabla_X^f df(Y) = \nabla_{df(X)}^{\tilde{g}} df(Y)$$

holds (see [33, p. 4]). The *second fundamental form* ∇df is defined by

$$(\nabla df)(X, Y) = \nabla_X^f df(Y) - df(\nabla_X Y).$$

The *tension field* $\tau(f)$ is a section of $f^*T\widetilde{M}$ defined by

$$\tau(f) = \text{tr}_g(\nabla df) = \sum_{i=1}^m (\nabla df)(e_i, e_i),$$

where $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal frame field of M ($m = \dim M$). Then f is said to be *harmonic* if its tension field vanishes. The harmonicity has a variational characterization. Indeed, f is harmonic if and only if it is a critical point of the *Dirichlet energy*:

$$E(f; \Omega) = \int_{\Omega} \frac{1}{2} |df|^2 dv_g = \int_{\Omega} \frac{1}{2} \sum_{i=1}^m \tilde{g}(df(e_i), df(e_i)) dv_g$$

over any relatively compact domain $\Omega \subset M$ with respect to compactly supported variations. The first variational formula for E is given by

$$\left. \frac{d}{dt} \right|_{t=0} E(f_t; \Omega) = - \int_{\Omega} \tilde{g}(\tau(f), V) dv_g.$$

Here $\{f_t\}$ is a variation through $f = f_0$ and V is the *variational vector field* of $\{f_t\}$, i.e.,

$$V = \left. \frac{\partial}{\partial t} \right|_{t=0} f_t.$$

Example 2.1. Let (M, g, J) and $(\widetilde{M}, \tilde{g}, \tilde{J})$ be Kähler manifolds. Then a smooth map $f : M \rightarrow \widetilde{M}$ is said to be *holomorphic* if it satisfies $df \circ J = \tilde{J} \circ df$. Lichnerowicz [61] proved that any holomorphic map is a Dirichlet energy minimizing harmonic map in its homotopy class.

The differential df is regarded as an $f^*T\widetilde{M}$ -valued 1-form. The symmetry of ∇df is expressed as $d^{\nabla^f}(df) = 0 \in A^2(f^*T\widetilde{M})$. On the other hand, the tension field $\tau(f)$ is expressed as $\tau(f) = -\delta^{\nabla^f}(df) \in A^0(f^*T\widetilde{M})$. Hence if f is a harmonic map, then its differential df is a $f^*T\widetilde{M}$ -valued harmonic 1-form. By Weitzenböck formula (2.3), we get

$$(\Delta^{\nabla^f}(df))(X) = (\bar{\Delta}_f df)(X) - \text{tr}_g R^{\nabla^f}(\cdot, X)df,$$

where the curvature term

$$\text{tr}_g R^{\nabla^f}(\cdot, X)df = \sum_{i=1}^m (R^{\nabla^f}(e_i, X)df)(e_i).$$

is computed as

$$\sum_{i=1}^m (R^{\nabla^f}(e_i, X)df)(e_i) = \sum_{i=1}^m \tilde{R}(df(e_i), df(X))df(e_i) - df\left(\sum_{i=1}^m R(X, e_i)e_i\right).$$

Note that we denote the rough Laplacian

$$\bar{\Delta}^{f^*TN} = -\sum_{i=1}^m \left(\nabla_{e_i}^f \nabla_{e_i}^f - \nabla_{\nabla_{e_i}^f e_i}^f \right)$$

of $f^*T\widetilde{M}$ by $\bar{\Delta}_f$.

The second variational formula for a harmonic map f is given by

$$\left. \frac{d^2}{dt^2} \right|_{t=0} E(f_t; \Omega) = \int_{\Omega} \tilde{g}(\mathcal{J}_f(V), V) dv_g.$$

Here the operator $\mathcal{J}_f$ acting on $\Gamma(f^*T\widetilde{M})$ is defined by

$$\mathcal{J}_f(V) = \bar{\Delta}_f V - \text{tr}_g \tilde{R}(V, df)df,$$

and called the *Jacobi operator* of a harmonic map f .

2.3. Biharmonic maps

As we mentioned in the Introduction, some mapping spaces do not contain any harmonic maps. To look for alternative representatives in each homotopy class, the notion of bienergy as well as biharmonicity were proposed [32, 33]. The *bienergy* of a smooth map $f : (M, g) \rightarrow (\widetilde{M}, \tilde{g})$ over a relatively compact domain Ω is defined by

$$E_2(f; \Omega) = \int_{\Omega} \frac{1}{2} \sum_{i=1}^m \tilde{g}(\tau(f), \tau(f)) dv_g.$$

The Euler-Lagrange equation of the bienergy is

$$\tau_2(f) = -\mathcal{J}_f(\tau(f)) = 0. \quad (2.4)$$

The section $\tau_2(f) \in \Gamma(f^*T\widetilde{M})$ is called the *bitension field* of f . A smooth map f is said to be *biharmonic* if its bitension field vanishes. Clearly, harmonic maps are biharmonic.

2.4. Hypersurfaces

When a map $f : (M, g) \rightarrow (\widetilde{M}, \tilde{g})$ is an isometric immersion, then ∇df coincides with the (vector valued) second fundamental form of f in the sense of submanifold geometry. The tension field $\tau(f)$ is expressed as $\tau(f) = mH$, where H is the mean curvature vector field. Thus, an isometric immersion f is a harmonic map if and only if it is a minimal immersion.

This article focuses on hypersurfaces. Let $f : (M^m, g) \rightarrow (\widetilde{M}^{m+1}, \tilde{g})$ be an orientable hypersurface immersion with unit normal vector field ν . Then the Levi-Civita connections ∇ and $\tilde{\nabla}$ are related by the *Gauss formula*:

$$\nabla_X^f df(Y) = df(\nabla_X Y) + g(AX, Y)\nu, \quad X, Y \in \Gamma(TM).$$

The endomorphism field A is called the *shape operator* derived from ν and defined by the *Weingarten formula*:

$$\nabla_X \nu = -df(AX), \quad X \in \Gamma(TM).$$

The second fundamental form of the immersion f is represented as $(\nabla df)(X, Y) = g(AX, Y)\nu$. An eigenvector X of the shape operator A is called a *principal curvature vector*. The corresponding eigenvalue λ of A is called a *principal curvature*. The function $H = \text{tr}_g A/m$ is called the *mean curvature* of M . The mean curvature vector field H is expressed as $H = H\nu$. For any section W of the normal bundle $T^\perp M$, we have the splitting

$$\nabla_X^f W = -df(A_W X) + \nabla_X^\perp W$$

of $\nabla_X^f W$ into its tangential part $-df(A_W X)$ and normal part $\nabla_X^\perp W$. This formula defines an endomorphism field A_W on M (called the *Weingarten map*) and the connection $\nabla^\perp$ on $T^\perp M$ (called the *normal connection*). A section $W \in \Gamma(T^\perp M)$ is said to be *parallel* if it satisfies $\nabla_X^\perp W = 0$ for any $X \in \Gamma(TM)$.

2.5. Ou's formula

Ou derived the following criterion for biharmonicity of hypersurfaces in Einstein manifolds.

Theorem 2.1 ([74]). *Let $f : (M^m, g) \rightarrow (\widetilde{M}^{m+1}, \tilde{g})$ be an orientable hypersurface with shape operator A . Assume that the ambient space N is an Einstein manifold with Ricci tensor field $\widetilde{\text{Ric}} = \lambda \tilde{g}$ and f has constant mean curvature H . Then the isometric immersion f is biharmonic if and only if either f is minimal or non-minimal with*

$$|A|^2 = \lambda.$$

Furthermore, in the latter case, both the hypersurface and the ambient space must have positive scalar curvatures:

$$\rho = (m-2)\lambda + m^2 H^2 > 0, \quad \tilde{\rho} = (m+1)\lambda > 0.$$

Ou's formula implies that the only biharmonic hypersurfaces of constant mean curvature in an Einstein manifold of *non-positive* scalar curvature are minimal ones.

For more information on biharmonic maps, we refer to the book by Ou and Chen [75].

3. CR-manifolds and contact metric manifolds

3.1. Contact structures

Let M be a manifold of odd dimension $m = 2n - 1 > 1$. A 1-form η is said to be a *contact form* if $\eta \wedge (d\eta)^{n-1} \neq 0$ on whole M . A $(2n - 1)$ -manifold M together with a contact form η is called a *contact manifold*.

On a contact manifold (M, η) , there exists a unique vector field ξ such that

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0.$$

The vector field ξ is called the *Reeb vector field* of a contact manifold (M, η) . A diffeomorphism f of M is said to be a *contactmorphism* if there exists a non-vanishing smooth function λ on M such that $f^*\eta = \lambda\eta$. In particular, a *strict contactmorphism* is a contactmorphism satisfying $f^*\eta = \eta$.

3.2. CR-manifolds

Let M be an m -manifold. An almost CR-structure (also called a *partial complex structure*) is a real vector subbundle $D \subset TM$ of the tangent bundle of M together with a bundle morphism J satisfying $J^2 = -I$. For an almost CR-structure (D, J) , we obtain a complex vector subbundle

$$\mathcal{S} = \{X - \sqrt{-1}JX \mid X \in D\} \quad (3.1)$$

of the complexified tangent bundle $T^{\mathbb{C}}M$. Conversely, let $\mathcal{S}$ be a complex vector subbundle of $T^{\mathbb{C}}M$ satisfying $\mathcal{S} \cap \overline{\mathcal{S}} = \{0\}$, then we obtain an almost CR-structure (D, J) satisfying (3.1). Thus $\mathcal{S}$ is also called an almost CR-structure. The pair (D, J) is called the *real expression* of $\mathcal{S}$.

A manifold M equipped with an almost CR-structure is called an *almost CR-manifold*. An almost CR-structure is said to be *integrable* if $\mathcal{S}$ satisfies the integrability condition $[\Gamma(\mathcal{S}), \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S})$. An almost CR-manifold $M = (M, \mathcal{S})$ is said to be a *CR-manifold* (Cauchy-Riemann manifold) if its almost CR-structure is integrable. A diffeomorphism $f : M \rightarrow M$ of a CR-manifold M is said to be a *CR-diffeomorphism* if its differential df preserves (D, J) , that is,

$$df_p(D_p) = D_{f(p)}, \quad df_p|_D \circ J_p = J_{f(p)} \circ df_p|_D$$

at any point $p \in M$. The group $\text{Aut}(M)$ of all CR-diffeomorphisms is called the CR-automorphism group of M .

Example 3.1 (Complex manifolds). Let (M, J) be an almost complex manifold. Then (TM, J) is an almost CR-structure. Obviously (TM, J) is integrable if and only if (M, J) is a complex manifold. On a complex manifold (M, J) , the complex vector subbundle

$$T'M = \{X - \sqrt{-1}JX \mid X \in TM\}$$

of $T^{\mathbb{C}}M$ is called the *holomorphic tangent bundle* of (M, J) .

Example 3.2 (Real hypersurfaces). Let M be a real hypersurface of a complex n -manifold $(\widetilde{M}, J)$ with holomorphic tangent bundle $T'\widetilde{M}$. Define a subbundle $\mathcal{S}$ of $T^{\mathbb{C}}M$ by

$$\mathcal{S}_p = T'_p \widetilde{M} \cap T_p^{\mathbb{C}}M, \quad p \in M.$$

Then $(M, \mathcal{S})$ is a CR-manifold.

Example 3.3 (CR-submanifolds). Let M be a submanifold of an almost Hermitian manifold $(\widetilde{M}, \tilde{g}, J)$. Then M is said to be a *CR-submanifold* (complex-real submanifold) in the sense of Bejancu [5] if there exists a non-trivial distribution D on M satisfying:

- $J_p(D_p) \subset D_p$ for any $p \in M$.
- $J_p(D_p^{\perp}) \subset T_p^{\perp}M$ for any $p \in M$.

On a CR-submanifold M , $(D, J|_D)$ is an almost CR-structure, in general. Blair and Chen [12] proved that any CR-submanifold of a Kähler manifold is a CR-manifold.

Assume that a CR-manifold M is orientable, $\dim M = 2n - 1$, $\text{rank } D = 2n - 2$ and there exists a 1-form ϑ annihilating D , that is, $D = \text{Ker } \vartheta$. Then, the *Levi-form* L_{ϑ} (with respect to ϑ) is defined by

$$L_{\vartheta}(X, Y) := -\frac{1}{2}d\vartheta(X, JY), \quad X, Y \in \Gamma(D).$$

An almost CR-manifold (M, D) is said to be *non-degenerate* if L_{ϑ} is non-degenerate for some ϑ (and in turn all). In case (M, D) is non-degenerate, ϑ is a *contact form* on M , i.e., $\vartheta \wedge (d\vartheta)^{n-1} \neq 0$. In particular, if L is positive definite, then (M, D, η) is said to be a *strongly pseudo-convex CR-manifold* [93]. On a strongly pseudo-convex CR-manifold M , L is extended to a Riemannian metric g_{ϑ} on M by $g_{\vartheta} = L_{\vartheta} + \vartheta \otimes \vartheta$. The metric g_{ϑ} is called the *Webster metric*.

3.3. Contact metric manifolds

Let M be a manifold of odd dimension $m = 2n - 1$. Then M is said to be an *almost contact manifold* if its structure group $\text{GL}(m, \mathbb{R})$ of the linear frame bundle is reducible to $\text{U}(n - 1) \times \{1\}$. This is equivalent to the existence of an endomorphism field ϕ , a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \quad (3.2)$$

From these conditions, one can deduce that $\phi\xi = 0$ and $\eta \circ \phi = 0$.

Moreover, since $\text{U}(n - 1) \times \{1\} \subset \text{SO}(2n - 1)$, M admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM)$. Such a metric is called an *associated metric* of the almost contact manifold $M = (M; \phi, \xi, \eta)$. With respect to the associated metric g , η is metrically dual to ξ , that is $g(X, \xi) = \eta(X)$ for all $X \in \Gamma(TM)$. A structure (ϕ, ξ, η, g) on M is called an *almost contact metric structure*, and a manifold M equipped with an almost contact metric structure is said to be an *almost contact metric manifold*.

A plane section Π at a point p of $(M; \phi, \xi, \eta, g)$ is said to be a ϕ -section if it is invariant under ϕ_p . The sectional curvature function of ϕ -sections are called the ϕ -sectional curvature.

On an almost contact metric manifold M , we define an endomorphism field h by $h = (\mathcal{L}_\xi \phi)/2$. Here $\mathcal{L}_\xi$ denotes the Lie differentiation by ξ .

The *fundamental 2-form* Φ of $(M; \phi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \phi Y), \quad X, Y \in \Gamma(TM).$$

An almost contact metric manifold M is said to be a *contact metric manifold* if $\Phi = d\eta/2$. On a contact metric manifold, η is a *contact form*, i.e., $(d\eta)^{n-1} \wedge \eta \neq 0$. Thus every contact metric manifold is orientable. More precisely, the volume element dv_g induced from the associated metric g coincides with the following volume element ([96, p. 200]):

$$dv_\eta = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \eta \wedge (d\eta)^{n-1} \quad (3.3)$$

determined only by η . Even if M is not contact metric, we may orient M by the volume element

$$dv_g = \frac{(-1)^{n-1}}{(n-1)!} \eta \wedge \Phi^{n-1}. \quad (3.4)$$

On an almost contact metric manifold $(M; \phi, \xi, \eta, g)$,

$$D = \{X \in TM \mid \eta(X) = 0\}, \quad J = \phi|_D$$

defines an almost CR-structure on M . This almost CR-structure is referred to as the *standard almost CR-structure* of $(M; \phi, \xi, \eta, g)$. One can see that the standard almost CR-structure is integrable if and only if

$$[\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi = 0$$

for any $X, Y \in \Gamma(D)$. More strongly, an almost contact metric manifold M is said to be *normal* if

$$[\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi = 0$$

for any $X, Y \in \Gamma(TM)$.

Definition 3.1. An almost contact metric manifold is said to be a *quasi-Sasakian manifold* if it is normal and $d\Phi = 0$.

Definition 3.2. A contact metric manifold is said to be a *Sasakian manifold* if it is normal. In particular, Sasakian manifolds of constant ϕ -sectional curvature are called *Sasakian space forms*.

Sasakian manifolds are quasi-Sasakian, since $\Phi = d\eta/2$. With respect to the Levi-Civita connection ∇ of g , Sasakian property is characterized as follows:

Proposition 3.1. An almost contact metric manifold M is Sasakian if it satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for all $X, Y \in \mathfrak{X}(M)$.

For more information on contact metric geometry, we refer to Blair's monograph [11].

3.4. Tanno tensor fields

Let $(M; \phi, \xi, \eta, g)$ be a *contact metric manifold*. Then the standard almost CR-structure of M is integrable if and only if the *Tanno tensor field*

$$\mathcal{Q}(X, Y) = (\nabla_Y \phi)X + \{(\nabla_Y \eta)(\phi X)\}\xi + \eta(X)\phi(\nabla_Y \xi), \quad X, Y \in \Gamma(TM)$$

vanishes [95].

Assume that the standard almost CR-structure on a contact metric manifold M is integrable, then we can choose $\vartheta = \eta$ and hence the Levi-form L_η with respect to η is

$$L_\eta(X, Y) = g(X, Y), \quad X, Y \in \Gamma(D).$$

Thus the standard CR-structure is strongly pseudo-convex. In particular, the Webster metric g_η coincides with g .

Conversely, let $M = (M, D, J)$ be a strongly pseudo-convex CR-manifold with contact form ϑ . Then by choosing $\eta = \vartheta$, we get a contact structure on M . Denote by $g = g_\eta$ the Webster metric. Let ξ be the Reeb vector field of η . Then we can extend J to a bundle morphism ϕ of TM by $\phi\xi = 0$ and $\phi X = JX$ for $X \in \Gamma(D)$. Then (ϕ, ξ, η, g) is a contact metric structure compatible to η . The integrability of the CR-structure yields the vanishing of the Tanno tensor field.

From these observations, we can regard strongly pseudo-convex CR-manifolds as contact metric manifolds with a vanishing Tanno tensor field.

Remark 3.1. On a strongly pseudo-convex CR-manifold $M = (M, \mathcal{S})$, the contact form ϑ has the opposite sign to the one used in [13]. Under the formulation of [13], η and ϑ is related by $\eta = -\vartheta$. Note that the Levi-form of [13] and ours are identical.

If the standard almost CR-structure of a contact metric manifold $(M; \phi, \xi, \eta, g)$ is integrable, then

$$\nabla_X \xi = -\phi(I + h)X, \quad X \in \Gamma(TM) \quad (3.5)$$

holds.

Proposition 3.2. The standard almost CR-structure of a contact metric manifold $(M; \phi, \xi, \eta, g)$ is integrable if and only if

$$(\nabla_X \phi)Y = g((I + h)X, Y)\xi - \eta(Y)(I + h)X$$

holds for all $X, Y \in \Gamma(TM)$.

Since the normality of (ϕ, ξ, η) is stronger than the integrability of the standard almost CR-structure, we obtain the following well-known fact.

Proposition 3.3. The standard almost CR-structure of a Sasakian manifold is integrable and strongly pseudo-convex. In particular, a strongly pseudo-convex CR-manifold is Sasakian if and only if $h = 0$.

On the other hand, in [93], the notion of normality for strongly pseudo-convex CR-manifolds is introduced as the condition:

$$[\xi, \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S}) \quad \text{and} \quad [\xi, JZ] = J[\xi, Z]$$

for all $Z \in \Gamma(\mathcal{S})$. One can confirm the following fundamental fact (cf. [93]).

Proposition 3.4. A strongly pseudo-convex CR-manifold is normal if and only if it is a Sasakian manifold.

Theorem 3.1 ([94]). Let $M = (M; \phi, \xi, \eta, g)$ be a contact metric manifold. If a diffeomorphism f of M is ϕ -holomorphic, i.e., $df \circ \phi = \phi \circ df$, then there exists a positive constant a such that

$$f^*g = ag + a(a - 1)\eta \otimes \eta, \quad df(\xi) = a\xi, \quad f^*\eta = a\eta.$$

On a strongly pseudo-convex CR-manifold M , a CR-diffeomorphism f is said to be a CR-isometry if it is an isometry with respect to the Webster metric. A local CR-isometry σ_p defined around a point $p \in M$ is called a local CR-symmetry at p if p is a fixed point of σ_p and satisfies $(d\sigma)_p|_{D_p} = -I_{D_p}$.

A strongly pseudo-convex CR-manifold M is said to be locally CR-symmetric if there exists a local CR-symmetry σ_p at any point $p \in M$.

More strongly, M is said to be CR-symmetric if there exists a globally defined CR-symmetry σ_p at any point $p \in M$ [56]. Dileo and Lotta obtained the following two fundamental results.

Theorem 3.2 ([28]). Let M be a non-normal strongly pseudo-convex CR-manifold of dimension $2n - 1 > 3$. Then M is locally CR-symmetric if and only if the underlying contact metric structure satisfies the (κ, μ) -condition:

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad X, Y \in \Gamma(TM). \quad (3.6)$$

Here $\kappa < 1$ and μ are constants.

Remark 3.2. A contact metric manifold M is said to be a *contact (κ, μ) -space* if it satisfies (3.6). The Boeckx invariant $\mathcal{I}$ of a contact (κ, μ) -space M is $\mathcal{I} = (1 - \frac{\mu}{2})/\sqrt{1 - \kappa}$.

The local CR-symmetry of Sasakian manifolds is characterized by the local ϕ -symmetry in the sense of Takahashi [91].

Theorem 3.3 ([28]). Let M be a normal strongly pseudo-convex CR-manifold of dimension $2n - 1 > 3$. Then M is locally CR-symmetric if and only if the underlying Sasakian structure is locally ϕ -symmetric:

$$\phi^2\{(\nabla_V R)(X, Y)Z\} = 0, \quad X, Y, Z, V \in \Gamma(D).$$

3.5. Tanaka-Webster connection

The *Tanaka-Webster connection* $\hat{\nabla}$ of a strongly pseudo-convex CR-manifold M together with associated contact metric structure (ϕ, ξ, η, g) is a linear connection defined by [92, 101]:

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\nabla_X \xi + \{(\nabla_X \eta)Y\}\xi, \quad X, Y \in \Gamma(TM).$$

The Tanaka-Webster connection satisfies

$$\hat{\nabla}\phi = 0, \quad \hat{\nabla}\xi = 0, \quad \hat{\nabla}\eta = 0, \quad \hat{\nabla}g = 0.$$

Take a section $X \in \Gamma(D)$, then by using the formula $\nabla\xi = -\phi(I + h)$, we obtain

$$\begin{aligned} (\nabla_X \eta)X &= g(\nabla_X \xi, X) = g(h\phi X, X), \\ (\nabla_{\phi X} \eta)\phi X &= g(\nabla_{\phi X} \xi, \phi X) = -g(\phi(I + h)\phi X, \phi X) = -g((I + h)\phi X, X) = -g(h\phi X, X). \end{aligned}$$

Hence we deduce that

$$(\nabla_X \eta)X + (\nabla_{\phi X} \eta)\phi X = 0 \tag{3.7}$$

holds for all $X \in \Gamma(D)$. The formula (3.7) implies that

$$\hat{\nabla}_X X + \hat{\nabla}_{\phi X}(\phi X) = \nabla_X X + \nabla_{\phi X}(\phi X). \tag{3.8}$$

Let M be a strongly pseudo-convex CR-manifold. We denote by $\hat{R}$ the curvature tensor field of Tanaka-Webster connection. For a ϕ -section Π , the sectional curvature of Π with respect to $\hat{R}$ is well-defined. More precisely, take an orthonormal basis $\{X, \phi X\}$ of Π , then

$$\hat{K}_\phi(\Pi) = L_\eta(\hat{R}(X, \phi X)\phi X, X)$$

is independent of the choice of $\{X, \phi X\}$. The sectional curvature of a ϕ -section with respect to $\hat{R}$ is called the *pseudohermitian curvature*. A strongly pseudo-convex CR-manifold M is called a *contact CR-space form* in the sense of [24] if the pseudohermitian curvature is a constant.

3.6. Three dimensional contact (κ, μ) -spaces

Blair, Koufogiorgos, and Papantoniou [14] classified 3-dimensional contact (κ, μ) -spaces. They proved that 3-dimensional contact (κ, μ) -spaces are either Sasakian or locally isomorphic to a 3-dimensional unimodular Lie group equipped with left invariant contact (κ, μ) -structure.

Let G be a 3-dimensional unimodular Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ of G such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad c_1, c_2, c_3 \in \mathbb{R}. \tag{3.9}$$

Three-dimensional unimodular Lie groups are classified by Milnor as follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+, +, +)$	$SU(2)$	compact and simple
$(-, -, +)$	$\widetilde{SL}_2\mathbb{R}$	non-compact and simple
$(0, +, +)$	$\widetilde{E}(2)$	solvable
$(0, -, +)$	$E(1, 1)$	solvable
$(0, 0, +)$	Heisenberg group Nil_3	nilpotent
$(0, 0, 0)$	$(\mathbb{R}^3, +)$	Abelian

To describe the Levi-Civita connection ∇ of G , we introduce the following constants:

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i, \quad i = 1, 2, 3.$$

Proposition 3.5. *The Levi-Civita connection is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= \mu_1 e_3, & \nabla_{e_1} e_3 &= -\mu_1 e_2 \\ \nabla_{e_2} e_1 &= -\mu_2 e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= \mu_2 e_1 \\ \nabla_{e_3} e_1 &= \mu_3 e_2, & \nabla_{e_3} e_2 &= -\mu_3 e_1, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

The Riemannian curvature R is given by

$$\begin{aligned} R(e_1, e_2)e_1 &= (\mu_1 \mu_2 - c_3 \mu_3)e_2, & R(e_1, e_2)e_2 &= -(\mu_1 \mu_2 - c_3 \mu_3)e_1, \\ R(e_2, e_3)e_2 &= (\mu_2 \mu_3 - c_1 \mu_1)e_3, & R(e_2, e_3)e_3 &= -(\mu_2 \mu_3 - c_1 \mu_1)e_2, \\ R(e_1, e_3)e_1 &= (\mu_3 \mu_1 - c_2 \mu_2)e_3, & R(e_1, e_3)e_3 &= -(\mu_3 \mu_1 - c_2 \mu_2)e_1. \end{aligned}$$

The basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci tensor field Ric . The principal Ricci curvatures are given by

$$\rho_1 = 2\mu_2 \mu_3, \quad \rho_2 = 2\mu_1 \mu_3, \quad \rho_3 = 2\mu_1 \mu_2.$$

According to a result due to Perrone [77], simply connected homogeneous contact metric 3-manifolds are classified by the Webster scalar curvature $W = (\rho - \text{Ric}(\xi, \xi) + 4)/8$ and the torsion invariant $|\tau|^2 = -2\text{Ric}(\xi, \xi) + 4$ as follows:

Theorem 3.4. *Let $(M^3, \varphi, \xi, \eta, g)$ be a simply connected homogeneous contact metric 3-manifold. Then M is a Lie group G together with a left invariant contact metric structure (ϕ, ξ, η, g) . If G is unimodular, then G is one of the following;*

1. the Heisenberg group Nil_3 if $W = |\tau| = 0$.
2. $\text{SU}(2)$ if $4\sqrt{2}W > |\tau|$.
3. $\tilde{\text{E}}(2)$ if $4\sqrt{2}W = |\tau| > 0$.
4. $\widetilde{\text{SL}}_2\mathbb{R}$ if $-|\tau| \neq 4\sqrt{2}W < |\tau|$.
5. $\text{E}(1, 1)$ if $4\sqrt{2}W = -|\tau| < 0$.

The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1, e_2, e_3\}$ as in (3.9) with $c_3 = 2$. The left invariant contact metric structure is determined by

$$\xi = e_3, \quad \phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \varphi \xi = 0.$$

Proposition 3.6 ([49]). *The endomorphism field h , the Webster scalar curvature and the torsion invariant of a unimodular Lie group $G = G(c_1, c_2)$ equipped with a left invariant homogeneous contact metric structure are given by*

$$he_1 = -\frac{1}{2}(c_1 - c_2)e_1, \quad he_2 = \frac{1}{2}(c_1 - c_2)e_2, \quad W = \frac{1}{4}(c_1 + c_2), \quad |\tau|^2 = (c_1 - c_2)^2.$$

The ϕ -sectional curvature K_ϕ of G is

$$K_\phi = -3 + \frac{1}{4}(c_1 - c_2)^2 + c_1 + c_2.$$

Corollary 3.1 ([49]). *If a unimodular Lie group G is non-Sasakian, i.e., $c_1 \neq c_2$, then G is a contact (κ, μ) -space with*

$$\kappa = 1 - \frac{1}{4}(c_1 - c_2)^2, \quad \mu = 2 - (c_1 + c_2).$$

Proposition 3.7 ([49]). *Let $G(c_1, c_2)$ be a 3-dimensional unimodular Lie group equipped with a left invariant contact metric structure. Then the Tanaka-Webster connection $\hat{\nabla}$ of $G(c_1, c_2)$ is described as*

$$\hat{\nabla}_{e_3} e_1 = \frac{1}{2}(c_1 + c_2)e_2, \quad \hat{\nabla}_{e_3} e_2 = -\frac{1}{2}(c_1 + c_2)e_1, \quad \text{all other } \hat{\nabla}_{e_i} e_j = 0.$$

From this table, the torsion $\hat{T}$ of the Tanaka-Webster connection $\hat{\nabla}$ is computed as

$$\hat{T}(e_1, e_2) = -2\xi, \quad \hat{T}(e_1, e_3) = -\frac{1}{2}(c_1 - c_2)e_2, \quad \hat{T}(e_2, e_3) = -\frac{1}{2}(c_1 - c_2)e_1.$$

The curvature tensor field $\hat{R}$ of $\hat{\nabla}$ is given by

$$\hat{R}(e_1, e_2)e_1 = -(c_1 + c_2)e_2, \quad \hat{R}(e_1, e_2)e_2 = (c_1 + c_2)e_1, \quad \text{all other } \hat{R}(e_i, e_j)e_k = 0.$$

The pseudohermitian curvature of G is $\hat{K}_\phi = c_1 + c_2$. Thus we get $\mu = 2 - \hat{K}_\phi$.

$$K_\phi = \hat{K}_\phi - 3 + \frac{1}{4}(c_1 - c_2)^2.$$

In particular, when $G(c_1, c_2)$ is Sasakian, then

$$K_\phi = -3 + \hat{K}_\phi.$$

When $\mu = 2$, we have $\hat{K}_\phi = 0$ (Compare with [24, Theorem 11]).

4. Real hypersurfaces

4.1. Hopf hypersurfaces

Let M be a real hypersurface of a Kähler manifold $\widetilde{M}_n = (\widetilde{M}_n, \tilde{g}, J)$ of complex dimension n . We can assume that M is orientable, as we are working in local theory. Take a unit normal vector field ν of M in $\widetilde{M}_n$. We denote by g the Riemannian metric (the first fundamental form) on M induced from Kähler metric $\tilde{g}$ of the ambient space $\widetilde{M}_n$. The second fundamental form $\nabla d\iota$ of the inclusion map $\iota : M \subset \widetilde{M}_n$ is given by

$$(\nabla d\iota)(X, Y) = g(AX, Y)\nu.$$

Define a vector field ξ on M by $\xi = -\epsilon J\nu$ with $\epsilon = \pm 1$. We call ϵ the sign of M relative to ν . The sign ϵ is chosen to align the orientation determined by ν and that from (3.4).

The vector field ξ is called the *structure vector field*. Next, define the 1-form η and the endomorphism field ϕ on M by

$$\eta(X) = g(\xi, X) = \epsilon \tilde{g}(JX, \nu), \quad g(\phi X, Y) = \tilde{g}(JX, Y), \quad X, Y \in \Gamma(TM),$$

respectively. Then one can see that (ϕ, ξ, η, g) is an almost contact metric structure on M , that is, it satisfies (3.2). The structure (ϕ, ξ, η, g) will be called the *induced almost contact metric structure*. Note that

$$Jd\iota(X) = d\iota(\phi X) + \epsilon \eta(X)\nu, \quad X \in \Gamma(TM) \quad (4.1)$$

holds. It follows that

$$(\nabla_X \phi)Y = \epsilon(\eta(Y)AX - g(AX, Y)\xi), \quad \nabla_X \xi = \epsilon \phi AX. \quad (4.2)$$

Proposition 4.1 ([9]). *Let M be an orientable real hypersurface of a Kähler manifold $\widetilde{M}_n$. Then the fundamental 2-form of M with respect to the almost contact metric structure induced from $\widetilde{M}_n$ is closed.*

A real hypersurface is said to be *Hopf* if its structure vector field ξ is a principal curvature vector field. Throughout this article, the principal curvature corresponding to ξ of a Hopf hypersurface is denoted by α .

4.2. Contact hypersurfaces

Let M be an orientable real hypersurface M of a Kähler manifold $\widetilde{M}_n$. Denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Then the standard almost CR-structure $(D, J = \phi|_D)$ of $(M; \phi, \xi, \eta, g)$ is automatically integrable. Thus, we can consider M to be a CR-manifold in this manner. It should be noted that $(M; \phi, \xi, \eta, g)$ is not necessarily strongly pseudo-convex.

Definition 4.1 ([9, 70]). An orientable real hypersurface of a Kähler manifold is said to be a *contact hypersurface* if its induced almost contact metric structure (ϕ, ξ, η, g) satisfies $d\eta = 2\gamma\Phi$ for some everywhere nonzero smooth function γ .

One can see that η is actually a contact form on a contact hypersurface. In particular, a contact hypersurface M is called a *contact metric hypersurface* if $d\eta = 2\Phi$.

Example 4.1. Let $\mathbb{S}^{2n-1}(c) \subset \mathbb{C}^n$ be the sphere of radius $1/\sqrt{c}$. Then we can take a unit normal vector field $\nu = \sqrt{c}x$, where x is the position vector field of $\mathbb{C}^n$. Then the shape operator is given by $AX = -\sqrt{c}X$ for any $X \in \Gamma(T\mathbb{S}^{2n-1}(c))$. Thus the second fundamental form is given by

$$(d\iota)(X, Y) = -\sqrt{c}g(X, Y)\nu.$$

Let us introduce a vector field ξ by $\xi = -J\nu$. Then the induced almost contact metric structure (ϕ, ξ, η, g) satisfies

$$d\eta = 2\sqrt{c}\Phi.$$

Thus $(\mathbb{S}^{2n-1}(c); \phi, \xi, \eta, g)$ is a contact hypersurface. In particular, the unit sphere $\mathbb{S}^{2n-1}$ is a contact metric hypersurface. Moreover $(\mathbb{S}^{2n-1}; \phi, \xi, \eta, g)$ is Sasakian. The CR-structure on $\mathbb{S}^{2n-1}$ introduced in this manner is called its *standard CR structure* of $\mathbb{S}^{2n-1}$.

According to Burns and Schneider [15], a strongly pseudo-convex CR-manifold M is said to be *spherical* if it is locally CR-equivalent to $\mathbb{S}^{2n+1}$ equipped with the standard CR-structure. A strongly pseudo-convex CR-manifold M is spherical if its Chern-Moser-Tanaka invariant vanishes [21, 92, 96]. Dileo and Lotta [28] classified complete, simply connected spherical CR-symmetric spaces of dimension greater than 3. Note that every 3-dimensional strongly pseudo-convex CR-manifold is spherical.

The following criterion is obtained independently by Berndt and Suh [9], Okumura [70, Lemma 2.1] and, Nagai and Kôjyô, [67, Theorem 2.1].

Proposition 4.2 ([9, 67, 70]). *Let M be an orientable real hypersurface in a Kähler manifold $\widetilde{M}_n$. Then M satisfies $d\eta = 2\gamma\Phi$ for some everywhere nonzero smooth function γ if and only if $A\phi + \phi A = -2\epsilon\gamma\phi$. In this case M is Hopf.*

Significant differences exist between the cases where $n = 2$ and $n > 2$.

Proposition 4.3 ([9, 70]). *Let M be an orientable real hypersurface in a Kähler manifold $\widetilde{M}_n$ of complex dimension $n > 2$. If M satisfies $d\eta = 2\gamma\Phi$ for some everywhere nonzero smooth function γ , then γ is a non-zero constant.*

Remark 4.1 ([9]). Let M be an orientable real hypersurface of a Kähler surface $\widetilde{M}_2$. Then M is a contact hypersurface if and only if M is a Hopf hypersurface and $\text{tr}_g A \neq \alpha$ everywhere.

Proposition 4.4 ([9]). *Let M be an orientable real hypersurface in a Kähler manifold $\widetilde{M}_n$ of complex dimension $n > 2$. Then the principal curvature α corresponding to ξ is constant if and only if M is of constant mean curvature.*

Assume that $d\eta = 2\gamma\Phi$ for some nonzero constant γ . Then η is a contact form on M . Put $\vartheta = \text{sgn}(\gamma)\eta$. Then the Levi-form $L = L_{\vartheta}$ with respect to ϑ is

$$L(X, Y) = |\gamma|g(X, Y), \quad X, Y \in \Gamma(D).$$

Thus the CR-structure $\mathcal{S}$ is strongly pseudo-convex. The resulting Webster metric is $|\gamma|g$.

5. Real hypersurfaces in complex space forms

Now let us assume that the ambient space $\widetilde{M}_n$ is a *complex space form* of constant holomorphic sectional curvature c and complex dimension n . As is well known, a complete and simply connected complex space form $\widetilde{M}_n(c)$ is a *complex projective space* $\mathbb{C}P^n(c)$, a *complex Euclidean space* $\mathbb{C}^n$ or a *complex hyperbolic space* $\mathbb{C}H^n(c)$, according as $c > 0$, $c = 0$ or $c < 0$. Our general references for real hypersurfaces in complex space forms are [19] and [68]. The *Gauss equation* which describes the Riemannian curvature R of the real hypersurface $M \subset \widetilde{M}_n(c)$ is given by

$$\begin{aligned} R(X, Y)Z = & \frac{c}{4}\{g(Y, Z)X - g(Z, X)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY, Z)AX - g(AX, Z)AY. \end{aligned} \quad (5.1)$$

The Ricci operator is expressed as

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta \otimes \xi\} + (2n-1)H A - A^2. \quad (5.2)$$

The next lemma is well known.

Lemma 5.1. *If M is a Hopf hypersurface in a non-flat complex space form. Then the principal curvature α corresponding to ξ is a constant.*

Okumura and Vernon proved the following result (see also [9]).

Proposition 5.1 ([70, 98]). *If M is a complete Hopf hypersurface in a complex space form $\widetilde{M}_n(c)$ with $n > 2$. Then M is homogeneous.*

In case $n = 2$, non-homogeneous contact hypersurfaces exist (see [9] for $\mathbb{C}^2$ case).

Sharma showed the following fundamental fact.

Theorem 5.1 ([84]). *Let M be the contact metric hypersurface of a complex space form $\widetilde{M}_n(c)$. Then either*

1. *M is a Sasakian manifold of constant ϕ -sectional curvature and is η -umbilical, or*
2. *M is locally isometric, up to a pseudo-homothetic deformation, to the unit tangent sphere bundle of some space of constant curvature different from 1.*

Remark 5.1. A real hypersurface M is said to be η -umbilical if its shape operator A has the form $A = \lambda I + \mu\eta \otimes \xi$ for some functions λ and μ [59] (cf. [66]).

Okumura proved the following fact (compare with Proposition 6.3).

Proposition 5.2 ([69]). *Let M be a contact hypersurface of a complex space form $\widetilde{M}_n(c)$ with $n > 2$. Then M has at most three distinct principal curvatures and all of those are constant.*

In the next section, we give more detailed description of these real hypersurfaces.

6. Homogeneous real hypersurfaces in complex projective space

6.1. Homogeneous orbits

Homogeneous real hypersurfaces in complex projective space are classified by Takagi.

Theorem 6.1. ([88]) *Let U/K be a compact Hermitian symmetric space of rank 2 and complex dimension $(n+1)$ with associated Cartan decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$. We identify $\mathfrak{p}$ with complex Euclidean $(n+1)$ -space $\mathbb{C}^{n+1}$ via the Killing metric. Denote by $\Pi : \mathbb{S}^{2n+1} \subset \mathfrak{p} \rightarrow \mathbb{C}P^n(4)$ the Hopf fibering. Take a unit regular element $A \in \mathfrak{p}$ and consider its $\text{Ad}(K)$ -orbit $\widehat{M} = \text{Ad}(K)A$. The orbit $\widehat{M}$ is a homogeneous hypersurface of $\mathbb{S}^{2n+1} \subset \mathfrak{p}$. Then the Hopf image*

$$M = \Pi(\widehat{M}) \subset \mathbb{C}P^n(4) = \mathbb{S}^{2n+1}/U(1)$$

is a homogeneous real hypersurface of $\mathbb{C}P^n(4)$.

	$\mathfrak{u}$	$\mathfrak{k}$	$\dim M$
A	$\mathfrak{su}(p+1) + \mathfrak{su}(q+1),$ $p+q = n-1$	$\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(1)) + \mathfrak{s}(\mathfrak{u}(q) \oplus \mathfrak{u}(1))$	$2n-1$
B	$\mathfrak{o}(p+2), p \geq 3, n = p-1,$	$\mathfrak{o}(p) + \mathbb{R}$	$2p-3$
C	$\mathfrak{su}(p+2), p \geq 3, n = 2p-1,$	$\mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(2))$	$4p-3$
D	$\mathfrak{o}(10)$	$\mathfrak{u}(5)$	17
E	$\mathfrak{e}_6$	$\mathfrak{o}(10) + \mathbb{R}$	29

Table 1. Homogeneous real hypersurfaces in $\mathbb{C}P^n(4)$, $n \geq 2$

Conversely, every homogeneous real hypersurface in $\mathbb{C}P^n(4)$ is congruent to these orbits.

Remark 6.1. The orbits $\widehat{M} \subset \mathbb{S}^{2n+1}$ are appeared in [44, Theorem 5, Table II].

Takagi [89, 90] gave the list of principal curvatures and their multiplicities of homogeneous real hypersurfaces.

By performing homothetic change of the Fubini-Study metric of $\mathbb{C}P^n(4)$, we obtain the following table:

	α	λ_1	λ_2	λ_3	λ_4
A_1	$\sqrt{c} \cot(\sqrt{c}r)$	$\frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}r}{2}$	—	—	—
A_2	$\sqrt{c} \cot(\sqrt{c}r)$	$\frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}r}{2}$	$-\frac{\sqrt{c}}{2} \tan \frac{\sqrt{c}r}{2}$	—	—
B	$\sqrt{c} \cot(\sqrt{c}r)$	$\frac{\sqrt{c}}{2} \cot(\frac{\sqrt{c}r}{2} - \frac{\pi}{4})$	$\frac{\sqrt{c}}{2} \cot(\frac{\sqrt{c}r}{2} + \frac{\pi}{4})$	—	—
C,D,E	$\sqrt{c} \cot(\sqrt{c}r)$	$\frac{\sqrt{c}}{2} \cot(\frac{\sqrt{c}r}{2} - \frac{\pi}{4})$	$\frac{\sqrt{c}}{2} \cot(\frac{\sqrt{c}r}{2} + \frac{\pi}{4})$	$\frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}r}{2}$	$-\frac{\sqrt{c}}{2} \tan \frac{\sqrt{c}r}{2}$

Table 2. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$, $n \geq 2$

6.2. Takagi's list

Cecil and Ryan extensively studied Hopf hypersurfaces, which are realized as tubes over certain Kähler submanifolds in $\mathbb{C}P^n(c)$ [18]. Kimura proved the equivalence of the extrinsic homogeneity and the constancy of principal curvatures in the class of all Hopf hypersurfaces in $\mathbb{C}P^n(c)$ [57]. As a result, we know the following classification table.

Theorem 6.2 ([89, 57]). *Let M be a Hopf hypersurface of $\mathbb{C}P^n(c)$. Then M has constant principal curvatures if and only if M is locally holomorphically congruent to one of the following real hypersurfaces:*

- (A₁) a geodesic sphere of radius r , where $0 < r < \pi/\sqrt{c}$,
- (A₂) a tube of radius r over a totally geodesic $\mathbb{C}P^\ell(c)$ ($1 \leq \ell \leq n-2$) via Segre imbedding, where $0 < r < \pi/(\sqrt{c})$,
- (B) a tube of radius r over a complex quadric $\mathcal{Q}_{n-1}$, where $0 < r < \pi/(2\sqrt{c})$,
- (C) a tube of radius r over a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and $n \geq 5$ is odd,
- (D) a tube of radius r over the Plücker imbedding of complex Grassmannian $\text{Gr}_2(\mathbb{C}^5) \subset \mathbb{C}P^9(c)$, where $0 < r < \pi/(2\sqrt{c})$,
- (E) a tube of radius r over a Hermitian symmetric space $\text{SO}(10)/\text{U}(5) \subset \mathbb{C}P^{15}(c)$, where $0 < r < \pi/(2\sqrt{c})$.

It should be remarked that a tube of radius r over a complex quadric $\mathcal{Q}_{n-1}$ is realized also as a tube of radius $\pi/(2\sqrt{c}) - r$ over a totally geodesic Lagrangian real projective space $\mathbb{R}P^n(c/4)$ [18]. This classification is referred to as *Takagi's list*. Note that the tube around $\mathbb{C}P^\ell(c)$ ($1 \leq \ell < n-2$) of radius $r = \pi/(2\sqrt{c})$ are *quadric* in the sense of Tanaka [92] and Yamaguchi [102, 103].

6.3. Montiel's list

Corresponding table for $\mathbb{C}H^n(c)$ was obtained by Berndt.

Theorem 6.3 ([7]). *Let M be a Hopf hypersurface of $\mathbb{C}H^n(c)$. Then M has constant principal curvatures if and only if M is locally holomorphically congruent to one of the following real hypersurfaces:*

- (A₀) a horosphere,
- (A₁) a geodesic sphere (A_{1,0}) or a tube over a complex hyperbolic hyperplane $\mathbb{C}H^{n-1}(c)$ (A_{1,1}),
- (A₂) a tube over a totally geodesic $\mathbb{C}H^\ell(c)$ ($1 \leq \ell \leq n-2$),
- (B) a tube over a totally geodesic Lagrangian real hyperbolic space $\mathbb{R}H^n(c/4)$.

We call simply type (A) for real hypersurfaces of type (A₁), (A₂) in $\mathbb{C}P^n(c)$ and ones of type (A₀), (A₁) or (A₂) in $\mathbb{C}H^n(c)$. In some literature, the above list is referred to as *Montiel's list* (e.g., [19]).

It should be emphasized that homogeneous real hypersurfaces are not exhausted by Montiel's list. Indeed, there exist homogeneous ruled hypersurfaces, see [13, 10].

6.4. Contact (κ, μ) -condition

From the Gauss equation (5.1) for a real hypersurface $M \subset \widetilde{M}_n(c)$, we have

$$R(X, Y)\xi = \frac{c}{4}(\eta(Y)X - \eta(X)Y) + g(AY, \xi)AX - g(AX, \xi)AY. \quad (6.1)$$

	α	λ_1	λ_2
A_0	$\sqrt{ c }$	$\frac{\sqrt{ c }}{2}$	—
$A_{1,0}$	$\sqrt{ c } \coth(\sqrt{ c } r)$	$\frac{\sqrt{ c }}{2} \coth \frac{\sqrt{ c } r}{2}$	—
$A_{1,1}$	$\sqrt{ c } \coth(\sqrt{ c } r)$	$\frac{\sqrt{ c }}{2} \tanh \frac{\sqrt{ c } r}{2}$	—
A_2	$\sqrt{ c } \coth(\sqrt{ c } r)$	$\frac{\sqrt{ c }}{2} \coth \frac{\sqrt{ c } r}{2}$	$\frac{\sqrt{ c }}{2} \tanh \frac{\sqrt{ c } r}{2}$
B	$\sqrt{ c } \tanh(\sqrt{ c } r)$	$\frac{\sqrt{ c }}{2} \coth \frac{\sqrt{ c } r}{2}$	$\frac{\sqrt{ c }}{2} \tanh \frac{\sqrt{ c } r}{2}$

Table 3. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$, $n \geq 2$

The operator h is computed as

$$hX = \frac{\epsilon}{2}(\eta(X)A\xi - (\phi A\phi)X - AX).$$

When M is a Hopf hypersurface satisfying $A\xi = \alpha\xi$, we have

$$R(X, Y)\xi = \left\{ \left(\frac{c}{4} + \frac{\gamma}{2}\alpha \right) I - \epsilon\alpha h \right\} (\eta(Y)X - \eta(X)Y). \quad (6.2)$$

From this formula, we obtain:

Theorem 6.4 ([22]). *Let $M \subset \widetilde{M}_n(c)$ be an orientable real hypersurface in a non-flat complex space form with sign ϵ . If M is a contact hypersurface satisfying the condition $A\phi + \phi A = \gamma\phi$ for some nonzero constant γ . Then M satisfies*

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y)$$

with $\kappa = c/4 + \gamma\alpha/2$ and $\mu = -\epsilon\alpha$.

In particular, contact metric hypersurfaces are (κ, μ) -spaces. More precisely, we obtain the following result.

Corollary 6.1. *Let $M \subset \widetilde{M}_n(c)$ be an orientable real hypersurface in a non-flat complex space form with sign ϵ . If M is contact metric, that is $d\eta = 2\Phi$, then M is a contact (κ, μ) -space with $\kappa = c/4 - \epsilon\alpha$ and $\mu = -\epsilon\alpha$.*

Real hypersurfaces with $A\phi + \phi A = \gamma\phi$ ($\gamma \in \mathbb{R}^\times$) are classified by Adachi, Kameda and Maeda (see also Suh [87, Lemma 3.1] for the case $c < 0$ and $n > 2$):

Lemma 6.1 ([2]). *Let $M \subset \widetilde{M}_n(c)$ be an orientable real hypersurface with $n \geq 2$ and $c \neq 0$. Then M satisfies $\phi A + A\phi = \gamma\phi$ for some nonzero constant γ if and only if M is of type (A_0) , (A_1) or (B).*

Remark 6.2. In [87], Hopf hypersurfaces in non-flat complex space form $\widetilde{M}_n(c)$ with η -parallel Ricci operator are investigated. However, as Maeda [62] pointed out, Suh's classification is true under the condition $n > 2$. Maeda obtained classification of Hopf hypersurfaces with η -parallel Ricci operator in $\mathbb{C}P^2(4)$ and $\mathbb{C}H^2(-4)$.

Let M be a real hypersurface of type A_1 in $\mathbb{C}P^n(c)$, type A_0 in $\mathbb{C}H^n(c)$ or type A_1 in $\mathbb{C}H^n(c)$. Then M is Hopf and has two distinct principal curvatures α and $\lambda := \lambda_1$. It is easy to see that M satisfies $A\phi + \phi A = 2\lambda\phi$. Berndt showed that these real hypersurfaces are Sasakian space forms up to homothety [6] (see also Ejiri [35] for A_1 case). Adachi, Kameda and Maeda classified all Sasakian hypersurfaces in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$.

Theorem 6.5 ([2]). *Let $M \subset \widetilde{M}_n(c)$ be an oriented real hypersurface with $n \geq 2$ and $c \neq 0$. Then with respect to the induced almost contact metric structure, the following conditions are mutually equivalent:*

1. M is a Sasakian manifold;
2. M is a Sasakian space form of ϕ -sectional curvature $K_\phi = c + 1$;
3. M is locally holomorphically congruent to one of the following homogeneous real hypersurfaces;
 - (A_1) a geodesic sphere of radius $r = (2/\sqrt{c}) \tan^{-1}(\sqrt{c}/2)$ in $\mathbb{C}P^n(c)$ with sign $\epsilon = -1$ ($K_\phi > 1$),
 - $(A_{1,0})$ a geodesic sphere of radius $r = (2/\sqrt{-c}) \tanh^{-1}(\sqrt{-c}/2)$ in $\mathbb{C}H^n(c)$ with sign $\epsilon = -1$, where $-4 < c < 0$ and $-3 < K_\phi < 1$,
 - (A_0) a horosphere in $\mathbb{C}H^n(-4)$ with sign $\epsilon = -1$ ($K_\phi = -3$),

(A_{1,1}) a tube around totally geodesic $\mathbb{C}H^{n-1}(c)$ of radius $r = (2/\sqrt{-c}) \coth^{-1}(\sqrt{-c}/2)$ in $\mathbb{C}H_n(c)$ with sign $\epsilon = -1$, where $c < -4$, ($K_\phi < -3$).

Let us pick up Sasakian geodesic sphere $M = M_r$ of radius $r = (2/\sqrt{c}) \tan^{-1}(\sqrt{c}/2)$ in $\mathbb{C}P^n(c)$. Then M has principal curvatures

$$\alpha = \sqrt{c} \cot(\sqrt{c}r), \quad \lambda = \frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r\right)$$

with multiplicities $m_\alpha = 1$, $m_\lambda = 2(n-1)$. One can confirm that M satisfies $A\phi + \phi A = 2\lambda\phi$.

Non-Sasakian contact metric hypersurfaces are classified as follows:

Theorem 6.6 ([2]). Let $M \subset \widetilde{M}_n(c)$ with $n \geq 2$ and $c \neq 0$ be a real hypersurface. If M is contact metric but not Sasakian, then M is locally holomorphically congruent to one of the following homogeneous real hypersurfaces of type B :

- a tube of $r = (2/\sqrt{c}) \tan^{-1}\{(\sqrt{c+4} - \sqrt{c})/2\} < \pi/(2\sqrt{c})$ around the complex quadric $\mathcal{Q}_{n-1} \subset \mathbb{C}P^n(c)$ with sign $\epsilon = 1$;
- a tube of radius $r = (1/\sqrt{-c}) \tanh^{-1}(\sqrt{-c}/2)$ around the totally geodesic Lagrangian real hyperbolic space $\mathbb{R}H^n(c/4) \subset \mathbb{C}H^n(c)$ with sign $\epsilon = -1$ and $-4 < c < 0$.

These tubes have three distinct principal curvatures unless $c = -3$. In case $c = -3$, the tubes have two distinct principal curvatures $3/2$ and 2 .

Corollary 6.1 implies that these hypersurfaces of type B are contact (κ, μ) -spaces, especially CR-symmetric spaces. We compute the values of κ and μ of these hypersurfaces [22].

Let M be a tube of radius r around $\mathbb{R}P^n(c/4)$. Then M has principal curvatures

$$\lambda_1 = -\frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r\right), \quad \lambda_2 = \frac{\sqrt{c}}{2} \tan\left(\frac{\sqrt{c}}{2}r\right), \quad \alpha = \sqrt{c} \tan(\sqrt{c}r)$$

with multiplicities

$$m_1 = m_2 = n-1, \quad m_\alpha = 1.$$

Note that if we regard M as a tube around $\mathcal{Q}_{n-1}$, then (see Table 2):

$$\lambda_1 = \frac{\sqrt{c}}{2} \cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right), \quad \lambda_2 = \frac{\sqrt{c}}{2} \tan\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right), \quad \alpha = \sqrt{c} \cot(\sqrt{c}r)$$

with multiplicities

$$m_1 = m_2 = n-1, \quad m_\alpha = 1.$$

Thus the tube M around $\mathcal{Q}_{n-1}$ satisfies $A\phi + \phi A = \gamma\phi$ if and only if $\gamma = \lambda_1 + \lambda_2$. If we put $t = \tan(\sqrt{c}r/2)$, then

$$t^2 - 1 = \frac{2\gamma}{\sqrt{c}}t.$$

Hence $A\phi + \phi A = 2\epsilon k\phi$ if and only if

$$r = \frac{2}{\sqrt{c}} \tan^{-1}\left(\frac{\sqrt{c+4k^2} - 2\epsilon k}{\sqrt{c}}\right).$$

Proposition 6.1. Let M be a tube of radius $r = (2/\sqrt{c}) \tan^{-1}\{(\sqrt{c+4} - \sqrt{c})/2\} < \pi/(2\sqrt{c})$ around the complex quadric $\mathcal{Q}_{n-1} \subset \mathbb{C}P^n(c)$ with sign $\epsilon = 1$. Then M is a contact metric Hopf hypersurface. The principal curvature α corresponding to ξ is $\alpha = c/2$. M is a non-Sasakian (κ, μ) -space with $\kappa = -c/4$, $\mu = -\sqrt{c}/2$. The Boeckx invariant is $\mathcal{I} = \sqrt{1+c/4} > 1$.

On the other hand we know the following fact for $\mathbb{C}H^n(c)$.

Proposition 6.2 ([22]). Let M be a tube of radius $r = (1/\sqrt{-c}) \tanh^{-1}(\sqrt{-c}/2)$ around the totally geodesic Lagrangian real hyperbolic space $\mathbb{R}H^n(c/4) \subset \mathbb{C}H^n(c)$ with sign $\epsilon = -1$ and $-4 < c < 0$. Then M is a contact metric Hopf hypersurface. The principal curvature corresponding to ξ is $\alpha = -c/2$. Hence M is a contact (κ, μ) -space with $\kappa = 3c/4 < 0$, $\mu = -c/2 > 0$. The Boeckx invariant is $0 < \mathcal{I} = (c+4)/(2\sqrt{4-3c}) < 1$.

Cho and Kimura [24, Theorem 3] characterized the contact (κ, μ) -space of Boeckx invariant $\mathcal{I} > 1$ as a tube of radius $\sqrt{2/c} \tan^{-1}(2\sqrt{2}/c) \in (0, \pi/\sqrt{2c})$ around $\mathbb{S}^{n+1}$ in the complex quadric $\mathcal{Q}_{n+1}(c)$ of maximal sectional curvature $c > 0$. On the other hand, the contact (κ, μ) -space of Boeckx invariant $\mathcal{I} \in (0, 1)$ as a tube of radius $\sqrt{2/|c|} \coth^{-1}(2\sqrt{2}/|c|)$ around $\mathbb{R}H^{n+1}$ in the complex quadric $\mathcal{Q}_{n+1}^*(c)$ of maximal sectional curvature $c \in (-8, 0)$. Here $\mathcal{Q}_{n+1}^*(c)$ is the dual Riemannian symmetric space of $\mathcal{Q}_{n+1}(c)$.

6.5. Normal real hypersurfaces

Okumura classified real hypersurfaces in $\mathbb{C}P^n(4)$ whose induced almost contact metric structures are normal. Together with a characterization theorem due to Maeda and Udagawa, we get the following result (see also Olszak [73]).

Theorem 6.7 ([63, 69, 71, 72, 73]). *Let M be a real hypersurface of $\mathbb{C}P^n(c)$ ($n \geq 2$). Then the following properties are mutually equivalent:*

1. M is locally holomorphically congruent to type A real hypersurface.
2. ξ is a Killing vector field.
3. $\mathcal{L}_\xi \phi = 0$ holds.
4. $A\phi = \phi A$ holds.
5. The induced almost contact metric structure is quasi-Sasakian.
6. The induced almost contact metric structure is normal.

Okumura proved the following fact (compare with Proposition 5.2).

Proposition 6.3 ([69]). *Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$ with $n > 2$. If the induced almost contact metric structure is normal, then M has at most three distinct principal curvatures and all of those are constant.*

7. Biharmonic homogeneous real hypersurfaces in complex projective space

Biharmonic homogeneous real hypersurfaces were investigated in [47, 48]. The second named author obtained the correct classification of those hypersurfaces [82]. In this section, we recall the classification of biharmonic homogeneous real hypersurfaces in $\mathbb{C}P^n(4)$. This procedure will be useful in later sections.

7.1. The biharmonic equation

Let us consider biharmonic real hypersurfaces with constant mean curvature in the complex projective n -space $\mathbb{C}P^n(4)$. Then by using Ou's formula, the following criterion is obtained (see also [47, 13 Theorem], [48, Theorem 4]).

Theorem 7.1. *A real hypersurface $M \subset \mathbb{C}P^n(4)$ with non-zero constant mean curvature is biharmonic if and only if $|A|^2 = 2(n+1)$.*

Corollary 7.1. *Let $M \subset \mathbb{C}P^n(4)$ be a real hypersurface with non-zero constant mean curvature. Then M is biharmonic if and only if M has constant positive scalar curvature*

$$\rho = 4(n^2 - 1) + (2n - 1)^2 H^2 - 2(n + 1).$$

Proof. From the Gauss equation, the scalar curvature ρ of a real hypersurface $M \subset \mathbb{C}P^n(4)$ is given by

$$\rho = 4(n^2 - 1) + (2n - 1)^2 H^2 - |A|^2.$$

Thus the result follows. □

Jiang obtained the following result.

Theorem 7.2 ([54]). *Let $f : M^m \rightarrow \mathbb{C}P^n(4)$ be a weakly stable biharmonic isometric immersion of a compact Riemannian m -manifold. If f satisfies $|H| > 6m$, then f is harmonic.*

7.2. Type A real hypersurfaces

Let us consider the Hermitian symmetric space $U/K = \{SU(p+1) \times SU(q+1)\}/\{S(U(p) \times U(1)) \times S(U(q) \times U(1))\}$. Then $\text{Ad}(K)A$ is the Riemannian product of odd-dimensional spheres:

$$\widehat{M}_{p,q}(r) := \mathbb{S}^{2p+1}(\cos r) \times \mathbb{S}^{2q+1}(\sin r) \subset \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1},$$

where $0 < r < \pi/2$, $p, q \geq 0$ and $p + q = n - 1$. Then $M_{p,q}(r) = \Pi(\widehat{M}_{p,q}(r))$ is a homogeneous real hypersurface in $\mathbb{C}P^n(4)$. The real hypersurface $M_{p,q}(r)$ is a tube of linear subspace $\mathbb{C}P^p$ with radius $r \in (0, \pi/2)$. In particular, $M_r := M_{0,n-1}(r)$ is a geodesic sphere of radius $r \in (0, \pi/2)$. Note that $M_{n-1,0}$ is also a geodesic sphere. A real hypersurface $M_{p,q}(r)$ is said to be of type A_1 if it is a geodesic sphere and of type A_2 otherwise.

Remark 7.1. The geodesic sphere $M_r \subset \mathbb{C}P^n(4)$ of radius $r = \pi/4$ is a Sasakian space form of constant ϕ -sectional curvature 5. Type A_2 real hypersurfaces are *non-Sasakian* quasi-Sasakian manifolds. See §6.4 or [2, 6, 22].

The real hypersurface $M_{p,q}(r)$ has constant principal curvatures $\lambda_1 = \cot r$ with multiplicity $m_1 = 2q$, $\lambda_2 = -\tan r$ with multiplicity $m_2 = 2p$ and $\alpha = 2 \cot(2r)$ with multiplicity 1.

Remark 7.2. If a real hypersurface $M \subset \mathbb{C}P^n(4)$ ($n \geq 2$) has two distinct constant principal curvatures, then M is holomorphically congruent to an open part of a geodesic sphere M_r [89]. Note that under the limit $r \rightarrow 0$, M_r collapses to a point. On the other hand, under the limit $t \rightarrow \pi/2$, M_r collapses to the projective line $\mathbb{C}P^1$. For $n > 2$, Cecil and Ryan generalized the above result for $n > 2$ by requiring that M has at most two distinct principal curvatures at each point [18].

Theorem 7.3. *The only minimal tube $M_{p,q}(r)$ is the tube of radius*

$$r = \tan^{-1} \sqrt{(2q+1)/(2p+1)}.$$

In particular the tube $M_{p,q}(\pi/4)$ is minimal if and only if $p = q$.

Proof. The mean curvature of $M_{p,q}(r)$ is

$$H = \frac{1}{2n-1} \{ (2q+1) \cot r - (2p+1) \tan r \}.$$

Thus the tube $M_{p,q}(r)$ is minimal if and only if its radius is $r = \tan^{-1} \sqrt{(2q+1)/(2p+1)}$.

In other words, the minimal tube $M_{p,q}(r)$ is the Hopf projection of

$$\mathbb{S}^{2p+1}(\sqrt{(2p+1)/(2n)}) \times \mathbb{S}^{2q+1}(\sqrt{(2q+1)/(2n)}).$$

□

Corollary 7.2. *The only minimal geodesic sphere is a geodesic sphere of radius*

$$r = \tan^{-1} \sqrt{2n-1}.$$

Non-minimal biharmonic tubes $M_{p,q}(r)$ are classified as follows.

Theorem 7.4. *A tube $M_{p,q}(r)$ is non-minimal biharmonic if and only if its radius is*

$$r = \cot^{-1} \sqrt{\frac{(n+2) \pm \sqrt{(p-q)^2 + 4(n+1)}}{1+2q}}.$$

Proof. The square norm of the second fundamental form of $M_{p,q}(r)$ is

$$|A|^2 := (2q+1) \cot^2 r + (2p+1) \tan^2 r - 2.$$

Assume that $|A|^2 = 2(n+1) = 2(p+q+2)$. Then $t = \cot r$ is a solution to:

$$(2q+1)t^4 - 2(p+q+3)t^2 + (2p+1) = 0.$$

Solving this algebraic equation, we get

$$\cot r = \sqrt{\frac{(n+2) \pm \sqrt{(p-q)^2 + 4(n+1)}}{1+2q}} > 0. \quad (7.1)$$

□

Corollary 7.3. *A geodesic sphere M_r in $\mathbb{C}P^n(4)$, ($n \geq 2$) is non-minimal biharmonic if and only if*

$$r = \cot^{-1} \sqrt{\frac{n+2 \pm \sqrt{n^2+2n+5}}{2n-1}}.$$

Remark 7.3. Type A_2 real hypersurfaces $M_{p,q}(\pi/4)$ satisfy $|A|^2 = 2(p+q) = 2(n-1)$. Thus the only biharmonic tube $M_{p,q}(\pi/4)$ is the minimal one $M_{p,p}(\pi/4)$.

7.3. Type B real hypersurfaces

Let us consider the Hermitian symmetric space $U/K = \widetilde{\text{Gr}}_{n+1}(\mathbb{R}^{n+3}) = \text{SO}(n+3)/\text{SO}(n+1) \times \text{SO}(2)$. The Hermitian symmetric space is identified with the complex quadric $Q_{n+1} \subset \mathbb{C}P^{n+2}$. Homogeneous real hypersurfaces of type B are obtained by the Hopf projection of

$$\{\text{SO}(n+1) \times \text{SO}(2)\} / \{\text{SO}(n-1) \times \mathbb{Z}_2\} \subset \mathbb{S}^{2n+1}.$$

The type B real hypersurface M_r is realized as a tube around a totally geodesic and Lagrangian embedded real projective space $\mathbb{R}P^n$ with radius $r \in (0, \pi/4)$ or a tube around a complex quadric Q_{n-1} with radius $\pi/4 - r$.

Remark 7.4. The tube M_r around the complex quadric with radius $r = \tan^{-1}(\sqrt{2}-1)$ is a contact (κ, μ) -space with $\kappa = \mu = -1$ (see §6.4 or [22]).

The principal curvatures of M_r are $\lambda_1 = -\cot r$ with multiplicity $m_1 = n-1$, $\lambda_2 = \tan r$ with $m_2 = n-1$ and $\alpha = 2 \tan(2r)$ with multiplicity 1.

Theorem 7.5 ([47, 48]). *A tube M_r of radius r around $\mathbb{R}P^n \subset \mathbb{C}P^n(4)$ is biharmonic if and only if it is minimal and of radius*

$$r = \cot^{-1} \frac{\sqrt{n}+1}{\sqrt{n-1}} = \tan^{-1} \frac{\sqrt{n}-1}{\sqrt{n-1}} < \frac{\pi}{4}.$$

In particular, a tube M_r of radius r around $\mathbb{R}P^n \subset \mathbb{C}P^2(4)$ is biharmonic if and only if it is minimal and of radius $\pi/8$.

Proof. Let us compute the mean curvature H of a tube M_r . Since the principal curvatures of M_r are rewritten as

$$\lambda_1 = -\cot r = -t, \quad \lambda_2 = \tan r = \frac{1}{t}, \quad \alpha = 2 \tan(2r) = \frac{4t}{t^2-1},$$

we have

$$(2n-1)H = -(n-1)t + \frac{n-1}{t} + \frac{4t}{t^2-1} = -\frac{(n-1)t^4 - 2(n+1)t^2 + n-1}{t(t^2-1)}.$$

Hence M is minimal if and only if $(n-1)t^4 - 2(n+1)t^2 + n-1 = 0$. Thus

$$t^2 = \frac{n+1 \pm 2\sqrt{n}}{n-1} = \left(\frac{\sqrt{n} \pm 1}{\sqrt{n-1}} \right)^2.$$

Here we notice that

$$\frac{\sqrt{n}+1}{\sqrt{n-1}} > 1, \quad \frac{\sqrt{n}-1}{\sqrt{n-1}} < 1.$$

Now, we look for biharmonic real hypersurfaces. The square norm of the second fundamental form is

$$|A|^2 = (n-1)t^2 + \frac{n-1}{t^2} + \frac{16t^2}{(t^2-1)^2} = \frac{(n-1)(t^2-1)^2(t^4+1) + 16t^4}{t^2(t^2-1)^2}.$$

Hence, M is biharmonic if and only if $t = \cot r > 1$ is a solution to

$$(n-1)(t^2-1)^2(t^4+1) + 16t^4 = 2(n+1)t^2(t^2-1)^2.$$

Equivalently

$$(n-1)t^8 - 4nt^6 + 6(n+3)t^4 - 4nt^2 + (n-1) = 0.$$

From this we get

$$2-n = \frac{t^8 - 8t^6 + 30t^4 - 8t^2 + 1}{(t^2-1)^4}.$$

Since the right hand side of this formula is positive for any $t > 1$. Hence there are no non-minimal biharmonic tubes. $\square$

7.4. Type C real hypersurfaces

Type C real hypersurfaces are derived from the complex Grassmannian manifold

$$U/K = \text{Gr}_2(\mathbb{C}^{p+2}) = \text{SU}(p+2)/\text{S}(\text{U}(2) \times \text{U}(p)), \quad p = \frac{n+1}{2} \geq 3.$$

The type C hypersurface M is the Hopf projection of

$$\text{S}(\text{U}(2) \times \text{U}((n+1)/2))/(\mathbb{T}^2 \times \text{SU}((n-3)/2)) \subset \mathbb{S}^{2n+1}, \quad n \geq 5.$$

These hypersurfaces are tubes over the Segre imbedding of $\mathbb{C}P^1 \times \mathbb{C}P^{k-1}$ with radius $r \in (0, \pi/4)$. The principal curvatures and their multiplicities are

$$\begin{aligned} \lambda_1 &= -\cot r, & m_1 &= n-3, \\ \lambda_2 &= \cot\left(\frac{\pi}{4} - r\right), & m_2 &= 2, \\ \lambda_3 &= \cot\left(\frac{\pi}{2} - r\right), & m_3 &= n-3, \\ \lambda_4 &= \cot\left(\frac{3\pi}{4} - r\right), & m_4 &= 2, \\ \alpha &= -2\cot(2r), & m_\alpha &= 1. \end{aligned}$$

Put $t = \cot r$. Then the principal curvatures are rewritten as

$$\lambda_1 = -t, \quad \lambda_2 = \frac{t+1}{t-1}, \quad \lambda_3 = \frac{1}{t}, \quad \lambda_4 = \frac{1-t}{t+1}, \quad \alpha = -t + \frac{1}{t}.$$

Theorem 7.6. *The only biharmonic tube around $\mathbb{C}P^1 \times \mathbb{C}P^{(n-1)/2}$ is the minimal tube of radius*

$$r = \cot^{-1} \frac{\sqrt{n} + \sqrt{2}}{\sqrt{n-2}}, \quad n = 5, 7, 9, \dots$$

Proof. The mean curvature H is computed as

$$\begin{aligned} (2n-1)H &= -(n-3)t + \frac{2(t+1)}{t-1} + \frac{n-3}{t} + \frac{2(1-t)}{t+1} - t + \frac{1}{t} \\ &= -(n-2)t + \frac{n-2}{t} + 2 \left\{ \frac{(t+1)^2 - (t-1)^2}{(t+1)(t-1)} \right\} \\ &= -(n-2)t + \frac{n-2}{t} + \frac{8t}{(t+1)(t-1)} \\ &= -\frac{(n-2)t^4 - 2(n+2)t^2 + n-2}{t(t+1)(t-1)}. \end{aligned}$$

Hence M_r is minimal if and only if

$$\cot r = \sqrt{\frac{n+2 \pm 2\sqrt{2n}}{n-2}} = \frac{\sqrt{n} \pm \sqrt{2}}{\sqrt{n-2}}.$$

Here we notice that

$$\frac{\sqrt{n} - \sqrt{2}}{\sqrt{n-2}} < 1, \quad \frac{\sqrt{n} + \sqrt{2}}{\sqrt{n-2}} > 1$$

for all $n \geq 3$. Thus the tube of radius

$$r = \cot^{-1} \frac{\sqrt{n} + \sqrt{2}}{\sqrt{n-2}}$$

is the only minimal tube.

The square norm $|A|^2$ is computed as

$$\begin{aligned} |A|^2 &= (n-3)t^2 + 2\left(\frac{t+1}{t-1}\right)^2 + \frac{n-3}{t^2} + 2\left(\frac{1-t}{t+1}\right)^2 + \left(-t + \frac{1}{t}\right)^2 \\ &= (n-2)t^2 + \frac{n-2}{t^2} + 2\left\{\left(\frac{t+1}{t-1}\right)^2 + \left(\frac{1-t}{t+1}\right)^2\right\} - 2 \\ &= (n-2)t^2 + \frac{n-2}{t^2} + \frac{4(t^4 + 6t^2 + 1)}{(t^2 - 1)^2(t^2 + 1)^2} - 2 \\ &= \frac{(n-2)t^4(t^2 - 1)^2 + (n-2)(t^2 - 1)^2 + 4t^2(t^4 + 6t^2 + 1) - 2t^2(t^2 - 1)^2}{t^2(t^2 - 1)^2}. \end{aligned}$$

Hence M is biharmonic if and only if

$$(n-2)t^4(t^2 - 1)^2 + (n-2)(t^2 - 1)^2 + 4t^2(t^4 + 6t^2 + 1) - 2t^2(t^2 - 1)^2 = 2(n+1)t^2(t^2 - 1)^2.$$

Namely

$$(n-2)t^4(t^2 - 1)^2 + (n-2)(t^2 - 1)^2 + 4t^2(t^4 + 6t^2 + 1) - 2(n+2)t^2(t^2 - 1)^2 = 0.$$

The left hand side of this equation is rewritten as

$$(n-2)(t^2 - 1)^4 - 4t^2(t^4 - 10t^2 + 1) = (t^2 - 1)^4 n - 2t^8 + 4t^6 + 28t^4 + 4t^2 - 2.$$

Thus we get

$$n = \frac{2t^8 - 4t^6 - 28t^4 - 4t^2 + 2}{(t^2 - 1)^4}$$

Hence we have

$$5 - n = \frac{3t^8 - 16t^6 + 58t^4 - 16t^2 + 3}{(t^2 - 1)^4} > 0$$

for any $n \geq 5$ and $t^2 > 1$. Thus this equation has no solutions for $t^2 > 1$. Thus the result follows. $\square$

7.5. Type D real hypersurfaces

The type D real hypersurfaces are associated to $U/K = \text{SO}(10)/\text{U}(5)$. The resulting real hypersurfaces are the Hopf image of

$$\text{U}(5)/(\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)) \subset \mathbb{S}^{19}.$$

One can see that the type D real hypersurfaces are tubes over the Plücker imbedding of the Grassmannian manifold $\text{Gr}_2(\mathbb{C}^5)$ into $\mathbb{C}P^9(4)$ with radius $r \in (0, \pi/4)$.

The principal curvatures and their multiplicities are

$$\begin{aligned} \lambda_1 &= -\cot r, & m_1 &= 4, \\ \lambda_2 &= \cot\left(\frac{\pi}{4} - r\right), & m_2 &= 4, \\ \lambda_3 &= \cot\left(\frac{\pi}{2} - r\right), & m_3 &= 4, \\ \lambda_4 &= \cot\left(\frac{3\pi}{4} - r\right), & m_4 &= 4, \\ \alpha &= -2\cot(2r), & m_\alpha &= 1. \end{aligned}$$

Theorem 7.7 ([82]). *A type D real hypersurface is biharmonic if and only if it is minimal and a tube around $\text{Gr}_2(\mathbb{C}^5)$ of radius $r = \tan^{-1}(1/\sqrt{5})$.*

Proof. The type D real hypersurface has constant mean curvature

$$H = \frac{1}{17} \left[4 \left\{ (-t) + \frac{t+1}{t-1} + \frac{1}{t} + \frac{1-t}{1+t} \right\} - t + \frac{1}{t} \right] = -\frac{5t^4 - 26t^2 + 5}{17t(t^2 - 1)} = -\frac{(5t^2 - 1)(t^2 - 5)}{17t(t^2 - 1)}, \quad t = \cot r.$$

Hence, a type D real hypersurface M is minimal if and only if $\cot r = \sqrt{5} > 1$, since $0 < r < \pi/4$.

Next, the square norm $|A|^2$ is

$$\begin{aligned} |A|^2 &= 4 \left\{ t^2 + \left(\frac{t+1}{t-1} \right)^2 + \frac{1}{t^2} + \left(\frac{1-t}{1+t} \right)^2 \right\} + \left(\frac{1}{t} - t \right)^2 \\ &= 5t^2 + \frac{5}{t^2} + \frac{8(t^4 + 6t^2 + 1)}{(t^2 - 1)^2} - 2 = \frac{5t^8 - 4t^6 + 62t^4 - 4t^2 + 5}{t^2(t^2 - 1)^2}. \end{aligned}$$

Thus M is biharmonic if and only if

$$\frac{5t^8 - 4t^6 + 62t^4 - 4t^2 + 5}{t^2(t^2 - 1)^2} = 20.$$

This is equivalent to

$$5t^8 - 24t^6 + 102t^4 - 24t^2 + 5 = 0.$$

The left hand side of this equation is rewritten as

$$t^4(5t^4 - 24t^2 + 51) + 51t^4 - 24t^2 + 5 > 0.$$

Thus there are no non-minimal biharmonic real hypersurfaces of type D. □

7.6. Type E real hypersurfaces

Type E real hypersurfaces are associated to $U/K = E_6/SO(10) \cdot SO(2)$. These hypersurfaces are tubes over the canonical imbedding of the Hermitian symmetric space $SO(10)/U(5) \subset \mathbb{C}P^{15}(4)$ with radius $r \in (0, \pi/4)$.

The type E real hypersurfaces are Hopf image of

$$U(1) \times \text{Spin}(10)/U(1) \times SU(4) \subset \mathbb{S}^{31}.$$

The principal curvatures of a type E real hypersurface are

$$\begin{aligned} \lambda_1 &= -\cot r, & m_1 &= 8, \\ \lambda_2 &= \cot \left(\frac{\pi}{4} - r \right), & m_2 &= 6, \\ \lambda_3 &= \cot \left(\frac{\pi}{2} - r \right), & m_3 &= 8, \\ \lambda_4 &= \cot \left(\frac{3\pi}{4} - r \right), & m_4 &= 6, \\ \alpha &= -2 \cot(2r), & m_\alpha &= 1. \end{aligned}$$

The mean curvature is given by

$$H = \frac{1}{29} \left[\left\{ 8(-t) + 6 \left(\frac{t+1}{t-1} \right) + 8 \left(\frac{1}{t} \right) + 6 \left(\frac{1-t}{1+t} \right) \right\} - t + \frac{1}{t} \right] = -\frac{3(3t^4 - 14t^2 + 3)}{29t(t^2 - 1)}, \quad t = \cot r.$$

Hence M is minimal if and only if

$$t^2 = \frac{7 + 2\sqrt{10}}{3} > 1.$$

Thus, the radius is determined as

$$r = \tan^{-1} \frac{\sqrt{3}}{\sqrt{2} + \sqrt{5}}.$$

We look for biharmonic ones.

$$\begin{aligned} |A|^2 &= 8t^2 + 6 \left(\frac{t+1}{t-1} \right)^2 + \frac{8}{t^2} + 6 \left(\frac{1-t}{1+t} \right)^2 + \left(-t + \frac{1}{t} \right)^2 \\ &= 9t^2 + \frac{9}{t^2} + \frac{12(t^4 + 6t^2 + 1)}{(t^2 - 1)^2} - 2 = \frac{9t^8 - 8t^6 + 94t^4 - 8t^2 + 9}{t^2(t^2 - 1)^2}. \end{aligned}$$

Thus the biharmonicity condition $|A|^2 = 2(n+1) = 2(15+1) = 32$ is

$$\frac{9t^8 - 8t^6 + 94t^4 - 8t^2 + 9}{t^2(t^2 - 1)^2} = 32.$$

Hence $t = \cot r$ is a solution to

$$9t^8 - 40t^6 + 158t^4 - 40t^2 + 9 = 0.$$

The left hand side of this equation is rewritten as

$$t^4(9t^2 - 40t^2 + 79) + 79t^4 - 40t^2 + 9 > 0.$$

Thus there are no non-minimal biharmonic real hypersurfaces of type E.

Theorem 7.8 ([82]). *A type E real hypersurface is biharmonic if and only if it is minimal and a tube around $SO(10)/U(5) \subset \mathbb{C}P^{15}(4)$ of radius $r = \tan^{-1}\{\sqrt{3}/(\sqrt{2} + \sqrt{5})\}$.*

Hence the correct classification of proper biharmonic homogeneous real hypersurfaces in $\mathbb{C}P^n(4)$ is described as follows:

Theorem 7.9 ([47, 48, 82]). *Let M be a homogeneous real hypersurface of $\mathbb{C}P^n(4)$ with $n \geq 2$. Then M is proper biharmonic if and only if it is holomorphically congruent to an open part of a tube around $\mathbb{C}P^m(4)$ ($0 \leq m \leq n-2$) of radius*

$$r = \cot^{-1} \sqrt{\frac{n+2 \pm \sqrt{(2m-n+1)^2 + 4(n+1)}}{2n-2m-1}}.$$

7.7. Ruled real hypersurfaces

According to [58], a real hypersurface $M \subset \mathbb{C}P^n(4)$ is said to be *ruled* if its holomorphic distribution

$$D = \{X \in TM \mid \eta(X) = 0\}$$

is integrable and each leaf of its maximal integral manifolds is locally congruent to the hyperplane $\mathbb{C}P^{n-1}(4)$. The second named author proved the following theorem.

Theorem 7.10 ([82]). *Let M be a ruled real hypersurface in $\mathbb{C}P^n(4)$, where $n \geq 2$. If M is biharmonic, then it is minimal.*

Ruled hypersurfaces in $\mathbb{C}P^n(4)$ are incomplete. Minimal ruled hypersurfaces of $\mathbb{C}P^n(4)$ are investigated in [1]. Pérez-Barral [76] proved that biharmonic ruled hypersurfaces of $\mathbb{C}H^n(-4)$ are minimal.

7.8. Three dimensional Hopf hypersurfaces

Here we restrict our attention to real hypersurfaces in $\mathbb{C}P^2(4)$. Wang proved the following fundamental fact.

Theorem 7.11 ([99]). *Let M be a real hypersurface of $\mathbb{C}P^2(4)$ with three distinct constant principal curvatures. Then M is holomorphically congruent to an open part of a tube of radius $r \in (0, \pi/4)$ around the complex quadric $Q_1 \subset \mathbb{C}P^2(4)$.*

The tube M_r is diffeomorphic to the lens space $L(4, 1) = \mathbb{S}^3/\mathbb{Z}_4$. Under the limit $r \rightarrow \pi/4$, the tube M_r around Q_1 collapses to $\mathbb{R}P^2$ (see [17]). In particular, we notice the following fact.

Proposition 7.1. *Let M be a real hypersurface of $\mathbb{C}P^2(4)$ with three distinct constant principal curvatures. Then the following properties are mutually equivalent:*

- M is contact metric.
- M is minimal.
- M is biharmonic.
- M is holomorphically congruent to an open part of a tube of radius $\pi/8$ around the complex quadric $Q_1 \subset \mathbb{C}P^2(4)$.

7.8.1. *Sasakian 3-sphere* As is well known, the unit 3-sphere $\mathbb{S}^3$ is identified with the special unitary group

$$\mathrm{SU}(2) = \{P \in \mathrm{SL}_2\mathbb{C} \mid \bar{i}P = 1\}$$

with bi-invariant Riemannian metric of constant curvature 1. Here 1 denotes the identity matrix. The bi-invariant metric $g^{(1)}$ of constant curvature 1 on $\mathrm{SU}(2)$ is induced by the following inner product $\langle \cdot, \cdot \rangle_1$ on the Lie algebra $T_1\mathrm{SU}(2) = \mathfrak{su}(2)$:

$$\langle X, Y \rangle_1 = -\frac{1}{8}B(X, Y) = -\frac{1}{2} \mathrm{tr}(XY), \quad X, Y \in \mathfrak{su}(2).$$

Here B denotes the Killing form of $\mathfrak{su}(2)$. We call $g^{(1)}$ the *normalized Killing metric*. Take a *quaternionic basis* of $\mathfrak{su}(2)$:

$$\mathbf{i} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

By using this basis, the Lie group $\mathrm{SU}(2)$ is described as

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} x_0 + \sqrt{-1}x_3 & -x_2 + \sqrt{-1}x_1 \\ x_2 + \sqrt{-1}x_1 & x_0 - \sqrt{-1}x_3 \end{pmatrix} \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

In the spinor representation of the Euclidean 3-space $\mathbb{E}^3$, we identify $\mathbb{E}^3$ with $\mathfrak{su}(2)$ via the correspondence

$$(x_1, x_2, x_3) \longleftrightarrow x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = \begin{pmatrix} \sqrt{-1}x_3 & -x_2 + \sqrt{-1}x_1 \\ x_2 + \sqrt{-1}x_1 & -\sqrt{-1}x_3 \end{pmatrix}.$$

Denote the left translated vector fields of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ by $\{E_1, E_2, E_3\}$. The commutation relations of $\{E_1, E_2, E_3\}$ are

$$[E_1, E_2] = 2E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 2E_2.$$

The left invariant 1-form

$$\eta^{(1)} = g^{(1)}(E_3, \cdot)$$

is a contact form with Reeb vector field

$$\xi^{(1)} := E_3.$$

The metric g_1 is compatible to η_1 . The contact metric condition

$$g^{(1)}(X, \phi Y) = \frac{1}{2}d\eta^{(1)}(X, Y), \quad X, Y \in \mathfrak{su}(2)$$

induces a left invariant endomorphism field ϕ as

$$\phi(E_1) = E_2, \quad \phi(E_2) = -E_1, \quad \phi(E_3) = 0.$$

It should be remarked that ξ_1 is a unit Killing vector field. The resulting left invariant contact metric structure $(\phi, \xi^{(1)}, \eta^{(1)}, g^{(1)})$ is Sasakian. The Lie group $\mathrm{SU}(2)$ acts isometrically on the Lie algebra $\mathfrak{su}(2)$ by the Ad-action.

$$\mathrm{Ad} : \mathrm{SU}(2) \times \mathfrak{su}(2) \rightarrow \mathfrak{su}(2); \quad \mathrm{Ad}(a)X = aXa^{-1}, \quad a \in \mathrm{SU}(2), \quad X \in \mathfrak{su}(2).$$

The Ad-orbit of $\mathbf{k}/2$ is a 2-sphere $\mathbb{S}^2(4)$ of curvature 4 in the Euclidean 3-space $\mathbb{E}^3 = \mathfrak{su}(2)$. The Ad-action of $\mathrm{SU}(2)$ on $\mathbb{S}^2(4)$ is isometric and transitive. The isotropy subgroup of $\mathrm{SU}(2)$ at $\mathbf{k}/2$ is

$$K_1 = \left\{ \exp(t\mathbf{k}) = \begin{pmatrix} e^{\sqrt{-1}t} & 0 \\ 0 & e^{-\sqrt{-1}t} \end{pmatrix} \mid t \in \mathbb{R} \right\} \cong \mathrm{U}(1) = \left\{ e^{\sqrt{-1}t} \mid t \in \mathbb{R} \right\}.$$

Hence $\mathbb{S}^2(4)$ is represented by $\mathrm{SU}(2)/\mathrm{U}(1)$ as a Riemannian symmetric space. The natural projection

$$\pi_1 : \mathbb{S}^3 \rightarrow \mathbb{S}^2(4), \quad \pi_1(a) = \mathrm{Ad}(a)(\mathbf{k}/2)$$

is a Riemannian submersion and defines a principal $\mathrm{U}(1)$ -bundle over $\mathbb{S}^2(4)$. This fibering is nothing but the well known *Hopf fibering*. Moreover it is the *Boothby-Wang fibering* of $\mathbb{S}^3$ as a regular contact 3-manifold.

7.8.2. *Berger 3-sphere* $\mathcal{M}^3(c)$ Next, let us deform the Riemannian metric $g^{(1)}$ of the unit 3-sphere $\mathbb{S}^3$ as

$$g^{(c)}(X, Y) = \frac{4}{c+3} \left(g^{(1)}(X, Y) - \frac{c-1}{c+3} \eta^{(1)}(X) \eta^{(1)}(Y) \right),$$

where $c > -3$ is a constant. The resulting Riemannian 3-manifold $\mathcal{M}^3(c) = (\mathbb{S}^3, g^{(c)})$ is called the *Berger sphere*.

Precisely speaking, the original one due to Berger is $(\mathbb{S}^3, \frac{c+3}{4} g^{(c)})$ and $c \neq 1$. Note that under the limit $c \rightarrow -3$ in Gromov-Hausdorff sense, $(\mathbb{S}^3, \frac{c+3}{4} g)$ converges to $\mathbb{S}^3$ equipped with the Carnot-Carathéodory metric. On the other hand, under the limit $c \rightarrow 1$, $(\mathbb{S}^3, \frac{c+3}{4} g)$ collapses to $\mathbb{S}^2$.

Let us deform the contact form η_1 and the Reeb vector field $\xi^{(1)}$ as

$$\eta^{(c)} := \frac{4}{c+3} \eta^{(1)}, \quad \xi^{(c)} := \frac{c+3}{4} \xi^{(1)}.$$

Then $g^{(c)}$ is compatible to $\eta^{(c)}$. The Berger sphere $\mathcal{M}^3(c)$ with $c \neq 0$ is no longer a space form, but the ϕ -sectional curvatures are constant c . In particular, $\mathcal{M}^3(c)$ is a Sasakian space form.

The Reeb vector field $\xi^{(c)}$ generates a one parameter group of transformations on $\mathcal{M}^3(c)$. Since ξ is a Killing vector field with respect to the Berger metric, this transformation group acts isometrically on $G = \text{SU}(2)$. The transformation group generated by $\xi^{(c)}$ is identified with the following Lie subgroup $K = K_c$ of G :

$$K_c = \left\{ \exp \left(\frac{(c+3)t}{4} \mathbf{k} \right) \mid t \in \mathbb{R} \right\} \cong \text{U}(1).$$

Furthermore, the action of the transformation group generated by ξ_c corresponds to the natural right action of K_c on $\text{SU}(2)$:

$$\text{SU}(2) \times K_c \rightarrow \text{SU}(2); (a, k) \mapsto ak.$$

By using the well-known curvature formula for Riemannian submersion due to O'Neill, one can see that the orbit space G/K_c is a 2-sphere $\mathbb{S}^2(c+3)$ of curvature $c+3$. The Riemannian metric g_c is not only $\text{SU}(2)$ -left invariant but also K_c -right invariant. Hence $\text{SU}(2) \times K_c$ acts isometrically on $\text{SU}(2)$. The Berger sphere $\mathcal{M}^3(c)$ is represented by $(\text{SU}(2) \times K_c)/K_c$ as a naturally reductive homogeneous space. For $c \neq 1$, $\mathcal{M}^3(c)$ has 4-dimensional isometry group. In particular, g is G -bi-invariant if and only if $c = 1$. In this case $\mathcal{M}^3(1)$ is represented by $(\text{SU}(2) \times \text{SU}(2))/\text{SU}(2)$ as a Riemannian symmetric space. Note that $\mathcal{M}^3(1)$ has 6-dimensional isometry group.

Consider an orthonormal frame field $\{e_1, e_2, e_3\}$ of $\mathcal{M}^3(c)$ by

$$e_1 := \frac{\sqrt{c+3}}{2} E_1, \quad e_2 := \frac{\sqrt{c+3}}{2} E_2, \quad e_3 := \frac{c+3}{4} E_3.$$

Then the commutation relations of this basis are

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = \frac{c+3}{2} e_1, \quad [e_3, e_1] = \frac{c+3}{2} e_2.$$

The Levi-Civita connection ∇ of $(\mathcal{M}^3(c), g^{(c)})$ is described by

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= e_3, & \nabla_{e_1} e_3 &= -e_2, \\ \nabla_{e_2} e_1 &= -e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= e_1, \\ \nabla_{e_3} e_1 &= \frac{c+1}{2} e_2, & \nabla_{e_3} e_2 &= -\frac{c+1}{2} e_1, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

The Riemannian curvature R of $(\mathcal{M}^3(c), g^{(c)}, \nabla)$ is described by

$$R_{1212} = c, \quad R_{1313} = R_{2323} = 1,$$

and the sectional curvatures are:

$$K_{12} = c, \quad K_{13} = K_{23} = 1.$$

The Ricci tensor field Ric and the scalar curvature ρ are computed to be

$$\text{Ric}_{11} = \text{Ric}_{22} = c+1, \quad \text{Ric}_{33} = 2, \quad \rho = 2(c+2).$$

7.8.3. *Generalized Berger 3-spheres* Let us consider immersions of $SU(2)$ into the complex projective plane $\mathbb{C}P^2(4)$. Let us take the quaternion basis $\{i, j, k\}$ of the Lie algebra $\mathfrak{su}(2)$. Denote by $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ the dual basis of $\{i, j, k\}$. The dual basis $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ is regarded as a left invariant frame field on $SU(2)$. The bi-invariant metric g_1 is represented as

$$g_1 = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

The Berger sphere metric g_c is represented as

$$g_c = \frac{4}{c+3} (\vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2) + \left(\frac{4}{c+3} \right)^2 \vartheta^3 \otimes \vartheta^3.$$

Let us equip a left invariant metric of the form

$$g_{\alpha_1, \alpha_2, \alpha_3} = (\alpha_1)^2 \vartheta^1 \otimes \vartheta^1 + (\alpha_2)^2 \vartheta^2 \otimes \vartheta^2 + (\alpha_3)^2 \vartheta^3 \otimes \vartheta^3,$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive constants. Then we obtain a unimodular frame $\{e_1^\alpha, e_2^\alpha, e_3^\alpha\}$ by

$$e_1^\alpha = \frac{1}{\alpha_1} E_1, \quad e_2^\alpha = \frac{1}{\alpha_2} E_2, \quad e_3^\alpha = \frac{1}{\alpha_3} E_3.$$

Then we have the commutation relations

$$[e_1^\alpha, e_2^\alpha] = c_3^\alpha e_3^\alpha, \quad [e_2^\alpha, e_3^\alpha] = c_1^\alpha e_1^\alpha, \quad [e_3^\alpha, e_1^\alpha] = c_2^\alpha e_2^\alpha,$$

where

$$c_1^\alpha = \frac{\alpha_1}{\alpha_2 \alpha_3}, \quad c_2^\alpha = \frac{\alpha_2}{\alpha_3 \alpha_1}, \quad c_3^\alpha = \frac{\alpha_3}{\alpha_1 \alpha_2}.$$

We introduce a left invariant almost contact structure $(\phi^\alpha, \xi^\alpha, \eta^\alpha)$ compatible to the metric $g_{\alpha_1, \alpha_2, \alpha_3}$ by

$$\xi^\alpha = e_3^\alpha, \quad \eta^\alpha = g_{\alpha_1, \alpha_2, \alpha_3}(\xi^\alpha, \cdot), \quad \phi e_1^\alpha = e_2^\alpha, \quad \phi e_2^\alpha = -e_1^\alpha, \quad \phi e_3^\alpha = 0.$$

Then one can see that $(\phi, \xi^\alpha, \eta^\alpha, g_{\alpha_1, \alpha_2, \alpha_3})$ is contact metric if and only if $\alpha_3 = \alpha_1 \alpha_2$. In such a case, $(SU(2); \phi, \xi^\alpha, \eta^\alpha, g_{\alpha_1, \alpha_2, \alpha_3})$ is a contact (κ, μ) -space with

$$\kappa = 1 - \frac{(\alpha_1^2 - \alpha_2^2)^2}{(\alpha_1 \alpha_2)^4}, \quad \mu = 2 - \frac{\alpha_1^2 + \alpha_2^2}{(\alpha_1 \alpha_2)^2}.$$

The Berger 3-sphere $\mathcal{M}^3(c)$ is obtained by the choice

$$\alpha_1 = \alpha_2 = \frac{2}{\sqrt{c+3}}, \quad \alpha_3 = \frac{4}{c+3}.$$

Li determined isometric immersions of $(SU(2), g_{\alpha_1, \alpha_2, \alpha_3})$ into $\mathbb{C}P^2(4)$ under the assumption that the resulting real hypersurfaces are Hopf.

Theorem 7.12 ([60]). *Let $f : (SU(2), g_{\alpha_1, \alpha_2, \alpha_3}) \rightarrow \mathbb{C}P^2(4)$ be an isometric immersion. Assume that the image $M = f(SU(2))$ is a Hopf hypersurface. Then M is holomorphically congruent to one of the following homogeneous hypersurfaces:*

- A geodesic sphere of radius $r \in (0, \pi/2)$. The induced metric is isometric to

$$g = \sin^2 r (\vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \cos^2 r \vartheta^3 \otimes \vartheta^3). \quad (7.2)$$

The metric g is a contact metric if and only if $r = \pi/4$ and the resulting metric is a Sasakian metric of constant ϕ -sectional curvature 5. The image M is minimal if and only if $r = \pi/3$.

- The tube M_r of radius $r \in (0, \pi/4)$ around the complex quadric $Q_1 = \mathbb{C}P^1$. The induced metric is isometric to

$$g = 4 \left(\sin^2 \left(r - \frac{\pi}{4} \right) \vartheta^1 \otimes \vartheta^1 + \sin^2 \left(r + \frac{\pi}{4} \right) \vartheta^2 \otimes \vartheta^2 + \sin^2 r \vartheta^3 \otimes \vartheta^3 \right). \quad (7.3)$$

The metric g is a contact metric if and only if $r = \pi/8$ and M is minimal in $\mathbb{C}P^2(4)$.

One can see that the Hopf hypersurfaces in Theorem 7.12 are equivariant. On the other hand, Hu, Yin and Li classified full equivariant CR minimal immersions $f : (\mathrm{SU}(2), g_{\alpha_1, \alpha_2, \alpha_3}) \rightarrow \mathbb{C}P^2(4)$.

Theorem 7.13 ([45]). *Let $f : (\mathrm{SU}(2), g_{\alpha_1, \alpha_2, \alpha_3}) \rightarrow \mathbb{C}P^2(4)$ be a full equivariant CR minimal immersion. Then up to an inner automorphism of $\mathrm{SU}(2)$ and a holomorphic isometry of $\mathbb{C}P^2(4)$, f is*

- a geodesic sphere of radius $r = \pi/3$ or
- a tube of radius $\pi/8$ around $\mathcal{Q}_1$.

The second named author obtained the following classification.

Theorem 7.14 ([82]). *Let M be a Hopf hypersurface in $\mathbb{C}P^2(4)$. Then M is proper biharmonic if and only if it is holomorphically congruent to an open part of a geodesic sphere of radius*

$$r = \cot^{-1} \sqrt{\frac{4 \pm \sqrt{13}}{3}}.$$

8. Subelliptic biharmonic maps

Harmonic maps from or into strongly pseudo-convex CR-manifolds have been studied by several authors. For instance, Ianus and Pastore [46] proved that every holomorphic map between contact metric manifolds is harmonic. For more information on the harmonicity of holomorphic maps, we refer to [37, 38, 39, 50].

8.1. Pseudo-harmonic maps

Now let $M = (M, \mathcal{S})$ be a $(2n - 1)$ -dimensional strongly pseudo-convex CR-manifold equipped with Tanaka-Webster connection $\hat{\nabla}$. We denote by $(\phi_\vartheta, T, \vartheta, g_\vartheta)$ the associated contact metric structure. The real expression of $\mathcal{S}$ is denoted by (D_ϑ, J) . Take a smooth map $f : (M, \mathcal{S}) \rightarrow (\widetilde{M}, \widetilde{g})$ of M into a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The pseudo-second fundamental form $\hat{\nabla}df$ of f is defined by Petit [80]:

$$(\hat{\nabla}_X df)Y = \nabla_X^f df(Y) - df(\hat{\nabla}_X Y), \quad X, Y \in \Gamma(TM).$$

The pseudo-energy $E_b(f)$ of f over a relatively compact domain Ω is defined by [4]:

$$E_b(f; \Omega) = \int_{\Omega} \frac{1}{2} \sum_{i=1}^{2n-2} (f^* \widetilde{g})(e_i, e_i) dv_\vartheta.$$

Here $\{e_1, e_2, \dots, e_{2n-1}\}$ is a local frame field of M which is orthonormal with respect to Webster metric g_ϑ and of the form

$$e_{n+i-1} = \phi_\vartheta e_i \quad (i = 1, 2, \dots, n-1), \quad e_{2n-1} = T. \quad (8.1)$$

The Euler-Lagrange equation of the pseudo-energy is obtained in [4]:

$$\tau_b(f) = \mathrm{tr}_{D_\vartheta}(\hat{\nabla}df) = \sum_{i=1}^{2n-2} (\hat{\nabla}df)(e_i, e_i) = 0.$$

The section $\tau_b(f) \in \Gamma(f^*TN)$ is called the pseudo-tension field. A smooth map f is said to be pseudo-harmonic (in the sense of [4]) if $\tau_b(f) = 0$. It should be remarked that

$$\tau_b(f) = \mathrm{tr}_{D_\vartheta}(\nabla df) = \sum_{i=1}^{2n-2} (\nabla df)(e_i, e_i), \quad (8.2)$$

because of (3.8) (see (2.10) of [97]).

8.2. Subelliptic biharmonic maps

The Jacobi operator $\mathcal{J}_{b,f}$ associated to a pseudo-harmonic map f is given by [4, p. 227]:

$$\mathcal{J}_{b,f}(V) = \bar{\Delta}_b^f V - \text{tr}_{D_\vartheta} \tilde{R}(V, df) df,$$

where

$$\bar{\Delta}_b^f V = - \sum_{i=1}^{2n-2} \left(\nabla_{e_i}^f \nabla_{e_i}^f - \nabla_{\hat{\nabla}_{e_i} e_i}^f \right) V, \quad \text{tr}_{D_\vartheta} \tilde{R}(V, df) df = \sum_{i=1}^{2n-2} \tilde{R}(V, df(e_i)) df(e_i).$$

Here $\hat{\nabla}$ is the Tanaka-Webster connection as before.

The *pseudo-bienergy* of a smooth map f is introduced by Dragomir and Montaldo [30]:

$$E_{b,2}(f; \Omega) = \int_{\Omega} \frac{1}{2} \tilde{g}(\tau_b(f), \tau_b(f)) dv_{\vartheta}.$$

The Euler-Lagrange equation of the pseudo-bienergy is [30]:

$$\tau_{b,2}(f) = -\mathcal{J}_{b,f}(\tau_b(f)) = 0.$$

A smooth map f is said to be *subelliptic biharmonic* if its *pseudo-bitension field* $\tau_{b,2}(f)$ vanishes.

8.3. Subelliptic biharmonic real hypersurfaces

Urakawa studied subelliptic biharmonicity of real hypersurfaces in complex projective space $\mathbb{C}P^n(c)$. For the sake of accuracy, here we explain Urakawa's setting in [97].

Let (M, S) be a strongly pseudo-convex CR-manifold of dimension $(2n-1)$ and $\widetilde{M}_n = (\widetilde{M}_n, \tilde{g}, J)$ a Kähler manifold of complex dimension $n \geq 2$.

We consider isometric immersions $f : M \rightarrow \widetilde{M}_n$. To distinguish the almost contact structure associated to the CR-structure (M, S) and the almost contact structure induced from $\widetilde{M}_n$, we denote the real expression of S by $(D_\vartheta, \phi_\vartheta)$. The associated contact form and Reeb vector field are denoted by ϑ and T , respectively.

Urakawa studied isometric immersions $f : M \rightarrow \widetilde{M}_n$ of a strongly pseudo-convex CR-manifold (M, S) into a Kähler manifold $\widetilde{M}_n$. Although the Kähler structure $(\tilde{g}, J)$ induces an almost contact metric structure $(\phi, \xi, \eta, f^* \tilde{g})$ by (see also (4.1)):

$$Jdf(X) = df(\phi X) + \epsilon \eta(X) \nu, \quad df(\xi) = -\epsilon J \nu, \quad g = f^* \tilde{g}, \quad (8.3)$$

Urakawa only demands the condition $f^* \tilde{g} = g_\vartheta$. In other words, he did *not* demand any relations between $(\phi_\vartheta, T, \vartheta)$ and (ϕ, ξ, η) .

Here we recall some terminologies.

The pseudo-tension field $\tau_b(f)$ and the tension field $\tau(f)$ are related by (cf. [4, Example 5.2]):

$$\tau_b(f) = \tau(f) - (\nabla df)(T, T).$$

The *pseudo-mean curvature vector field* H_b of f is defined by

$$H_b = \frac{1}{2n-1} \tau_b(f).$$

Equivalently, H_b and the mean curvature vector field H are related as

$$H_b = H - \frac{1}{2n-1} (\nabla df)(T, T).$$

Note that $\tau_b(f)$ is called the *pseudo mean curvature vector* in [97].

According to Urakawa [97, Definition 5.1], an isometric immersion $f : M \rightarrow \widetilde{M}_n$ of a $(2n-2)$ -dimensional strongly pseudo-convex CR-manifold $M = (M; \phi_\vartheta, T, \vartheta, g_\vartheta)$ into a Kähler manifold $(\widetilde{M}_n, \tilde{g}, J)$ of complex dimension n is said to be *admissible* if its second fundamental form satisfies

$$(\nabla df)(X, T) = 0$$

for any $X \in \Gamma(D_\vartheta)$. Urakawa showed that f is admissible if and only if T is a principal vector field.

Since f is an isometric immersion, the second fundamental form ∇df is expressed as $(\nabla df)(X, Y) = g_\vartheta(AX, Y)\nu$. Hence the tension field and pseudo-tension field are rewritten as (cf. [97, p. 160])

$$\tau(f) = (2n-1)H\nu, \quad \tau_b(f) = (2n-1)H_b\nu,$$

where

$$H = \frac{1}{2n-1} \operatorname{tr}_{g_\vartheta} A, \quad H_b = \frac{1}{2n-1} \sum_{i=1}^{2n-2} g_\vartheta(Ae_i, e_i).$$

Here $\{e_1, e_2, \dots, e_{2n-1}\}$ is a local orthonormal frame field of the form (8.1). Set

$$A_{ij} = g_\vartheta(Ae_i, e_j), \quad i = 1, 2, \dots, 2n-1$$

and

$$\alpha_\vartheta = A_{2n-1 \ 2n-1}.$$

Then we have

$$(2n-1)H_b = \sum_{i=1}^{2n-2} A_{ii}, \quad (2n-1)H = \sum_{i=1}^{2n-1} A_{ii} = (2n-1)H_b + \alpha_\vartheta.$$

Since f is admissible, T is a principal vector field with corresponding principal curvature α_ϑ .

Since the pseudo-mean curvature vector field H_b is normal to the immersion f , we have

$$\nabla_X^f H_b = -df(A_{H_b} X) + \nabla_X^\perp H_b.$$

Here $\nabla^\perp$ is the normal connection and A_{H_b} is the Weingarten operator derived from H_b .

The pseudo-mean curvature vector field H_b is said to be *pseudo-parallel* in the sense of [97] if

$$\nabla_X^\perp H_b = 0$$

for any $X \in \Gamma(D_\vartheta)$ (see [97, p. 159]). The derivative $\nabla^f H_b$ is computed as

$$\nabla_X^f H_b = (dH_b)(X)\nu + H_b \nabla_X^f \nu = (dH_b)(X)\nu - H_b df(AX).$$

Hence H_b is pseudo-parallel if and only if $(dH_b)(X) = 0$ for any $X \in \Gamma(D_\vartheta)$. Hence we deduce the following fundamental fact:

Proposition 8.1. *If all principal curvatures other than α_ϑ of an admissible isometric immersion $f : M \rightarrow \widetilde{M}_n$ are constant, then the pseudo-mean curvature vector field is pseudo-parallel.*

Now, let $f : M \rightarrow \widetilde{M}_n(c)$ be an isometric immersion of a $(2n-1)$ -dimensional strongly pseudo-convex CR-manifold $M = (M, \phi_\vartheta, T, \vartheta, g)$ into a complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature c . Urakawa obtained the following result.

Theorem 8.1. *Let M be a $(2n-1)$ -dimensional strongly pseudo-convex CR-manifold and $f : M \rightarrow \widetilde{M}_n(c)$ is an admissible isometric immersion with unit normal vector field ν . Assume that the pseudo mean curvature vector field is pseudo-parallel but does not vanish. Then M is subelliptic biharmonic if and only if either*

1. $\tilde{g}(Jdf(T), \nu) = 0$ and

$$|(\nabla df)|_{D_\vartheta}|^2 = \frac{c(2n+1)}{4} \quad \text{or}$$

2. $Jdf(T) = \tilde{g}(Jdf(T), \nu)\nu$ and

$$|(\nabla df)|_{D_\vartheta}|^2 = \frac{c(n-1)}{2}.$$

This result allows us to only consider real hypersurfaces in complex projective n -space $\mathbb{C}P^n(c)$. Let us compare the contact metric structure $(\phi_\vartheta, T, \vartheta, g_\vartheta)$ associated to $(M, \mathcal{S})$ and the almost contact metric structure (ϕ, ξ, η, g) induced from $\mathbb{C}P^n(c)$.

The case 1: Since f is an isometric immersion, we have $g = f^* \tilde{g} = g_\vartheta$. Next, ξ is defined by $df(\xi) = -\varepsilon J\nu$, hence

$$\tilde{g}(Jdf(T), \nu) = -\tilde{g}(df(T), J\nu) = -\tilde{g}(df(T), -\varepsilon df(\xi)) = \varepsilon \eta(T).$$

Thus the condition (1) of Theorem 8.1 is equivalent to $\eta(T) = 0$.

The case 2: On the other hand, the condition (2) of Theorem 8.1 is

$$Jdf(T) = \varepsilon \eta(T) \nu.$$

Here we notice that

$$\tilde{g}(Jdf(T), Jdf(T)) = \tilde{g}(df(T), df(T)) = g(T, T) = 1.$$

On the other hand,

$$\tilde{g}(\varepsilon \eta(T) \nu, \varepsilon \eta(T) \nu) = \eta(T)^2 \tilde{g}(\nu, \nu) = \eta(T)^2.$$

Hence $\eta(T) = \pm 1$. Without loss of generality we may assume that $\eta(T) = 1$. Hence

$$T = \xi, \quad \vartheta = \eta, \quad \phi_\vartheta = \phi.$$

In this case, M is a contact metric hypersurface.

Let us study subelliptic biharmonicity of contact metric hypersurfaces in $\mathbb{C}P^n(c)$. According to Theorem 6.5 and Theorem 6.6, contact metric hypersurfaces are one of the following list:

- a geodesic sphere of radius $r = (2/\sqrt{c}) \tan^{-1}(\sqrt{c}/2)$ with sign $\epsilon = -1$.
- a tube of $r = (2/\sqrt{c}) \tan^{-1}\{(\sqrt{c+4} - \sqrt{c})/2\} < \pi/(2\sqrt{c})$ around the complex quadric $Q_{n-1} \subset \mathbb{C}P_n(c)$ with sign $\epsilon = 1$.

For simplicity of description we normalize the holomorphic sectional curvature as $c = 4$.

Case (1) *Sasakian geodesic sphere*: Now let us investigate subelliptic biharmonicity of geodesic sphere of radius $r = \pi/4$. This geodesic sphere has principal curvatures

$$\alpha = 2 \cot(2r), \quad \lambda = -\tan r$$

with multiplicities $m_\alpha = 1$ and $m_\lambda = 2(n-1)$.

The pseudo-mean curvature is given by

$$\frac{1}{2(n-1)} (m_\lambda \lambda) = -\frac{2(n-1)}{2(n-1)} \tan r \neq 0.$$

Hence this geodesic sphere is not pseudo-harmonic. Next we have

$$|(\nabla df)|_D|^2 = m_\lambda \lambda^2 = 2(n-1) \tan^2 r = 2(n-1) \tan^2 \frac{\pi}{4} = 2(n-1).$$

This shows that the geodesic sphere of radius $\pi/4$ in $\mathbb{C}P^n(4)$ is subelliptic biharmonic.

Case (2). *non-Sasakian contact metric tube*: Next we consider a tube M of radius r around complex quadrics. The tube M is contact metric if and only if $r = \tan^{-1}(\sqrt{2}-1)$.

The principal curvatures are

$$\lambda_1 = -\cot r =: -t, \quad \lambda_2 = \tan r = \frac{1}{t}, \quad \alpha = 2 \tan(2r)$$

with multiplicities

$$m_1 = m_2 = n-1, \quad m_\alpha = 1.$$

The pseudo-mean curvature is computed as

$$\frac{1}{2(n-1)} (m_1 \lambda_1 + m_2 \lambda_2) = \frac{n-1}{2(n-1)} \left(-t + \frac{1}{t} \right).$$

This shows that the tube is pseudo-harmonic if and only if $t = \pm 1$, that is, $r = \pi/4$. This contradicts $r = \tan^{-1}(\sqrt{2} - 1)$. Hence M is not pseudo-harmonic.

Next, we check the subelliptic biharmonicity. The subelliptic biharmonicity equation of M is

$$|(\nabla df)|_D|^2 = m_1\lambda_1^2 + m_2\lambda_2^2 = (n-1)\left(t^2 + \frac{1}{t^2}\right) = 2(n-1).$$

From this equation, we deduce that $r = \pi/4$ again. Thus M is not subelliptic biharmonic. Summing up above discussions, we get the following result.

Theorem 8.2. *Let $M \subset \mathbb{C}P^n(4)$ be a contact metric hypersurface in the complex projective space of constant holomorphic sectional curvature 4. Then the inclusion map of M is subelliptic biharmonic if and only if it is locally holomorphically congruent to a geodesic sphere of radius $\pi/4$. The geodesic sphere is a Sasakian space form of constant ϕ -sectional curvature 5 and not pseudo-harmonic. In addition there are no pseudo-harmonic contact metric hypersurfaces in $\mathbb{C}P^n(4)$.*

This theorem is interpreted as a variational characterization of Sasakian hypersurface in $\mathbb{C}P^n(4)$.

Okumura [72] proved the following pinching theorem:

Theorem 8.3. *Let M be a compact orientable real hypersurface of $\mathbb{C}P^n(4)$ ($n \geq 2$). If M satisfies the inequality*

$$|A|^2 \leq 2(n-1) + (2n-1)Hg(A\xi, \xi),$$

then $|A|^2 = 2(n-1) + (2n-1)Hg(A\xi, \xi)$ holds and M is holomorphically congruent to $M_{p,q}$ for some p and q .

Motivated from Okumura's theorem, we propose the following problem:

Problem 1. Prove or disprove the following statement:

Let M be a compact strongly pseudo-convex CR-manifold and $f : M \rightarrow \mathbb{C}P^n(4)$ be an isometric immersion into the complex projective space of constant holomorphic sectional curvature 4. Assume that M satisfies the inequality

$$|(\nabla df)|_{D_\phi}|^2 \leq 2(n-1).$$

then $|(\nabla df)|_{D_\phi}|^2 = 2(n-1)$ holds and M is holomorphically congruent to the geodesic sphere of radius $\pi/4$.

Problem 2. Can we construct explicit examples of the case 1 of Theorem 8.1? For instance, let M be a 3-dimensional 3-Sasakian manifold with mutually orthogonal Reeb vector fields ξ_1, ξ_2, ξ_3 (see e.g., [78, 83]). Can we find any isometric immersion $f : M \rightarrow \mathbb{C}P^2(4)$ such that $df(\xi_3) = -\epsilon J\nu$ and $|(\nabla df)|_{D_{\eta_1}}|^2 = 5$?

For recent studies on pseudo-harmonic maps due to Dong, Ren and their collaborators, see [25, 29, 81].

9. Levi-harmonicity of homogeneous real hypersurfaces

9.1. Levi-harmonic maps

Let $(M; \phi, \xi, \eta, g)$ be an almost contact metric manifold and $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold. A smooth map $f : M \rightarrow \widetilde{M}$ is said to be *Levi-harmonic* in the sense of Dragomir and Perrone [31, 79] if it satisfies

$$\tau_D(f) := \text{tr}_D(\nabla df) = \sum_{i=1}^{2n-2} (\nabla df)(e_i, e_i) = 0.$$

For a smooth map $f : (M; \phi, \xi, \eta, g) \rightarrow (\widetilde{M}, \widetilde{g})$, Dragomir and Perrone introduced the following functional E_L for f over a relatively compact region $\Omega \subset M$

$$E_L(f; \Omega) = \int_{\Omega} \frac{1}{2} \sum_{i=1}^{2n-2} (f^* \widetilde{g})(e_i, e_i) dv_g.$$

Then f is a critical point of E_L if and only if it satisfies the Euler-Lagrange equation:

$$\tau_D(f) - df(\nabla_{\xi} \xi + (\text{div } \xi)\xi) = 0.$$

Thus if M satisfies $\nabla_\xi \xi = 0$ and $\operatorname{div} \xi = 0$, then the Levi-harmonic maps are characterized as critical points of E_L .

Here we investigate the Levi-harmonicity of the inclusion maps of homogeneous Hopf hypersurfaces in $\mathbb{C}P^n(4)$ and $\mathbb{C}H^n(-4)$.

Let M be a homogeneous Hopf hypersurface of $\widetilde{M}_n(c)$. Then we have

$$\nabla_\xi \xi = \epsilon \phi A \xi = \epsilon \phi(\alpha \xi) = 0.$$

Next, take a locally defined unit principal vector field V orthogonal to ξ corresponding to the principal curvature λ_i . Then we have

$$\tilde{g}(\nabla_V \xi, V) = \tilde{g}(\epsilon \phi A V, V) = \epsilon \lambda_i \tilde{g}(\phi V, V) = 0.$$

Thus we deduce that $\operatorname{div} \xi = 0$.

Proposition 9.1. *Let M be a Hopf hypersurface of a complex space form $\widetilde{M}_n(c)$. Then the inclusion map ι is Levi-harmonic if and only if it is a critical point of E_L .*

9.2. Type A hypersurfaces in $\mathbb{C}P^n(4)$

Let $M_{p,q}$ be a type A real hypersurface of $\mathbb{C}P^n(4)$, then the inclusion map ι satisfies

$$\tilde{g}(\tau_D(\iota), \nu) = m_1 \lambda_1 + m_2 \lambda_2 = 2(q \cot r - p \tan r).$$

First we consider geodesics spheres. In case $p = 0$ or $q = 0$, we have $\tilde{g}(\tau_D(\iota), \nu) \neq 0$. Thus geodesics spheres can not be Levi-harmonic. Next, in case $p \neq 0$ and $q \neq 0$, $\tilde{g}(\tau_D(\iota), \nu) = 0$ holds if and only if

$$r = \tan^{-1} \sqrt{\frac{q}{p}}.$$

Proposition 9.2. *A real hypersurface $M_{p,q}$ of type A in $\mathbb{C}P^n(4)$ is Levi-harmonic if and only if its radius is $r = \tan^{-1} \sqrt{q/p}$.*

9.3. Type B hypersurfaces in $\mathbb{C}P^n(4)$

Let M_r be a real hypersurface of type B in $\mathbb{C}P^n(4)$. Then we have

$$\tilde{g}(\tau_D(\iota), \nu) = m_1 \lambda_1 + m_2 \lambda_2 = (n-1)(-\cot r + \tan r).$$

The solution t of $\tilde{g}(\tau_D(\iota), \nu) = 0$ is $t = \cot r = \pm 1$. This is impossible, since $r \in (0, \pi/4)$. Thus we obtain the following result.

Proposition 9.3. *A real hypersurface M_r of type B in $\mathbb{C}P^n(4)$ can not be Levi-harmonic.*

9.4. Type C hypersurfaces in $\mathbb{C}P^n(4)$

Let M be a tube of radius $r \in (0, \pi/4)$ around $\operatorname{Gr}_2(\mathbb{C}^{p+2})$ with $p = (n+1)/2$. Then we have

$$\begin{aligned} \tilde{g}(\tau_D(\iota), \nu) &= m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3 + m_4 \lambda_4 \\ &= -(n-3)t + 2\frac{t+1}{t-1} + \frac{n-3}{t} + \frac{2(1-t)}{t+1} = -\frac{(n-3)t^4 - 2(n+1)t^2 + (n-3)}{t(t+1)(t-1)}, \end{aligned}$$

where $t = \cot r$ as before. Thus $\tilde{g}(\tau_D(\iota), \nu) = 0$ if and only if

$$t^2 = \frac{n+1 \pm 2\sqrt{2(n-1)}}{n-3}.$$

We notice that

$$\begin{aligned} \frac{n+1+2\sqrt{2(n-1)}}{n-3} &> 1 \quad (n \geq 5), \quad \lim_{n \rightarrow \infty} \frac{n+1+2\sqrt{2(n-1)}}{n-3} = 1. \\ \frac{n+1-2\sqrt{2(n-1)}}{n-3} &< 1 \quad (n \geq 5), \quad \lim_{n \rightarrow \infty} \frac{n+1-2\sqrt{2(n-1)}}{n-3} = 1. \end{aligned}$$

Hence M is Levi-harmonic if and only if

$$r = \cot^{-1} \sqrt{\frac{n+1+2\sqrt{2(n-1)}}{n-3}}.$$

Proposition 9.4. *A real hypersurface M_r of type C in $\mathbb{C}P^n(4)$ is Levi-harmonic if and only if its radius is*

$$r = \cot^{-1} \sqrt{\frac{n+1+2\sqrt{2(n-1)}}{n-3}}.$$

9.5. Type D hypersurfaces in $\mathbb{C}P^n(4)$

Let M be a tube of radius $r \in (0, \pi/4)$ around the Plücker image of $\text{Gr}_2(\mathbb{C}^5)$. Then

$$\begin{aligned} \tilde{g}(\tau_D(\iota), \nu) &= m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 \\ &= 4(-t) + \frac{4(t+1)}{t-1} + \frac{4}{t} + \frac{4(1-t)}{t+1} = \frac{-4t^4 + 24t^2 - 4}{t^3 - t} = \frac{-4(t^4 - 6t^2 + 1)}{t(t+1)(t-1)}, \end{aligned}$$

where $t = \cot r$ as before. Thus $\tilde{g}(\tau_D(\iota), \nu) = 0$ if and only if $t^2 = 3 \pm 2\sqrt{2}$. Since $t^2 > 1$, we get $t^2 = 3 + 2\sqrt{2}$ and hence $t = 1 + \sqrt{2}$. The possible solution is $t = 1 + \sqrt{2}$, that is, $r = \pi/8$.

Proposition 9.5. *The only Levi-harmonic real hypersurface M_r of type D in $\mathbb{C}P^n(4)$ is the tube of radius $\pi/8$ around the Plücker image of $\text{Gr}_2(\mathbb{C}^5)$.*

9.6. Type E hypersurfaces in $\mathbb{C}P^n(4)$

Let M be a tube of radius $r \in (0, \pi/4)$ around $\text{SO}(10)/\text{U}(5)$. Then

$$\begin{aligned} \tilde{g}(\tau_D(\iota), \nu) &= m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 \\ &= -8t + \frac{6(t+1)}{t-1} + \frac{8}{t} + \frac{6(1-t)}{t+1} = \frac{-8(t^4 - 5t^2 + 1)}{t(t+1)(t-1)}, \end{aligned}$$

where $t = \cot r$ as before. Thus the solutions to $\tilde{g}(\tau_D(\iota), \nu) = 0$ are

$$t^2 = \frac{5 \pm \sqrt{21}}{2}.$$

Since $t^2 > 1$, we have $t^2 = (5 + \sqrt{21})/2$ and hence $t = (\sqrt{7} + \sqrt{3})/2$.

Proposition 9.6. *The only Levi-harmonic real hypersurface M_r of type D in $\mathbb{C}P^n(4)$ is the tube of radius*

$$r = \cot^{-1} \frac{\sqrt{7} + \sqrt{3}}{2}$$

around $\text{SO}(10)/\text{U}(5)$.

9.7. Type A_0 hypersurfaces in $\mathbb{C}H^n(-4)$

From now on, we study Levi-harmonicity of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n(-4) = \text{SU}(1, n)/\text{S}(\text{U}(1) \times \text{U}(n))$. Let $\text{SU}(1, n) = NAK$ be the Iwasawa decomposition of $\text{SU}(1, n)$ with $K = \text{S}(\text{U}(1) \times \text{U}(n))$. The nilpotent part N is the Heisenberg group. The complex hyperbolic space $\mathbb{C}H^n(-4)$ is identified with the solvable part $\mathcal{S} = NA$. The orbit of the Heisenberg group N is a homogeneous real hypersurface of type A_0 called the *horosphere* of $\mathbb{C}H^n(-4)$.

The horosphere M has principal curvatures $\alpha = 2$ of multiplicity $m_\alpha = 1$ and $\lambda_1 = 1$ of multiplicity of $2n - 2$. Obviously, M is neither minimal nor Levi-harmonic.

9.8. Type $A_{1,0}$ hypersurfaces in $\mathbb{C}H^n(-4)$

The type $A_{1,0}$ hypersurfaces are geodesic spheres and have principal curvatures $\alpha = 2 \coth(2r)$ of multiplicity $m_\alpha = 1$ and $\lambda_1 = \coth r$ of multiplicity $m_1 = 2n - 2$. Put $t = \coth r$, then $\alpha = (1 + t^2)/t$. The mean curvature H is given by

$$(2n - 1)H = m_1\lambda_1 + m_\alpha\alpha = \frac{(2n - 1)t^2 + 1}{t} > 0.$$

Hence M can not be minimal. On the other hand,

$$\tilde{g}(\tau_D(\iota), \nu) = m_1\lambda_1 = 2(n - 1)t > 0.$$

Hence the geodesic sphere M is not Levi-harmonic.

9.9. Type $A_{1,1}$ hypersurfaces in $\mathbb{C}H^n(-4)$

A tube $M = M_r$ of radius $r > 0$ around $\mathbb{C}H^{n-1}$ has principal curvatures $\alpha = 2 \coth(2r)$ of multiplicity $m_\alpha = 1$ and $\lambda_1 = \tanh r$ of multiplicity $m_1 = 2n - 2$. The mean curvature H is given by

$$(2n - 1)H = m_1\lambda_1 + m_\alpha\alpha = \frac{t^2 + 2n - 1}{t} > 0, \quad t = \coth r.$$

Hence M can not be minimal. On the other hand,

$$\tilde{g}(\tau_D(\iota), \nu) = m_1\lambda_1 = \frac{2(n - 1)}{t} > 0.$$

Hence the tube M_r is not Levi-harmonic.

9.10. Type A_2 hypersurfaces in $\mathbb{C}H^n(-4)$

A tube $M = M_r$ of radius $r > 0$ around $\mathbb{C}H^\ell$ ($1 \leq \ell \leq n - 2$). A tube $M = M_r$ of radius $r > 0$ around $\mathbb{C}H^{n-1}$ has principal curvatures $\alpha = 2 \coth(2r)$ of multiplicity $m_\alpha = 1$, $\lambda_1 = \coth(r)$ of multiplicity $m_1 = 2(n - 1 - \ell)$ and $\lambda_2 = \tanh(r)$ of multiplicity $m_2 = 2\ell$. The mean curvature H is given by

$$(2n - 1)H = m_1\lambda_1 + m_2\lambda_2 + m_\alpha\alpha = \frac{(2n - 2\ell - 1)t^2 + 2\ell + 1}{t} > 0, \quad t = \coth r.$$

Hence M can not be minimal. On the other hand,

$$\tilde{g}(\tau_D(\iota), \nu) = m_1\lambda_1 + m_2\lambda_2 = \frac{2(n - \ell - 1)t^2 + 2\ell}{t} > 0.$$

Hence the tube M_r is not Levi-harmonic.

Note that tubes M_r around $\mathbb{C}H^\ell$ ($0 \leq \ell \leq n - 1$) are orbits of $S(U(1, \ell) \times U(n - \ell))$.

9.11. Type B hypersurfaces in $\mathbb{C}H^n(-4)$

A tube M_r of radius r around the totally geodesic Lagrangian real hyperbolic space $\mathbb{R}H^n(c/4)$ is an orbit of $SO^+(1, n)$. The tube M_r has principal curvatures $\alpha = 2 \tanh(2r)$ of multiplicity $m_\alpha = 1$, $\lambda_1 = \coth(r)$ of multiplicity $m_1 = n - 1$ and $\lambda_2 = \tanh(r)$ of multiplicity $m_2 = n - 1$. The mean curvature H is given by

$$(2n - 1)H = m_1\lambda_1 + m_2\lambda_2 + m_\alpha\alpha = \frac{4t^2 + (n - 1)(t^2 + 1)^2}{t(1 + t^2)} > 0, \quad t = \coth r.$$

Hence M can not be minimal. On the other hand,

$$\tilde{g}(\tau_D(\iota), \nu) = m_1\lambda_1 + m_2\lambda_2 = \frac{(n - 1)(1 + t^2)}{t} > 0.$$

Hence the tube M_r is not Levi-harmonic.

It should be remarked that when $r = \coth^{-1} \sqrt{3} = \log(2 + \sqrt{3})/2$, $\lambda_1 = \alpha$ holds and hence $M_{\coth^{-1} \sqrt{3}}$ has two distinct principal curvatures $\lambda_1 = \sqrt{3}$ of multiplicity $m_1 = n$ and $\lambda_2 = 1/\sqrt{3}$ of multiplicity $m_2 = n - 1$. Moreover we have

$$(2n - 1)H = \frac{(n - 1) + nt^2}{t} > 0, \quad \tilde{g}(\tau_D(\iota), \nu) = (n - 1) \left(t + \frac{1}{t} \right) > 0.$$

As a conclusion, we obtain

Proposition 9.7. *There are no Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(-4)$ which are minimal or Levi-harmonic.*

For recent study on minimal real hypersurfaces in complex hyperbolic space, we refer to [40] and references therein.

10. Biharmonic real hypersurfaces in quaternion projective space

10.1. The quaternion projective space

The quaternion projective n -space $\mathbb{H}P^n$ is represented by $\mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ as a homogeneous space. We equip $\mathbb{H}P^n$ with the metric $\tilde{g}$ induced from a negative constant multiple of the Killing form. Then the resulting homogeneous Riemannian space is a real $4n$ -dimensional compact Riemannian symmetric space of rank 1. Moreover $\mathbb{H}P^n$ is a quaternionic symmetric space. We normalize the metric of $\mathbb{H}P^n$ so that the maximal sectional curvature of $\mathbb{H}P^n$ is 4 and denote the resulting Riemannian symmetric space by $\mathbb{H}P^n(4)$. Since $\mathbb{H}P^1$ is the 4-sphere $\mathbb{S}^4(4)$, hereafter we assume that $n > 1$. The biharmonicity criterion is given as follows ([47, 15 Theorem], [48, Theorem 6]).

Theorem 10.1. *Let $M \subset \mathbb{H}P^n(4)$ be a real hypersurface of a quaternion projective space of maximal sectional curvature 4 ($n \geq 2$). Assume that M has non-zero constant mean curvature. Then M is biharmonic if and only if $|A|^2 = 4(n+2)$. In such a case, M has scalar curvature*

$$\rho = 4(4n-3)(n+2) + (4n-1)^2 H^2.$$

10.2. Homogeneous real hypersurfaces

The homogeneous hypersurfaces in quaternion projective space were essentially classified by Iwata [53] and D'Atri [27]. Any such hypersurface is either a tube around a totally geodesic subspace $\mathbb{H}P^k \subset \mathbb{H}P^n(4)$ for some $k = 1, 2, \dots, n-1$ or a tube around a totally geodesic complex projective space $\mathbb{C}P^n \subset \mathbb{H}P^n(4)$.

Berndt proved that a hypersurface in $\mathbb{H}P^n(4)$ is curvature-adapted if and only if it is an open part of a homogeneous hypersurface in $\mathbb{H}P^n(4)$.

10.3. Geodesic spheres

Let M_r be a geodesic sphere of radius r in $\mathbb{H}P^n(4)$ ($0 < r < \pi/2$). Then M_r has principal curvatures $\lambda_1 = \cot r$ of multiplicity $m_1 = 4(n-1)$ and $\lambda_2 = 2 \cot(2r)$ of multiplicity $m_2 = 3$.

Theorem 10.2. *A geodesic sphere $M_r \subset \mathbb{H}P^n(4)$, ($n > 1$) is minimal if and only if its radius r is*

$$r = \tan^{-1} \sqrt{\frac{4n-1}{3}}.$$

Proof. The mean curvature H of M_r is

$$(4n-1)H = 4(n-1)t + 3 \left(t - \frac{1}{t} \right).$$

Here we put $t := \cot r$. Hence M_r is minimal if and only if

$$t = \sqrt{\frac{3}{4n-1}}. \quad (10.1)$$

Thus the geodesic sphere of radius

$$r = \tan^{-1} \sqrt{\frac{4n-1}{3}}$$

is minimal. □

Next we classify proper biharmonic geodesic spheres.

Theorem 10.3. *Geodesic spheres of radii*

$$r = \cot^{-1} \sqrt{\frac{2n+7 \pm 2\sqrt{n^2+4n+13}}{4n-1}}$$

are the only proper biharmonic geodesic spheres in $\mathbb{H}P^n(4)$, $n > 1$.

Proof. The square norm of the second fundamental form is given by

$$|A|^2 = 4(n-1)t^2 + 3\left(t - \frac{1}{t}\right)^2 = (4n-1)t^2 + \frac{3}{t^2} - 6. \quad (10.2)$$

Thus M_r is non-minimal and biharmonic if and only if

$$(4n-1)t^4 - 2(2n+7)t^2 + 3 = 0.$$

Solving this equation, we get

$$t^2 = \frac{2n+7 \pm 2\sqrt{n^2+4n+13}}{4n-1} > 0.$$

Thus geodesic spheres of radii

$$r = \cot^{-1} \sqrt{\frac{2n+7 \pm 2\sqrt{n^2+4n+13}}{4n-1}}$$

are proper biharmonic. $\square$

Remark 10.1. Although $\mathbb{C}P^n(4)$ has no Einstein real hypersurfaces, $\mathbb{H}P^n(4)$ has. In fact, the geodesic sphere M_r of radius $r = \cot^{-1}(1/\sqrt{2n})$ is the only Einstein real hypersurface in $\mathbb{H}P^n(4)$. The computation above shows that the Einstein geodesic sphere is non-biharmonic.

10.4. Tubes of quaternionic subspaces

The tube $M_k(r)$ of totally geodesic and quaternionic subspace $\mathbb{H}P^k \subset \mathbb{H}P^n(4)$ of radius $r \in (0, \pi/2)$ has principal curvatures

$$\lambda = \cot r, \quad \mu = -\tan r, \quad \alpha_1 = 2 \cot(2r)$$

with multiplicities

$$m_\lambda = 4(n-k-1), \quad m_\mu = 4k, \quad m_{\alpha_1} = 3.$$

Theorem 10.4. *The only minimal tubes around $\mathbb{H}P^k \subset \mathbb{H}P^n(4)$ are tubes of radius*

$$r = \tan^{-1} \sqrt{\frac{4n-4k-1}{4k+3}}.$$

Proof. The mean curvature H of $M_k(r)$ is computed as

$$\begin{aligned} (4n-1)H &= 4(n-k-1)t + 4k\left(-\frac{1}{t}\right) + 3\left(t - \frac{1}{t}\right) \\ &= (4n-4k-1)t - \frac{4k+3}{t}. \end{aligned}$$

Hence M is minimal if and only if

$$r = \tan^{-1} \sqrt{\frac{4n-4k-1}{4k+3}}.$$

Theorem 10.5. *The only non-minimal biharmonic tubes M_r around $\mathbb{H}P^k$ are tubes of radii*

$$r = \cot^{-1} \sqrt{\frac{2n+7 \pm 2\sqrt{(n-2k)^2+4(n+k)+13}}{4n-4k-1}}.$$

Proof. The square norm of the second fundamental form is given by

$$|A|^2 = 4(n-k-1)t^2 + \frac{4k}{t^2} + 3\left(t - \frac{1}{t}\right)^2 = (4n-4k-1)t^2 + \frac{4k+3}{t^2} - 6.$$

Thus $M_k(r)$ is proper biharmonic if and only if

$$(4n - 4k - 1)t^4 - 2(2n + 7)t^2 + 4k + 3 = 0. \quad (10.3)$$

Thus

$$\begin{aligned} t_{\pm}^2 &= \frac{2n + 7 \pm \sqrt{(2n + 7)^2 - (4k + 3)(4n - 4k - 1)}}{4n - 4k - 1} \\ &= \frac{2n + 7 \pm 2\sqrt{(n - 2k)^2 + 4(n + k) + 13}}{4n - 4k - 1} > 0. \end{aligned}$$

Thus there exist two proper biharmonic tubes. □

10.5. Tubes of complex projective space

Let M_r be a tube of $\mathbb{C}P^n$ of radius $r \in (0, \pi/4)$. Then the principal curvatures of M_r are

$$\lambda = \cot r, \quad \mu = -\tan r, \quad \alpha_1 = 2 \cot(2r), \quad \alpha_2 = -2 \tan(2r)$$

with multiplicities

$$m_\lambda = 2(n - 1), \quad m_\mu = 2(n - 1), \quad m_{\alpha_1} = 1, \quad m_{\alpha_2} = 2.$$

Theorem 10.6. *The only biharmonic tubes around $\mathbb{C}P^n$ are minimal tubes of radius*

$$r = \cot^{-1} \sqrt{\frac{2n + 3 + 2\sqrt{2(2n + 1)}}{2n - 1}}.$$

Proof. First we look for minimal tubes. The mean curvature H of a tube M_r around $\mathbb{C}P^n$ is given by

$$\begin{aligned} (4n - 1)H &= 2(n - 1)t + 2(n - 1) \left(-\frac{1}{t} \right) + \left(t - \frac{1}{t} \right) + 2 \left(\frac{-4t}{t^2 - 1} \right) \\ &= (2n - 1)t - \frac{2n - 1}{t} - \frac{8t}{t^2 - 1} \\ &= \frac{(2n - 1)t^4 - 2(2n + 3)t^2 + 2n - 1}{t(t^2 - 1)}. \end{aligned}$$

Thus M_r is minimal if and only if

$$(2n - 1)t^4 - 2(2n + 3)t^2 + 2n - 1 = 0. \quad (10.4)$$

Hence we get

$$t^2 = \frac{2n + 3 \pm \sqrt{(2n + 3)^2 - (2n - 1)^2}}{2n - 1} = \frac{2n + 3 \pm 2\sqrt{2(2n + 1)}}{2n - 1} > 0.$$

We notice that

$$\frac{2n + 3 + 2\sqrt{2(2n + 1)}}{2n - 1} > 1, \quad 0 < \frac{2n + 3 - 2\sqrt{2(2n + 1)}}{2n - 1} < 1.$$

Thus the only minimal tube is the one with radius

$$r = \cot^{-1} \sqrt{\frac{2n + 3 + 2\sqrt{2(2n + 1)}}{2n - 1}}.$$

Next, we seek biharmonic tubes.

$$|A|^2 = 2(n - 1)t^2 + \frac{2(n - 1)}{t^2} + t^2 - 2 + \frac{1}{t^2} + \frac{32t^2}{(t^2 - 1)^2} = (2n - 1)t^2 + \frac{2n - 1}{t^2} - 2 + \frac{32t^2}{(t^2 - 1)^2}.$$

Hence $|A|^2 = 4(n + 2)$ if and only if

$$(2n - 1)t^8 - (8n + 8)t^6 + (50 + 12n)t^4 - (8n + 8)t^2 + 2n - 1 = 0. \quad (10.5)$$

From this we have

$$n = \frac{t^8 + 8t^6 - 50t^4 + 8t^2 + 1}{2(t^2 - 1)^4}.$$

This shows that

$$2 - n = \frac{3t^8 - 24t^6 + 74t^4 - 24t^2 + 3}{2(t^2 - 1)^4} > 0.$$

Thus M_r can not be proper biharmonic. $\square$

Martinez [64] studied ruled real hypersurfaces in $\mathbb{H}P^n(4)$ (see also [3]). Here we propose the following problem:

Problem 3. Classify biharmonic ruled hypersurfaces in $\mathbb{H}P^n(4)$.

10.6. Some remarks

Remark 10.2. C. Brandão [16] studied biharmonic totally real submanifolds with parallel mean curvature vector field of $\mathbb{H}P^n(4)$ as well as biharmonic anti quaternionic submanifolds with parallel mean curvature vector field of $\mathbb{H}P^n(4)$. On the other hand, Kacimi and Cherif [55] studied biharmonic totally real submanifolds of $\mathbb{H}P^n(4)$ with constant mean curvature.

The classification [51, Theorem 10] is now corrected as follows:

Theorem 10.7. *The proper biharmonic homogeneous hypersurfaces in simply connected compact Riemannian symmetric spaces of rank 1 are given as follows:*

- *Totally umbilical small hyperspheres of radius $r = 1/\sqrt{2}$ in the unit sphere $\mathbb{S}^n$.*
- *The product immersion $\mathbb{S}^{n-p}(1/\sqrt{2}) \times \mathbb{S}^{p-1}(1/\sqrt{2}) \subset \mathbb{S}^n$ with $n - p \neq p - 1$.*
- *The tubes around $\mathbb{C}P^m(4)$ ($0 \leq m \leq n - 2$) of radii*

$$r = \cot^{-1} \sqrt{\frac{n + 2 \pm \sqrt{(2m - n + 1)^2 + 4(n + 1)}}{2n - 2m - 1}}$$

in $\mathbb{C}P^n(4)$ of constant holomorphic sectional curvature 4.

- *The geodesic spheres of radii*

$$r = \cot^{-1} \sqrt{\frac{2n + 7 \pm 2\sqrt{n^2 + 4n + 13}}{4n - 1}}$$

in the quaternion projective space $\mathbb{H}P^n(4)$ of maximal sectional curvature 4.

- *The tubes around $\mathbb{H}P^k$ ($1 \leq k \leq n - 1$) of radii*

$$r = \cot^{-1} \sqrt{\frac{2n + 7 \pm 2\sqrt{(n - 2k)^2 + 4(n + k) + 13}}{4n - 4k - 1}}$$

in the quaternion projective space $\mathbb{H}P^n(4)$ of maximal sectional curvature 4.

- *Geodesic spheres of radii $r = \tan^{-1} \sqrt{(25 \pm 2\sqrt{130})/7}$ in Cayley projective plane $\mathfrak{O}P^2(4)$ of maximal sectional curvature 4.*

Remark 10.3. The geodesic sphere M_r of radius $r \in (0, \pi/2)$ in the Cayley projective plane $\mathfrak{O}P^2(4)$ of maximal sectional curvature 4 is

- Einstein if and only if $r = \tan^{-1}(2\sqrt{2}/\sqrt{3})$.
- a 1-type submanifold in the sense of Chen [20] via the first standard imbedding if and only if $r = \tan^{-1}(\sqrt{17}/\sqrt{7})$.
- minimal if and only if $r = \tan^{-1}(\sqrt{15}/\sqrt{7})$.

Remark 10.4 (Volume stability). As we mentioned in the Introduction, the harmonicity of an isometric immersion is equivalent to the minimality of the immersion. Let us consider a compact oriented minimal real hypersurface M of the simply connected compact Riemannian symmetric space G/K of rank 1 and set $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathfrak{D}$. More explicitly we have $d(\mathbb{R}) = 1$, $d(\mathbb{C}) = 2$, $d(\mathbb{H}) = 4$, and $d(\mathfrak{D}) = 8$.

Denote by $\text{null}_{\text{Vol}}(M)$ the nullity of M with respect to the volume functional. Simons [85] and Gotoh [41, 42, 43] proved that the volume nullity of $M \subset G/K$ satisfies the inequality $\text{null}_{\text{Vol}}(M) \geq d(\mathbb{F})n$. Moreover, in case $G/K = \mathbb{S}^n(1)$, the equality holds if and only if M is congruent to the totally geodesic sphere. Next, in case $G/K = \mathbb{C}P^n(4)$, $\mathbb{H}P^n(4)$ or $\mathcal{O}P^2(4)$, the equality holds if and only if M is congruent to the minimal geodesic sphere.

Remark 10.5. The Grassmannian manifold $\widetilde{\text{Gr}}_3(\mathbb{R}^7)$ of oriented 3-planes in $\mathbb{R}^7$ has the following totally geodesic singular orbits under cohomogeneity one actions [10]:

- $\widetilde{\text{Gr}}_2(\mathbb{R}^6)$ and $\widetilde{\text{Gr}}_3(\mathbb{R}^6)$ (reflective)
- The space $G_2/\text{SO}(4)$ of all quaternionic subalgebras of the Cayley algebra $\mathcal{O}$ (reflective). Here $\mathbb{R}^7$ is regarded as the imaginary part $\text{Im } \mathcal{O}$ of $\mathcal{O}$.

For a tube of radius r around $\widetilde{\text{Gr}}_2(\mathbb{R}^6) \subset \widetilde{\text{Gr}}_3(\mathbb{R}^7)$ is

- minimal if and only if (see [51, 52]):

$$r = \sqrt{10} \tan^{-1} \sqrt{\frac{3}{2}}.$$

- proper biharmonic if and only if

$$r = \frac{\pi}{4} \sqrt{10}.$$

The associative calibration $\Psi : \mathcal{O} \times \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ is regarded as a smooth function on the Grassmannian manifold $\widetilde{\text{Gr}}_3(\text{Im } \mathcal{O})$ and its range is the bounded closed interval $[-1, 1]$. Let us denote by $M(t)$ the level set of the associative calibration ($t \in [-1, 1]$). Then $M(\pm 1)$ are totally geodesic singular orbits under the cohomogeneity one action. Moreover $M(\pm 1)$ is identified with the Grassmannian manifold $\widetilde{\text{Gr}}_{\text{ass}}(\text{Im } \mathcal{O})$ of associative subspaces. Moreover $M(\pm 1)$ is the quaternionic symmetric space $G_2/\text{SO}(4)$ which is the Grassmannian manifold of quaternionic subalgebras of $\mathcal{O}$. On the other hand, for any $t \in (-1, 1)$, the level set $M(t)$ is a principal orbit of the cohomogeneity one action of G_2 . The level set is identified with $G_2/\text{SO}(3)$ and it is a reflective submanifold of $\widetilde{\text{Gr}}_3(\text{Im } \mathcal{O})$. Note that $G_2/\text{SO}(4) \subset \widetilde{\text{Gr}}_3(\text{Im } \mathcal{O})$ is non-reflective. Enyoshi [36] studied minimality and biharmonicity of $M(t)$. She proved that

- $M(t)$ is minimal if and only if $t = 0$. In such a case $M(0)$ is austere.
- $M(t)$ is proper biharmonic if and only if $t = \pm 1/\sqrt{10}$.

Remark 10.6. Urakawa proposed the following CR version of the generalized Chen conjecture:

Let M be a complete strongly pseudoconvex CR manifold, and (N, h) is a Riemannian manifold of non-positive curvature. Then, every subelliptic biharmonic isometric immersion $f : M \rightarrow N$ must be pseudo-harmonic.

On the other hand, in [23], we studied harmonicity and biharmonicity for smooth maps $f : (N, h) \rightarrow M$. It is not known whether the following statement is true.

Let M be a Sasakian manifold of constant ϕ -sectional curvature $c \leq -3$. Then, every pseudo-Hermitian biharmonic submanifolds in M is pseudo-Hermitian harmonic.

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Author's contributions

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