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Research Article

Hermite-Hadamard Inequalities Involving Fractional Conformable Integral Operators for the Product of Two Interval-Valued LR-Convex Functions

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Abstract
In this research, new Hermite-Hadamard-type inequalities are obtained for the product of two LR-convex interval-valued functions utilizing fractional conformable integrals. Specific parameter choices are employed to generalize existing results and to acquire new findings in the field. Examples are provided to illustrate the truth of the established inequalities, and graphical representations are included to support the understanding of these examples further.

Keywords: Fractional conformable integral, Hermite-Hadamard type inequalities, Interval-valued functions, LR-convexity
2020 MSC: 26B15, 26B25, 26D07, 26D10, 26D15

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1. Introduction

The Hermite–Hadamard inequality, as discovered by C. Hermite and J. Hadamard (as presented in [1]), stands as one of the most firmly established principles within the realm of convex function theory. This inequality not only possesses a geometric interpretation but also finds numerous practical applications. These inequalities articulate that when considering a convex function $f : I \to \mathbb{R}$ defined on a real number interval I, and selecting two distinct points a and b within I such that $a < b$, the following relationships hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

If f takes a concave form, the inequalities exhibit an inverse correlation. A multitude of mathematicians have played a role in solidifying the Hermite–Hadamard inequalities.

Interval analysis is a significant tool in mathematics and computational modeling, particularly for addressing interval uncertainty. Although its origins trace back to Archimedes' approximation of a circle's circumference, systematic research on this topic emerged only in the mid-20th century. The first comprehensive treatment of interval analysis was provided by Ramon E. Moore in 1966, who is regarded as the pioneer of interval calculus [2]. Since then, numerous studies have explored both theoretical aspects and practical applications of this field.

In this study, the notation $\mathbb{R}^+_{\mathscr{I}}$ denotes the set of all positive intervals in the real numbers. The collection of interval-valued functions that are Riemann integrable over $[a,b]$ is represented as $\mathscr{I}\mathscr{R}_{([a,b])}$, while $\mathscr{R}_{([a,b])}$ represents the set of real-valued Riemann integrable functions over the same interval. A relationship between (IR)-integrable functions and $\mathbb{R}$-integrable functions are established in the subsequent theorem.

For two intervals $\left[\underline{\mathscr{U}},\overline{\mathscr{U}}\right]$ and $\left[\underline{\mathscr{V}},\overline{\mathscr{V}}\right]$ in $\mathbb{R}^+_{\mathscr{I}}$, the inclusion relation $\subseteq$ is used. The interval $\left[\underline{\mathscr{U}},\overline{\mathscr{U}}\right]$ is considered a subset of $\left[\underline{\mathscr{V}},\overline{\mathscr{V}}\right]$ if and only if $\underline{\mathscr{V}} \leq \underline{\mathscr{U}}$ and $\overline{\mathscr{U}} \leq \overline{\mathscr{V}}$ hold.

Recent research has focused on integral inequalities involving interval-valued functions. Notably, Sadowska [3] generalized the Hermite-Hadamard inequality for set-valued functions, providing a framework that extends to interval-valued mappings.

Theorem 1.1. *[3] Let $F:[a,b]\to\mathbb{R}^+_{\mathscr{I}}$ be an interval-valued convex function, where $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. Then, the following inclusions hold:*

$$F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{b-a}\int_a^b F(x)\,dx\supseteq\frac{F(a)+F(b)}{2}.$$

Several well-known inequalities, including those of Ostrowski, Minkowski, and Beckenbach, along with their applications to interval-valued functions, were explored in [4–6]. Budak et al. [7] derived inequalities involving interval-valued Riemann-Liouville fractional integrals. Liu et al. [8] introduced the concept of interval-valued harmonically convex functions and established Hermite-Hadamard-type inequalities incorporating interval fractional integrals. Variants of Jensen's inequality for interval-valued functions via fuzzy integrals, along with associated integral inequalities, were presented in [9, 10]. Hermite-Hadamard-type inequalities for set-valued functions were proved by Mitroi et al. in [11], while general forms of interval-valued convex functions were used to establish such inequalities in [12]. Román Flores et al. [13] obtained Grönwall-type inequalities for interval-valued functions. Additionally, Zhao et al. [14, 15] proved various integral inequalities for interval-valued functions.

In recent studies [16–18], the concept of interval-valued convexity was extended by various researchers, leading to the definition of several types of LR-convexity for interval-valued functions. These studies also established numerous Hermite-Hadamard-type inequalities for LR-interval-valued convex functions.

Motivated by these advancements, we introduce fractional conformable integrals for interval-valued functions to derive Hermite-Hadamard-type inequalities for convex functions. A key benefit of the proposed inequalities is their flexibility they encompass Riemann-Liouville fractional Hermite-Hadamard inequalities and classical Hermite-Hadamard integral inequalities for LR-convex functions, eliminating the need to prove each case separately.

The organization of this paper is as follows:

2. Preliminaries

This section provides fundamental definitions, key results, concepts and properties that will be utilized throughout the paper.

2.1 Fractional integrals: Real-valued function perspective

Definition 2.1. *[19] Let $f\in L_1[a,b]$. The Riemann–Liouville fractional integrals of order $\alpha>0$, denoted by $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$, are defined as follows:*

$$J^\alpha_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\quad x>a,$$

and

$$J^\alpha_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\quad x<b,$$

respectively. Here, $\Gamma(\alpha)$ represents the Gamma function, defined as:

$$\Gamma(\alpha)=\int_0^\infty e^{-u}u^{\alpha-1}\,du.$$

It is also worth noting that for $\alpha=0$, the integrals reduce to the original function, i.e., $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

Various forms of the Hermite-Hadamard inequality derived using Riemann-Liouville fractional integrals are presented below.

Theorem 2.2. *[20] Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then, for $\alpha>0$, the following Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}.$$

Theorem 2.3. *[21] Under the assumptions of Theorem 2.2, the following inequality is established:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}.$$

Theorem 2.4. *[22] Assuming the same conditions as in Theorem 2.2, the following inequality is obtained:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\leq\frac{f(a)+f(b)}{2}.$$

In 2017, Jarad et al. [23] introduced the following fractional conformable integral operators. They also explored their properties and established connections between these operators and other fractional operators in the existing literature. The fractional conformable integral operators are defined as follows:

Definition 2.5. *[23] Let $f\in L^{1}[a,b]$. The fractional conformable integral operators of order $\beta\in\mathbb{C}$, with $\mathrm{Re}(\beta)>0$ and $\alpha\in(0,1]$, are defined as follows:*

$$\begin{aligned}{}_{a+}^{\beta}\mathbb{J}^{\alpha}f(x)&=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}\,dt,\\ {}_{b-}^{\beta}\mathbb{J}^{\alpha}f(x)&=\frac{1}{\Gamma(\beta)}\int_{x}^{b}\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}\,dt.\end{aligned}$$

Theorem 2.6. *[24] Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then, the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}_{a+}^{\beta}\mathbb{J}^{\alpha}f(b)+{}_{b-}^{\beta}\mathbb{J}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}.$$

Here, $\beta>0$, $\alpha\in(0,1]$, and Γ denotes the Euler Gamma function.

Theorem 2.7. *[25] Under the conditions of Theorem 2.6, the following inequality is obtained:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{\frac{a+b}{2}+}^{\beta}\mathbb{J}^{\alpha}f(b)+{}_{\frac{a+b}{2}-}^{\beta}\mathbb{J}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{2.1}$$

Theorem 2.8. *[26] Assuming the conditions of Theorem 2.6, the following inequality is established:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a+}^{\beta}\mathbb{J}^{\alpha}f\left(\frac{a+b}{2}\right)+{}_{b-}^{\beta}\mathbb{J}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\leq\frac{f(a)+f(b)}{2}.$$

2.2 Fractional integrals: Interval-valued function perspective

A positive interval is defined as an interval in which both endpoints are positive. The set of all positive intervals in $\mathbb{R}$ is denoted by $\mathbb{R}_{\mathscr{I}}^{+}$. The Hausdorff distance between two intervals $[\underline{X},\overline{X}]$ and $[\underline{Y},\overline{Y}]$ is given by:

$$d\left([\underline{X},\overline{X}],[\underline{Y},\overline{Y}]\right)=\max\left\{|\underline{X}-\underline{Y}|,|\overline{X}-\overline{Y}|\right\}.$$

The pair $(\mathbb{R}_{\mathscr{I}},d)$ forms a complete metric space. For further details and foundational concepts on interval-valued functions, see [27,28].

We summarize the fundamental properties of interval analysis operations for the intervals $\mathscr{U}$ and $\mathscr{V}$ as follows:

- Addition:

$$\mathscr{U}+\mathscr{V}=\left[\underline{\mathscr{U}}+\underline{\mathscr{V}},\overline{\mathscr{U}}+\overline{\mathscr{V}}\right].$$

- Subtraction:

$$\mathscr{U} - \mathscr{V} = \left[\underline{\mathscr{U}} - \overline{\mathscr{V}}, \overline{\mathscr{U}} - \underline{\mathscr{V}}\right].$$

- Multiplication:

$$\mathscr{U} \cdot \mathscr{V} = [\min \Lambda, \max \Lambda],$$

where

$$\Lambda = \left\{\underline{\mathscr{U}}\,\underline{\mathscr{V}}, \underline{\mathscr{U}}\,\overline{\mathscr{V}}, \overline{\mathscr{U}}\,\underline{\mathscr{V}}, \overline{\mathscr{U}}\,\overline{\mathscr{V}}\right\}.$$

- Division:

$$\mathscr{U} / \mathscr{V} = [\min \Delta, \max \Delta],$$

where

$$\Delta = \left\{\frac{\underline{\mathscr{U}}}{\underline{\mathscr{V}}}, \frac{\underline{\mathscr{U}}}{\overline{\mathscr{V}}}, \frac{\overline{\mathscr{U}}}{\underline{\mathscr{V}}}, \frac{\overline{\mathscr{U}}}{\overline{\mathscr{V}}}\right\}, \quad 0 \notin \mathscr{V}.$$

- Scalar Multiplication:

$$\theta \mathscr{U} = \theta\left[\underline{\mathscr{U}}, \overline{\mathscr{U}}\right] = \begin{cases} \left[\theta \underline{\mathscr{U}}, \theta \overline{\mathscr{U}}\right], & \theta > 0, \\ \{0\}, & \theta = 0, \\ \left[\theta \overline{\mathscr{U}}, \theta \underline{\mathscr{U}}\right], & \theta < 0, \end{cases}$$

where $\theta \in \mathbb{R}$.

For intervals $\left[\underline{\mathscr{U}}, \overline{\mathscr{U}}\right], \left[\underline{\mathscr{V}}, \overline{\mathscr{V}}\right] \in \mathbb{R}^+_{\mathscr{I}}$, the inclusion relation "$\subseteq$" is defined as follows:

$$\left[\underline{\mathscr{U}}, \overline{\mathscr{U}}\right] \subseteq \left[\underline{\mathscr{V}}, \overline{\mathscr{V}}\right] \quad \text{if and only if} \quad \underline{\mathscr{V}} \leq \underline{\mathscr{U}} \quad \text{and} \quad \overline{\mathscr{U}} \leq \overline{\mathscr{V}}.$$

1. The relation "$\leq_p$" is defined on $\mathbb{R}^+_{\mathscr{I}}$ by:

$$\left[\underline{\mathscr{U}}, \overline{\mathscr{U}}\right] \leq_p \left[\underline{\mathscr{V}}, \overline{\mathscr{V}}\right] \quad \text{if and only if} \quad \underline{\mathscr{U}} \leq \underline{\mathscr{V}} \quad \text{and} \quad \overline{\mathscr{U}} \leq \overline{\mathscr{V}}.$$

This condition holds for all $\left[\underline{\mathscr{U}}, \overline{\mathscr{U}}\right], \left[\underline{\mathscr{V}}, \overline{\mathscr{V}}\right] \in \mathbb{R}^+_{\mathscr{I}}$. The relation "$\leq_p$" is referred to as a pseudo-order relation.

2. The relation "$\leq_p$" can be interpreted as a "left and right" ordering on the real line $\mathbb{R}$. For this reason, it is commonly referred to as the "left and right" order, abbreviated as "LR" order.

It is noteworthy that Moore [2] introduced the concept of the Riemann integral for interval-valued functions. The set of all Riemann integrable interval-valued functions on $[a,b]$ is denoted by $\mathscr{IR}_{([a,b])}$, while the set of Riemann integrable real-valued functions on $[a,b]$ is denoted by $\mathscr{R}_{([a,b])}$. The relationship between (IR)-integrable functions and Riemann integrable (R-integrable) functions is established in the following theorem (see [27], p. 131):

Theorem 2.9. *Let $F : [a,b] \to \mathbb{R}_{\mathscr{I}}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. Then, $F \in \mathscr{IR}_{([a,b])}$ if and only if $\underline{F}(t), \overline{F}(t) \in \mathscr{R}_{([a,b])}$, and the integral of F is given by:*

$$\int_a^b F(t)\,dt = \left[\int_a^b \underline{F}(t)\,dt, \int_a^b \overline{F}(t)\,dt\right].$$

Furthermore, in [29], Lupulescu introduced the following definition of the interval-valued left-sided Riemann-Liouville fractional integral.

Definition 2.10. *Let $F : [a,b] \to \mathbb{R}_{\mathscr{I}}$ be an interval-valued function represented as $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, and let $\alpha > 0$. The interval-valued left-sided Riemann–Liouville fractional integral of F is defined as:*

$$\mathscr{J}^{\alpha}_{a+} F(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} F(s)\,ds, \quad x > a,$$

where $\Gamma(\alpha)$ denotes the Euler Gamma function.

Using Lupulescu's framework, Budak et al. in [7] introduced the interval-valued right-sided Riemann–Liouville fractional integral for a function F, defined as:

$$\mathscr{J}_{b-}^{\alpha}F(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}F(s)\,ds, \quad x<b,$$

where $\Gamma(\alpha)$ again represents the Euler Gamma function.

Definition 2.11. *[30] An interval-valued function $F: I\to\mathbb{R}_{\mathscr{J}}^{+}$ is called an LR-convex interval-valued function on the convex set I if, for all $a,b\in I$ and $t\in[0,1]$, the following inequality holds:*

$$F(ta+(1-t)b)\leq_p tF(a)+(1-t)F(b), \tag{2.2}$$

where $\leq_p$ represents the LR order. If the inequality in (2.2) is reversed, F is said to be LR-concave on I.

Theorem 2.12. *[31] Let $F:[a,b]\to\mathbb{R}_{\mathscr{J}}^{+}$ be an LR-convex interval-valued function on $[a,b]$, given by $F(x)=[\underline{F}(x),\overline{F}(x)]$ for all $x\in[a,b]$. If $F\in L([a,b],\mathbb{R}_{\mathscr{J}}^{+})$, then the following inequality holds:*

$$F\left(\frac{a+b}{2}\right)\leq_p \frac{1}{b-a}\int_a^b F(x)\,dx\leq_p \frac{F(a)+F(b)}{2}.$$

Theorem 2.13. *[32] Let $F:[a,b]\to\mathbb{R}_{\mathscr{J}}^{+}$ be an LR-convex interval-valued function on $[a,b]$ and let $F(x)=[\underline{F}(x),\overline{F}(x)]$ for all $x\in[a,b]$ and $F\in L([a,b],\mathbb{R}_{\mathscr{J}}^{+})$. Then, the following inequality holds:*

$$F\left(\frac{a+b}{2}\right)\leq_p \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}_{\frac{a+b}{2}-}^{\alpha}F(a)+\mathscr{J}_{\frac{a+b}{2}+}^{\alpha}F(b)\right]\leq_p \frac{F(a)+F(b)}{2}. \tag{2.3}$$

Theorem 2.14. *[30] Let I be a convex set, and let $F: I\to\mathbb{R}_{\mathscr{J}}^{+}$ be an interval-valued function defined as:*

$$F(t)=[\underline{F}(t),\overline{F}(t)], \quad \forall t\in I.$$

Then, F is an LR-convex interval-valued function on I if and only if both $\underline{F}(t)$ and $\overline{F}(t)$ are convex functions.

Definition 2.15. *[33] Let $F:[a,b]\to\mathbb{R}_{\mathscr{J}}^{+}$ be an interval-valued function defined as $F(x)=[\underline{F}(x),\overline{F}(x)]$. If $F\in L([a,b],\mathbb{R}_{\mathscr{J}}^{+})$, the conformable fractional integral operators for interval-valued functions of order $\beta>0$ and $\alpha\in(0,1]$ are defined as follows:*

For the left-sided conformable fractional integral:

$${}_{a+}^{\beta}\Upsilon^{\alpha}F(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{F(t)}{(t-a)^{1-\alpha}}\,dt.$$

For the right-sided conformable fractional integral:

$${}_{b-}^{\beta}\Upsilon^{\alpha}F(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{F(t)}{(b-t)^{1-\alpha}}\,dt.$$

Theorem 2.16. *[34] Let $F:[a,b]\to\mathbb{R}_{\mathscr{J}}^{+}$ be an LR-convex interval-valued function on $[a,b]$, defined as $F(x)=[\underline{F}(x),\overline{F}(x)]$ for all $x\in[a,b]$. Assume $F\in L([a,b],\mathbb{R}_{\mathscr{J}}^{+})$. Then, the following inequality holds:*

$$F\left(\frac{a+b}{2}\right)\leq_p \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{\frac{a+b}{2}+}^{\beta}\Upsilon^{\alpha}F(b)+{}^{\beta}\Upsilon_{\frac{a+b}{2}-}^{\alpha}F(a)\right]\leq_p \frac{F(a)+F(b)}{2}. \tag{2.4}$$

3. Fractional Hermite-Hadamard Inequalities for the Product of Two Interval-Valued *LR*-Convex Functions

In this section, Hermite-Hadamard inequalities involving fractional conformable integral operators for the product of two interval-valued LR-convex functions are established. By making specific choices for the parameters, both generalizations of existing results in the literature and new findings are obtained.

Theorem 3.1. *Let $F,G:[a,b]\to\mathbb{R}^+_{\mathscr{I}}$ be two LR-convex interval-valued functions on $[a,b]$, defined as $F(x)=[\underline{F}(x),\overline{F}(x)]$ and $G(x)=[\underline{G}(x),\overline{G}(x)]$ for all $x\in[a,b]$. Then, the following inequality for conformable fractional integral operators holds:*

$$\begin{aligned}&\frac{\alpha^\beta 2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left({}^{\beta}\Upsilon^{\alpha}_{\frac{a+b}{2}-}F(a)G(a)+{}^{\beta}_{\frac{a+b}{2}+}\Upsilon^{\alpha}F(b)G(b)\right)\\&\leq_p\frac{\beta}{4}\left[M(a,b)\left(\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{1}{\beta}\right)+N(a,b)\left(\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right)\right],\end{aligned}\tag{3.1}$$

where $\beta>0$ and $\alpha\in(0,1]$, and the terms $M(a,b)$ and $N(a,b)$ are defined as:

$$\begin{aligned}&M(a,b)=F(a)G(a)+F(b)G(b),\\&N(a,b)=F(a)G(b)+F(b)G(a).\end{aligned}$$

Proof. Since the functions F and G are LR-convex interval-valued functions, the following inequalities hold:

$$F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq_p\frac{2-t}{2}F(a)+\frac{t}{2}F(b)$$

and

$$G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq_p\frac{2-t}{2}G(a)+\frac{t}{2}G(b).$$

It follows that

$$\begin{aligned}&F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\\&\leq_p\frac{1}{4}\left[(2-t)^2F(a)G(a)+t^2F(b)G(b)+(2-t)t(F(a)G(b)+F(b)G(a))\right].\end{aligned}\tag{3.2}$$

Similarly, we obtain

$$F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\leq_p\frac{t}{2}F(a)+\frac{2-t}{2}F(b)$$

and

$$G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\leq_p\frac{t}{2}G(a)+\frac{2-t}{2}G(b).$$

Thus, we have

$$\begin{aligned}F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\leq_p\frac{1}{4}&\left[t^2F(a)G(a)+(2-t)^2F(b)G(b)\right.\\&\left.+(2-t)t(F(a)G(b)+F(b)G(a))\right].\end{aligned}\tag{3.3}$$

Combining inequalities (3.2) and (3.3), we obtain

$$\begin{aligned}&F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\\&\leq_p\frac{1}{4}\left[(t^2+(2-t)^2)M(a,b)+2t(2-t)N(a,b)\right].\end{aligned}\tag{3.4}$$

Multiplying both sides of (3.4) by $\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}$ and integrating over $[0,1]$, we get

$$\begin{aligned}&\int_0^1F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt\\&+\int_0^1F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt\end{aligned}$$

$$\leq_p \frac{1}{4}M(a,b)\int_0^1 \left(t^2+(2-t)^2\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt$$
$$+\frac{1}{4}N(a,b)\int_0^1 2t(2-t)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt. \tag{3.5}$$

Using changing variable and Definition 2.15, then we obtain

$$\int_0^1 F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt$$
$$=\left(\frac{2}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\,{}^{\beta}_{\frac{a+b}{2}-}\Upsilon^\alpha F(a)G(a) \tag{3.6}$$

and

$$\int_0^1 F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt$$
$$=\left(\frac{2}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\,{}^{\beta}_{\frac{a+b}{2}+}\Upsilon^\alpha F(b)G(b).$$

On the other hand, we have

$$\int_0^1 \left(t^2+(2-t)^2\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt=\frac{2}{\alpha^\beta}\left[\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{1}{\beta}\right] \tag{3.7}$$

and

$$\int_0^1 2t(2-t)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt=\frac{2}{\alpha^\beta}\left[\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right]. \tag{3.8}$$

Substituting (3.6)-(3.8) into (3.5), the result follows as

$$\left(\frac{2}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left({}^{\beta}_{\frac{a+b}{2}-}\Upsilon^\alpha F(a)G(a)+{}^{\beta}_{\frac{a+b}{2}+}\Upsilon^\alpha F(b)G(b)\right)\leq_p \frac{1}{2\alpha^\beta}\left[M(a,b)\left(\left(\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{1}{\beta}\right)\right)\right.$$
$$\left.+N(a,b)\left(\left(\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right)\right)\right]$$

Thus the proof is completed. □

Remark 3.2. *By assigning $G(x)=[1,1]$ for all $x\in[a,b]$ in Theorem 3.1, the inequality (3.1) simplifies to the right-hand side of the inequality (2.4).*

Corollary 3.3. *If we choose $\alpha=1$ in Theorem 3.1, the inequality for Riemann-Liouville fractional integral operators of interval-valued functions becomes*

$$\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left(\mathscr{J}^{\beta}_{\frac{a+b}{2}-}F(a)G(a)+\mathscr{J}^{\beta}_{\frac{a+b}{2}+}F(b)G(b)\right)$$
$$\leq_p M(a,b)\left(\frac{2+(\beta+1)(\beta+2)}{4(\beta+1)(\beta+2)}\right)+N(a,b)\left(\frac{(\beta+1)(\beta+2)-2}{4(\beta+1)(\beta+2)}\right), \tag{3.9}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 3.1.

Remark 3.4. *If we take $G(x)=[1,1]$ for all $x\in[a,b]$ in Corollary 3.3, the inequality (3.9) simplifies to the right-hand side of inequality (2.3).*

Remark 3.5. *If we choose $\alpha=1$ and $\beta=1$ in Theorem 3.1, the inequality (3.1) reduces to*

$$\frac{1}{b-a}\int_a^b F(x)G(x)\,dx\leq_p \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

which was established by Khan et al. in [31, Theorem 5].

Corollary 3.6. *If we take $\underline{F}(x)=\overline{F}(x)=f(x)$ and $\underline{G}(x)=\overline{G}(x)=g(x)$ for all $x\in[a,b]$ in Theorem 3.1, the Hermite-Hadamard inequality for conformable fractional integrals of real-valued functions is obtained as*

$$\begin{aligned}&\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\frac{a+b}{2}+}\mathbb{J}^{\alpha}f(b)g(b)+{}^{\beta}_{\frac{a+b}{2}-}\mathbb{J}^{\alpha}f(a)g(a)\right]\\&\le\frac{\beta}{4}\left[C(a,b)\left(\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{1}{\beta}\right)+D(a,b)\left(\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right)\right],\end{aligned}\tag{3.10}$$

where

$$C(a,b)=f(a)g(a)+f(b)g(b),$$

$$D(a,b)=f(a)g(b)+f(b)g(a).$$

Remark 3.7. *If we assign $g(x)=1$ for all $x\in[a,b]$ in Corollary 3.6, the inequality (3.10) reduces to the right-hand side of inequality (2.1).*

Corollary 3.8. *If we choose $\alpha=1$ is in Corollary 3.6, the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals of real-valued functions is obtained as*

$$\begin{aligned}&\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}+}f(b)g(b)+J^{\beta}_{\frac{a+b}{2}-}f(a)g(a)\right]\\&\le C(a,b)\left(\frac{2+(\beta+1)(\beta+2)}{4(\beta+1)(\beta+2)}\right)+D(a,b)\left(\frac{(\beta+1)(\beta+2)-2}{4(\beta+1)(\beta+2)}\right).\end{aligned}\tag{3.11}$$

Remark 3.9. *If we choose $g(x)=1$ for all $x\in[a,b]$ in Corollary 3.8, the inequality (3.11) reduce to the right-hand side of inequality (2.1).*

Remark 3.10. *If we take $\alpha=1$ and $\beta=1$ in Corollary 3.6, the inequality (3.10) reduces to*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\le\frac{1}{3}C(a,b)+\frac{1}{6}D(a,b),$$

which was established by Pachpatte in [35].

Example 3.11. *Consider the interval-valued functions $F,G:[1,2]\to\mathbb{R}^+_{\mathscr{I}}$ defined as $F(x)=\left[\frac{x^2}{2},5\right]$ and $G(x)=[x,x^2]$. It can be verified that F and G are LR-convex. Using Definition 2.15, we compute*

$$\begin{aligned}{}^{\beta}_{\frac{a+b}{2}-}\Upsilon^{\alpha}F(a)G(a)&={}^{\beta}_{\frac{3}{2}-}\Upsilon^{\alpha}F(1)G(1)\\&=\frac{1}{\alpha^{\beta-1}\Gamma(\beta)}\int_1^{\frac{3}{2}}\left(\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{3}{2}-t\right)^{\alpha}\right)^{\beta-1}\left(\frac{3}{2}-t\right)^{\alpha-1}\left[\frac{t^2}{2},5\right][t,t^2]\,dt\\&=\frac{1}{\alpha^{\beta-1}\Gamma(\beta)}\int_1^{\frac{3}{2}}\left(\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{3}{2}-t\right)^{\alpha}\right)^{\beta-1}\left(\frac{3}{2}-t\right)^{\alpha-1}\left[\frac{t^3}{2},5t^2\right]dt\\&=\frac{1}{\alpha^{\beta}\Gamma(\beta)2^{\alpha\beta+4}}\left[-\mathscr{B}\left(\beta,\frac{3}{\alpha}+1\right)+9\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)-27\mathscr{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{27}{\beta},\right.\\&\qquad\left.20\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)-120\mathscr{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{180}{\beta}\right].\end{aligned}\tag{3.12}$$

Similarly,

$$\begin{aligned}{}^{\beta}\Upsilon^{\alpha}_{\frac{a+b}{2}+}F(b)G(b)&={}^{\beta}\Upsilon^{\alpha}_{\frac{3}{2}+}F(2)G(2)\\&=\frac{1}{\alpha^{\beta}\Gamma(\beta)2^{\alpha\beta+4}}\left[\mathscr{B}\left(\beta,\frac{3}{\alpha}+1\right)+9\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+27\mathscr{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{27}{\beta},\right.\\&\qquad\left.20\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+120\mathscr{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{180}{\beta}\right].\end{aligned}\tag{3.13}$$

Combining (3.12) and (3.13), the left term of inequality (3.1) becomes

$$\frac{\alpha^{\beta}2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\Upsilon^{\alpha}_{\frac{a+b}{2}+}F(b)G(b)+{}^{\beta}_{\frac{a+b}{2}-}\Upsilon^{\alpha}F(a)G(a)\right]=[\Omega_3(\alpha,\beta),\Omega_4(\alpha,\beta)],$$

where

$$\Omega_3(\alpha,\beta)=\frac{9}{16}\beta\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{27}{16},$$
$$\Omega_4(\alpha,\beta)=\frac{5}{4}\beta\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{45}{4}.$$

On the other hand, the values of $M(a,b)$ and $N(a,b)$ are

$$M(a,b)=\left[\frac{9}{2},25\right],\tag{3.14}$$
$$N(a,b)=[3,25].$$

Substituting these values into the right term of (3.1) gives

$$\frac{\beta}{4}\left[M(a,b)\left(\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{1}{\beta}\right)+N(a,b)\left(\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right)\right]=\left[\frac{3\beta}{8}\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{15}{8},\frac{25}{2}\right].$$

Thus, the inequality (3.1) becomes

$$[\Omega_3(\alpha,\beta),\Omega_4(\alpha,\beta)]\leq_p\left[\frac{3\beta}{8}\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{15}{8},\frac{25}{2}\right].\tag{3.15}$$

The validity of inequality (3.15) is illustrated in Figure 3.1 and Figure 3.2 for $\alpha\in[0,1]$ and $\beta\in[0,5]$.

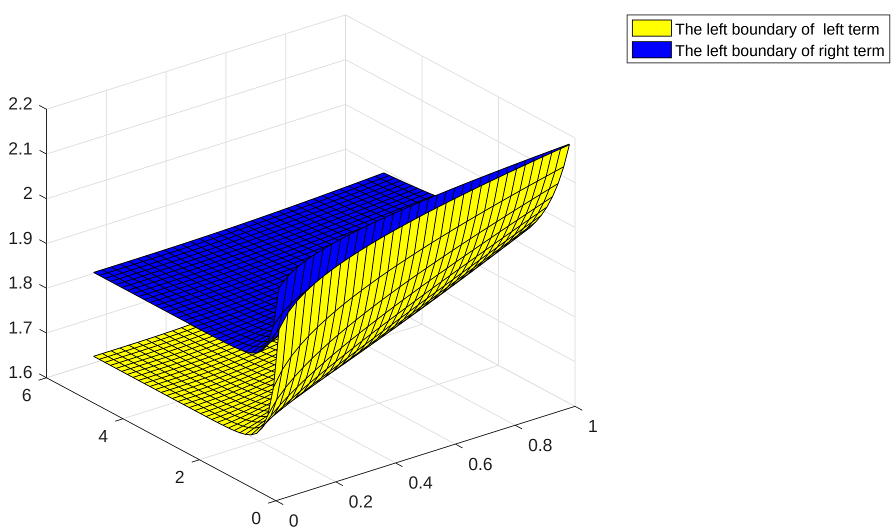

Figure 3.1. Comparison of left boundaries for Example 3.11

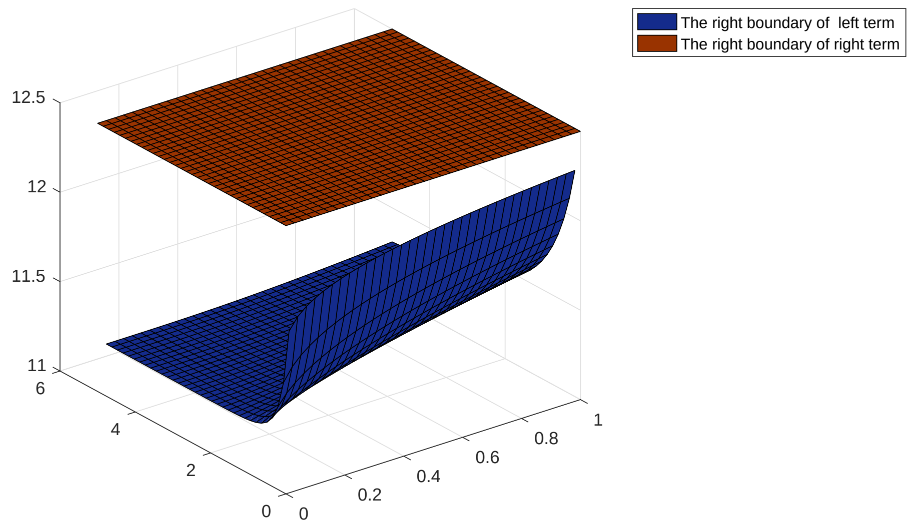

Figure 3.2. Comparison of right boundaries for Example 3.11

Theorem 3.12. *Let $F,G:[a,b]\to\mathbb{R}^+_{\mathscr{I}}$ be two LR-convex interval-valued functions on $[a,b]$, defined as $F(x)=[\underline{F}(x),\overline{F}(x)]$ and $G(x)=[\underline{G}(x),\overline{G}(x)]$ for all $x\in[a,b]$. The following inequality for conformable fractional integral operators of interval-valued functions holds:*

$$\begin{aligned}\frac{1}{\beta\alpha^\beta}F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\leq_p{}&\frac{2^{\alpha\beta}\Gamma(\beta)}{4(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\frac{a+b}{2}-}\Upsilon^{\alpha}F(a)G(a)+{}^{\beta}\Upsilon^{\alpha}_{\frac{a+b}{2}+}F(b)G(b)\right]\\&+\left[\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right]\frac{M(a,b)}{8\alpha^\beta}\\&+\left[\frac{1}{\beta}+\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right]\frac{N(a,b)}{8\alpha^\beta},\end{aligned}\tag{3.16}$$

where $\beta>0$, $\alpha\in(0,1]$, and $M(a,b)$ and $N(a,b)$ are defined as in Theorem 3.1.

Proof. Since F and G are LR-convex interval-valued functions, we have

$$\begin{aligned}F\left(\frac{a+b}{2}\right)&=F\left(\frac{1}{2}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+\frac{1}{2}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right)\\&\leq_p\frac{1}{2}\left[F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right],\end{aligned}$$

and similarly,

$$G\left(\frac{a+b}{2}\right)\leq_p\frac{1}{2}\left[G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right].$$

Using these, it follows that

$$\begin{aligned}F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\leq_p\frac{1}{4}\Big[&F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\\&+F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\\&+F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\\&+F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\Big].\end{aligned}$$

From LR-convexity of F and G, we get

$$F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) \leq_p \frac{1}{4}\Big[F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right) + F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right) + \frac{t(2-t)}{2}M(a,b)+\frac{t^2+(2-t)^2}{4}N(a,b)\Big]. \tag{3.17}$$

Multiplying both sides of (3.17) by $\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}$ and integrating over $[0,1]$, we obtain

$$\begin{aligned} &F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt \\ &\leq_p \frac{1}{4}\Bigg[\int_0^1 F\left(\frac{2-t}{2}a+\frac{t}{2}b\right)G\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt \\ &\quad+\int_0^1 F\left(\frac{t}{2}a+\frac{2-t}{2}b\right)G\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt \\ &\quad+M(a,b)\int_0^1\frac{t(2-t)}{2}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt \\ &\quad+N(a,b)\int_0^1\frac{t^2+(2-t)^2}{4}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt\Bigg]. \end{aligned} \tag{3.18}$$

Using the identity

$$\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}dt=\frac{1}{\beta\alpha^\beta},$$

and combining (3.7), (3.8), (3.18), $M(a,b)$ and $N(a,b)$, then we have

$$\begin{aligned}\frac{1}{\beta\alpha^\beta}F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) \leq_p\ & \frac{2^{\alpha\beta}\Gamma(\beta)}{4(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\frac{a+b}{2}-}\Upsilon^{\alpha}F(a)G(a)+{}^{\beta}\Upsilon^{\alpha}_{\frac{a+b}{2}+}F(b)G(b)\right] \\ &+\frac{M(a,b)}{8\alpha^\beta}\left(\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right) \\ &+\frac{N(a,b)}{8\alpha^\beta}\left(\frac{1}{\beta}+\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right).\end{aligned}$$

This completes the proof. □

Corollary 3.13. *If we choose $G(x)=[1,1]$ for all $x\in[a,b]$ in Theorem 3.12, the following inequality holds:*

$$2F\left(\frac{a+b}{2}\right)\leq_p\frac{\alpha^\beta 2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\frac{a+b}{2}-}\Upsilon^{\alpha}F(a)+{}^{\beta}\Upsilon^{\alpha}_{\frac{a+b}{2}+}F(b)\right]+\frac{F(a)+F(b)}{2}.$$

Corollary 3.14. *If we take $\alpha=1$ in Theorem 3.12, the inequality for Riemann-Liouville fractional integral operators of interval-valued functions becomes:*

$$\begin{aligned}2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\leq_p\ &\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathscr{J}^{\beta}_{\frac{a+b}{2}-}F(a)G(a)+\mathscr{J}^{\beta}_{\frac{a+b}{2}+}F(b)G(b)\right] \\ &+\left[\frac{(\beta+2)(\beta+1)-2}{4(\beta+2)(\beta+1)}\right]M(a,b) \\ &+\left[\frac{(\beta+2)(\beta+1)+2}{4(\beta+2)(\beta+1)}\right]N(a,b).\end{aligned}$$

Corollary 3.15. *If we assign $G(x) = [1,1]$ for all $x \in [a,b]$ in Corollary 3.14, the following inequality is obtained:*

$$2F\left(\frac{a+b}{2}\right) \leq_p \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathscr{J}^{\beta}_{\frac{a+b}{2}-}F(a)+\mathscr{J}^{\beta}_{\frac{a+b}{2}+}F(b)\right]+\frac{F(a)+F(b)}{2}.$$

Remark 3.16. *If we choose $\alpha = 1$ and $\beta = 1$ in Theorem 3.12, the inequality (3.16) reduces to:*

$$2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) \leq_p \frac{1}{b-a}\int_a^b F(x)G(x)\,dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),$$

which was established by Khan et al. in [31, Theorem 6].

Corollary 3.17. *If $\underline{F}(x) = \overline{F}(x) = f(x)$ and $\underline{G}(x) = \overline{G}(x) = g(x)$ are chosen for all $x \in [a,b]$ in Theorem 3.12, the Hermite-Hadamard inequality for conformable fractional integrals of real-valued functions is given by:*

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq_p{}& \frac{\alpha^\beta 2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathbb{J}^{\alpha}_{\frac{a+b}{2}-}f(a)g(a)+{}^{\beta}_{\frac{a+b}{2}+}\mathbb{J}^{\alpha}f(b)g(b)\right]\\&+\beta\left(\frac{1}{\beta}-\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right)\frac{C(a,b)}{4}\\&+\beta\left(\frac{1}{\beta}+\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)\right)\frac{D(a,b)}{4},\end{aligned}$$

where $C(a,b)$ and $D(a,b)$ are as defined in Corollary 3.6.

Corollary 3.18. *If we take $g(x) = 1$ for all $x \in [a,b]$ in Corollary 3.17, the following inequality is obtained:*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{\alpha^\beta 2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\frac{a+b}{2}-}\mathbb{J}^{\alpha}f(a)+{}^{\beta}_{\frac{a+b}{2}+}\mathbb{J}^{\alpha}f(b)\right]+\frac{f(a)+f(b)}{2}.$$

Corollary 3.19. *If we assign $\alpha = 1$ in Corollary 3.17, the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals of real-valued functions become:*

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq_p{}& \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}-}f(a)g(a)+J^{\beta}_{\frac{a+b}{2}+}f(b)g(b)\right]\\&+\left[\frac{(\beta+2)(\beta+1)-2}{4(\beta+2)(\beta+1)}\right]C(a,b)\\&+\left[\frac{(\beta+2)(\beta+1)+2}{4(\beta+2)(\beta+1)}\right]D(a,b).\end{aligned}$$

Corollary 3.20. *If we choose $g(x) = 1$ for all $x \in [a,b]$ in Corollary 3.19, the inequality simplifies to:*

$$2f\left(\frac{a+b}{2}\right) \leq_p \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J^{\beta}_{\frac{a+b}{2}-}f(a)+J^{\beta}_{\frac{a+b}{2}+}f(b)\right]+\frac{f(a)+f(b)}{2}. \tag{3.19}$$

Remark 3.21. *When $\alpha = 1$ and $\beta = 1$ are substituted into Corollary 3.17, the inequality (3.19) simplifies to:*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{6}C(a,b)+\frac{1}{3}D(a,b),$$

which was previously established by Pachpatte in [35].

Example 3.22. *Consider the functions F and G as defined in Example 3.11. For these functions, we calculate:*

$$2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) = 2F\left(\frac{3}{2}\right)G\left(\frac{3}{2}\right) = 2\left[\frac{9}{8},5\right]\left[\frac{3}{2},\frac{9}{4}\right] = \left[\frac{27}{8},\frac{45}{2}\right]. \tag{3.20}$$

Using the results from (3.12), (3.13), (3.14), and (3.20) in inequality (3.16), we obtain:

$$\left[\frac{27}{8},\frac{45}{2}\right] \leq_p \left[\Omega_5(\alpha,\beta),\Omega_6(\alpha,\beta)\right], \tag{3.21}$$

where

$$\Omega_5(\alpha,\beta) = \frac{3}{16}\beta\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{57}{16},$$
$$\Omega_6(\alpha,\beta) = \frac{5}{4}\beta\mathscr{B}\left(\beta,\frac{2}{\alpha}+1\right)+\frac{95}{4}.$$

The validity of the inequality (3.21) can be visualized in Figure 3.3 and Figure 3.4 for $\alpha \in [0,1]$ *and* $\beta \in [0,5]$.

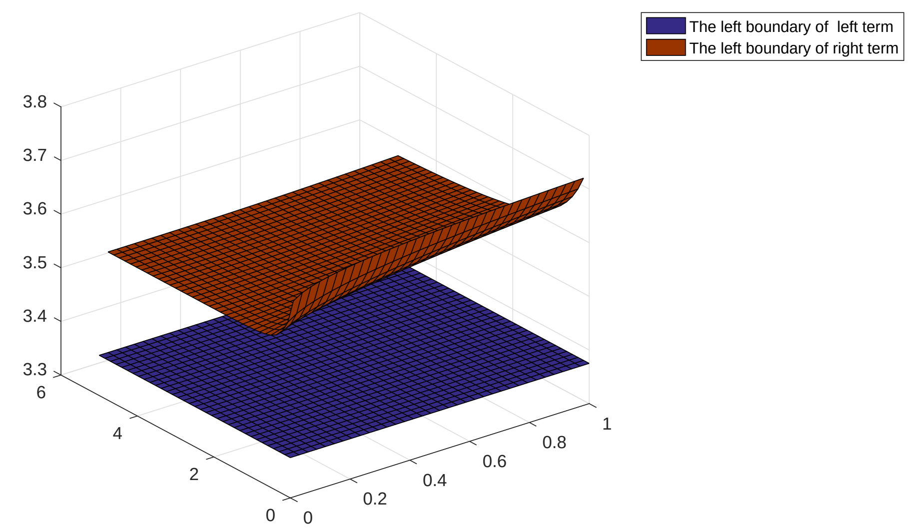

Figure 3.3. Comparison of left boundaries for Example 3.22

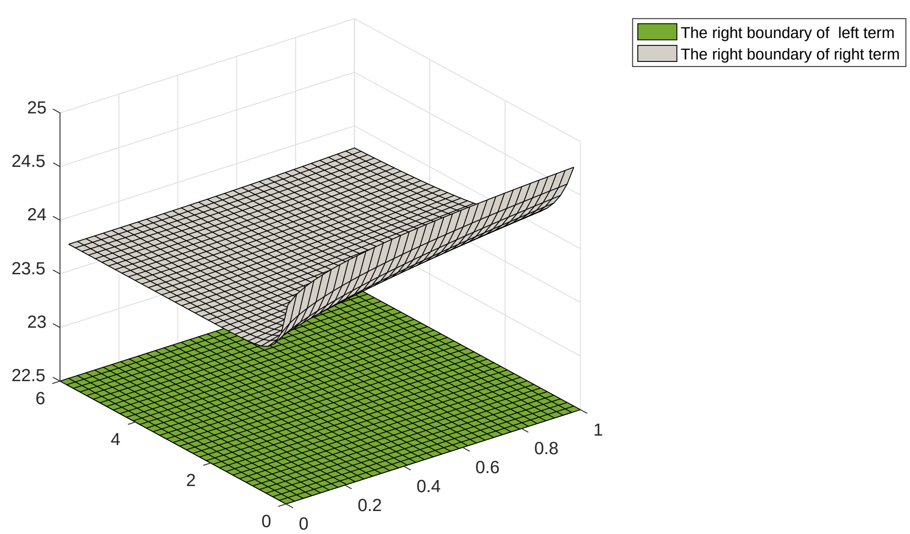

Figure 3.4. Comparison of right boundaries for Example 3.22

4. Conclusion

In this paper, several new Hermite-Hadamard type inequalities for the product of two interval-valued *LR*-convex functions were established using fractional conformable integral operators. These results provide generalized forms of classical inequalities and extend their applicability to interval-valued functions. Future research may explore analogous inequalities for *CR*-convex functions or other generalizations of convexity. Additionally, the techniques employed here could be adapted to investigate similar inequalities for different types of fractional integral operators. The approach can also be applied to derive new results for alternative inequality types, offering a broad range of possibilities for further mathematical exploration.

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