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Analysis of Diabetes Complications Model With Time Delays

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ABSTRACT

In this article, diabetes, a disease known to be influenced by genetic and environmental factors, is divided into two types: diabetes without complications (D) and diabetes with complications (C). The stability of the equilibrium points of this two-dimensional diabetes complication (DC) model, the D and C equations are studied by time delay. In addition, the behavior of the model is obtained with the help of the MATLAB package program.

Keywords: Diabetes complication (DC) model, Delay, Equilibrium point, Stability

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Introduction

Various phenomena in fields such as medicine, biology and physics are studied using mathematical models that form differential equation systems. In this context, epidemiological models include studies on aspects of various communicable and non-communicable diseases. Mathematical modeling is used to study the epidemiology of a disease. Epidemiology is diseases that have spread to a living organism. With the development of science, mathematical modeling is used to study not only the spread of communicable diseases, but also non-communicable diseases. Diabetes is a chronic disease that is manifested by high blood sugar levels. It can not work optimally in the pancreatic organ, which must produce the hormone insulin. Without insulin, the body's cells cannot absorb glucose and convert it into energy, so that the blood sugar level rises (type 1). It arises from the inability of the body to produce a sufficient amount of insulin (type 2). The methods of treatment used in diabetes lead to many complications [1]. In 2004 Boutayeb et al. they introduce the diabetes complication (DC) model to find out how much change is between uncomplicated (D) and complicated (C) diabetics. In this model, the number of new patients with diabetes is not constant and the number of events is determined by taking into account genetic and environmental factors. The rate of occurrence of complications, α, decreases with the number of individuals in compartment D and increases with the number of individuals in compartment C. In this way, those who suffer from diabetes can recover from their illness (γ), remain disabled (ν) and die (μ). Since diabetic patients who recovered from complications were assumed to still have diabetes, diabetic patients who died or were disabled were excluded from the population. In this model, natural deaths in both compartments are taken into account. The number of individuals in d and C decreases by the same amount due to natural death. This model is classified as uncomplicated diabetes, where the incidence number A is assumed to be constant, and the DC model is obtained[2-3]. Some systems of differential equations cannot accurately model real-life events, so one of the methods often used to improve accuracy is time delays. A more complex analysis is made in the system by introducing a time delay to the epidemic model and the analysis of the system in its past states is taken into consideration [4-5].

Delay models are becoming increasingly common in biological modeling. By introducing time delay into the equation systems, a more complex examination of the system is carried out and the analysis of the system in its past states is taken into account. A delayed differential system model is a system of differential equations that depend on the current state of the system and a certain constant time τ. The advantage of modeling delayed differential equations is that it is a powerful numerical method for determining the stability of the system or for the characteristic equations for the eigenvalues [6].Recently, there are new studies on time delays [7-13].

In this study, delayed differential equation systems are obtained using the diabetes complication (DC) model. In the DC equation system, time delays are used to model the dynamic behavior of diabetes patients in different ways. The equilibrium points of the time-delayed equation system and the stability of the equilibria are analyzed. In chapter 2, the DC model is introduced and in chapter 3, the results of the delayed model are given.

Diabetes Complications (DC) Model

The population in this model is a differential equation that is divided into two parts according to their health status. According to Boutayeb et al the DC model is given as:

$$\begin{aligned}\frac{dD}{dt} &= I-(\alpha+\mu)D+\gamma C\\ \frac{dC}{dt} &= \alpha D-(\gamma+\delta+\nu+\mu)C\end{aligned} \quad (1)$$

The variables of the system show diabetics with and without complications at any time of t in D and C. Figure 1 shows the kinematic diagram of the DC model it is shown in .

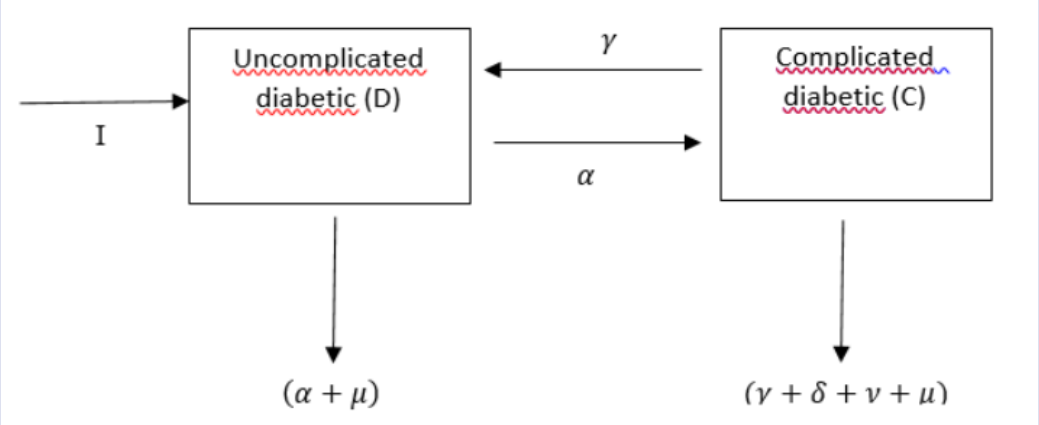

Figure 1. Kinematic diagram of the diabetes complication (DC) model

In the kinematic diagram, diabetics without complications are shown with D and diabetics with complications with C. The rate of occurrence of complications is indicated to him by α, and it decreases as the number of individuals in compartment D, I, decreases. The number of individuals in compartment C increases at the same rate as in compartment αD. He or she may have survived the complication disease, died or suffered disability. The recovery rate of complications, mortality due to complications, and disability rate were denoted by γ, δ and ν respectively, so the number of individuals in compartment C is reduced by as much as $\gamma C, \delta C$ and νC. At the same time, due to complication healing, the number of individuals in the D compartment increases by γC. The system (1) parameters are given in table 1 below.

Table 1. Parameter values of model (1)

Parametreler	Descriptions	Values
γ	The rate of recovery of complications	0.37141
ν	Disability rate	0.01623
δ	Complication-related mortality rate	0.0068
α	The rate of occurrence of complications	0.67758
μ	Mortality rate	0.00764
I	Number of diabetes cases	60

Here the parameters are $I, \alpha, \gamma, \delta, \nu, \mu > 0. D(0) = 9.65; C(0) = 11.05$ [2].

Time-Delayed DC Model

Initial state

If the time delay denoted τ is included in the second term of the first equation in the (1) system, we get:

$$\begin{aligned}\frac{dD}{dt} &= I-(\alpha+\mu)D(t)+\gamma C(t-\tau)\\ \frac{dC}{dt} &= \alpha D(t)-(\gamma+\delta+\nu+\mu)C(t)\end{aligned} \quad (2)$$

Where $I, \alpha, \gamma, \delta, \nu, \mu$ differ from zero. Assume $P(D_{1_0}, C_{1_0})$ be the positive equilibrium point of the (2) system. Let's linearize system (2) at equilibrium point P

$X(t) = D(t) - D_{1_0}, Y(t) = C(t) - C_{1_0}$. Using variable substitution

$$\frac{dX}{dt}=\frac{dD}{dt},\ \frac{dY}{dt}=\frac{dC}{dt}$$

choosing the f and g functions

$$f(D,C)=I-(\alpha+\mu)D(t)+\gamma C(t-\tau)$$
$$g(D,C)=\alpha D(t)-(\gamma+\delta+\nu+\mu)C(t)$$

Let's say. Let's linearize system (2) at point $P(D_{1_0},C_{1_0})$ here:

$$\begin{aligned}\frac{dX}{dt}&=\frac{\partial f}{\partial D}(P^*)X(t)+\frac{\partial f}{\partial C}(P^*)Y(t-\tau)\\ \frac{dY}{dt}&=\frac{\partial g}{\partial D}(P^*)X(t)+\frac{\partial g}{\partial C}(P^*)Y(t)\end{aligned}\tag{3}$$

Where

$$\begin{aligned}\frac{dX}{dt}&=-(\alpha+\mu)X(t)+\gamma Y(t-\tau)\\ \frac{dY}{dt}&=\alpha X(t)-(\gamma+\delta+\nu+\mu)Y(t)\end{aligned}\tag{4}$$

The characteristic equation of the system (4),

$$\Delta(\lambda,\tau)=\det(\lambda I-A_0-\sum_{j=1}^{1}A_j e^{-\lambda\tau j})$$

where

$$A_0=\begin{pmatrix}-(\alpha+\mu) & 0\\ \alpha & -(\gamma+\delta+\nu+\mu)\end{pmatrix}\text{ ve }A_1=\begin{pmatrix}0 & \gamma\\ 0 & 0\end{pmatrix}$$

$$\begin{aligned}\Delta(\lambda,\tau)&=\det\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}-\begin{pmatrix}-(\alpha+\mu) & 0\\ \alpha & -(\gamma+\delta+\nu+\mu)\end{pmatrix}-\begin{pmatrix}0 & \gamma\\ 0 & 0\end{pmatrix}e^{-\lambda\tau j}\right)\\ &=\det\begin{pmatrix}\lambda+(\alpha+\mu) & -\gamma e^{-\lambda\tau}\\ -\alpha & \lambda+(\gamma+\delta+\nu+\mu)\end{pmatrix}\\ Q(\lambda)&=\lambda^2+(\gamma+\delta+\nu+\alpha+2\mu)\lambda+(\alpha+\mu)(\gamma+\delta+\nu+\mu)-\alpha\gamma e^{-\lambda\tau}\end{aligned}\tag{5}$$

$$Q(\lambda)=\lambda^2+a\lambda+c+be^{-\lambda\tau}\tag{6}$$

From here,

$$a=(\gamma+\delta+\nu+\alpha+2\mu)+,c=(\alpha+\mu)(\gamma+\delta+\nu+\mu),b=-\alpha\gamma,d=c+b$$

Let the characteristic equation of (6) be $\tau=0$.

$$\lambda^2+a\lambda+d=0\tag{7}$$

and the roots of the equation

$$\lambda_{1,2}=\frac{-a\pm\sqrt{a^2-4d}}{2}\tag{8}$$

(H1) for $a=0,\lambda_1=-i\sqrt{d}$ and $\lambda\_2=i\sqrt{d}$. Then, λ_1 becomes $\lambda_{1,2}=\pm i\sqrt{d}$. So $P\left(D_{1_0},C_{1_0}\right)$ is the fixed center point for the equilibrium point $a=0$.

(H2) $a<0$, its eigenvalues are $\lambda_1=\frac{-a-\sqrt{a^2-4d}}{2}$ and $\lambda_2=\frac{-a+\sqrt{a^2-4d}}{2}$, from which $a=0,\lambda_{1,2}=\pm i\beta,\beta\neq 0$ is obtained. Then, if we generalize $a\in R$, the equilibrium point of P is the fixed center point, since $a\in R$, $\lambda_{1,2}=\pm i$ß, $\beta\neq 0$.
(7) the negative real parts of the equation

(H3) $a>0,d>0,|a|<2\sqrt{d}$, $\lambda_{1,2}=\alpha\pm i\beta,\alpha<0$, since $\alpha<0$, the equilibrium point P becomes a stable spiral.

(H4) $a<0,d>0,|a|<2\sqrt{d}$, $\lambda_{1,2}=\alpha\pm i\beta,\alpha>0$, since $\alpha>0$,the equilibrium point P becomes an unstable spiral

(H5) When $a=0,d>0,\lambda_{1,2}=\pm i\beta,\beta\neq 0$,$P$ the equilibrium point becomes the stable center point.

(H6) $a=0,d<0,\ \lambda_{1,2}=\pm i\sqrt{d},\lambda_1<0<\lambda_2$ the equilibrium point becomes the unstable node.

$$e^{-\lambda\tau}=e^{-i\omega t}=\cos(\bar{\omega}\tau)-isin(\bar{\omega}\tau)$$

obtained. From here,

$$(i\bar{\omega})^2+ai\bar{\omega}+c+b[\cos(\bar{\omega}\tau)-isin(\bar{\omega}\tau)]=0$$

Then,

$$-\bar{\omega}^2 + ai\bar{\omega} + c + b\cos(\bar{\omega}\tau) - bisin(\bar{\omega}\tau) = 0 \quad (9)$$

let's consider the real and virtual parts of the equation separately:

$$-\bar{\omega}^2 + c + b\cos(\bar{\omega}\tau) = 0 \quad (10)$$

and

$$a\bar{\omega} - bsin(\bar{\omega}\tau) = 0 \quad (11)$$

By making the necessary arrangements here

$$\cos(\bar{\omega}\tau) = \frac{\bar{\omega}^2 - c}{b}, sin(\bar{\omega}\tau) = \frac{a\bar{\omega}}{b} \quad (12)$$

(12) if the square of both sides of the equations is taken

$$\bar{\omega}^4 + \bar{\omega}^2(a^2 - 2c) + c^2 - b^2 = 0 \quad (13)$$

the roots of equation (13) are

$$\bar{\omega}_{\pm}^2 = \frac{-(a^2-2c) \pm \sqrt{(a^2-2c)^2 - 4(c^2-b^2)}}{2} \quad (14)$$

Thus, equation (14) has a positive and a negative root, and let's determine the values of τ_k as follows:

Let's divide the equations of (12)

$$\tan(\bar{\omega}\tau_k) = \frac{\cos(\bar{\omega}\tau)}{sin(\bar{\omega}\tau)} = \frac{\frac{\bar{\omega}^2 - c}{b}}{\frac{a\bar{\omega}}{b}} = \frac{a\bar{\omega}}{\bar{\omega}^2 - c} \quad (15)$$

if tan^{-1} of both sides of equation (15) is taken,

$$\bar{\omega}\tau_k = \tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + 2k\pi, k = 0,1,2,\ldots \quad (16)$$

$$\tau_k = \frac{1}{\bar{\omega}}\tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + 2k\pi, k = 0,1,2,\ldots \quad (17)$$

obtained.

Example. The parameter of the equation (2) is $\gamma = 0.37141, \nu = 0.01623, \delta = 0.0068, \lambda = 0.67758, \mu = 0.00764, I = 60$. When we consider the equation (2), the positive equilibrium point becomes $(45.7626, 77.1185)$. for $\tau = 0$, the eigenvalues $\{-1.0649, -0.0224\}$ of the jacobian matrix of the system (2) at the point $(45.7626, 77.1185)$ are obtained. Since the eigenvalues are negative $(45.7626, 77.1185)$, it becomes the stable equilibrium point.

Second Case

If the time delay is added to the first term of the second equation of equation (1), it is obtained that:

$$\begin{aligned} \frac{dD}{dt} &= I - (\alpha + \mu)D(t) + \gamma C(t) \\ \frac{dC}{dt} &= \alpha D(t - \tau) - (\gamma + \delta + \nu + \mu)C(t) \end{aligned} \quad (18)$$

Where $I, \alpha, \gamma, \delta, \nu, \mu$ differ from zero. Let's assume that $P(D_{1_0}, C_{1_0})$ (2) is the positive equilibrium point of the system. (18) let's linearize the system at the equilibrium point P:

$X(t) = D(t) - D_{1_0}, Y(t) = C(t) - C_{1_0}$. Using variable substitution

$$\frac{dX}{dt} = \frac{dD}{dt}, \quad \frac{dY}{dt} = \frac{dC}{dt}$$

choosing the f and g functions

$$f(D, C) = I - (\alpha + \mu)D(t) + \gamma C(t)$$
$$g(D, C) = \alpha D(t - \tau) - (\gamma + \delta + \nu + \mu)C(t)$$

Let's say. Let's linearize system (18) at point $P(D_{1_0}, C_{1_0})$ here:

$$\begin{aligned} \frac{dX}{dt} &= \frac{\partial f}{\partial D}(P^*)X(t) + \frac{\partial f}{\partial C}(P^*)Y(t) \\ \frac{dY}{dt} &= \frac{\partial g}{\partial D}(P^*)X(t - \tau) + \frac{\partial g}{\partial C}(P^*)Y(t) \end{aligned} \quad (19)$$

Where

$$\frac{dX}{dt} = -(\alpha+\mu)X(t) + \gamma Y(t)$$
$$\frac{dY}{dt} = \alpha X(t-\tau) - (\gamma+\delta+\nu+\mu)Y(t) \quad (20)$$

The characteristic equation of the system (20),

$$\Delta(\lambda,\tau) = \det(\lambda I - A_0 - \sum_{j=1}^{1} A_j e^{-\lambda\tau j})$$

where

$$A_0 = \begin{pmatrix} -(\alpha+\mu) & 0 \\ \alpha & -(\gamma+\delta+\nu+\mu) \end{pmatrix} \text{ ve } A_1 = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$$

$$\Delta(\lambda,\tau) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -(\alpha+\mu) & 0 \\ \alpha & -(\gamma+\delta+\nu+\mu) \end{pmatrix} - \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} e^{-\lambda\tau j}\right)$$
$$= \det\begin{pmatrix} \lambda+(\alpha+\mu) & -\gamma e^{-\lambda\tau} \\ -\alpha & \lambda+(\gamma+\delta+\nu+\mu) \end{pmatrix} \quad (21)$$
$$Q(\lambda) = \lambda^2 + (\gamma+\delta+\nu+\alpha+2\mu)\lambda + (\alpha+\mu)(\gamma+\delta+\nu+\mu) - \alpha\gamma e^{-\lambda\tau}$$

$$Q(\lambda) = \lambda^2 + a\lambda + c + be^{-\lambda\tau} \quad (22)$$

where,

$a = (\gamma+\delta+\nu+\alpha+2\mu)+, c = (\alpha+\mu)(\gamma+\delta+\nu+\mu), b = -\alpha\gamma, d = c+b$
Let the characteristic equation of (22) be $\tau = 0$.

$$\lambda^2 + a\lambda + d = 0 \quad (23)$$

and the roots of the equation

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2-4d}}{2} \quad (24)$$

the negative real parts of equation (23) are
(H7) $a > 0$
(H8) $d > 0$
Since it is valid under both conditions, the equilibrium point becomes locally asymptotically stable.
We determine whether the real part of some roots increases so that it reaches zero, and in the end it becomes positive as τ changes. (22) assume $\tau \neq 0$ in the characteristic equation. let $\lambda = i\bar{\omega}$ be a root of the characteristic equation $\bar{\omega} > 0$ (22) and

$$e^{-\lambda\tau} = e^{-i\omega t} = \cos(\bar{\omega}\tau) - isin(\bar{\omega}\tau)$$

obtained. Here, if λ is substituted in equation (22)

$$(i\bar{\omega})^2 + ai\bar{\omega} + c + b[\cos(\bar{\omega}\tau) - isin(\bar{\omega}\tau)] = 0$$

then,

$$-\bar{\omega}^2 + ai\bar{\omega} + c + b\cos(\bar{\omega}\tau) - bisin(\bar{\omega}\tau) = 0 \quad (25)$$

let us consider the real and virtual parts of equation (25) separately:

$$-\bar{\omega}^2 + c + b\cos(\bar{\omega}\tau) = 0 \quad (26)$$

and

$$a\bar{\omega} - bsin(\bar{\omega}\tau) = 0 \quad (27)$$

By making the necessary arrangements here

$$\cos(\bar{\omega}\tau) = \frac{\bar{\omega}^2 - c}{b} \quad (28)$$

$$sin(\bar{\omega}\tau) = \frac{a\bar{\omega}}{b} \quad (29)$$

taking the square on both sides of equations (28) and (29), let's sum it from side to side,

$$\begin{cases} \cos^2(\bar{\omega}\tau) = \dfrac{(\bar{\omega}^2 - c)^2}{b^2} \\ \sin^2(\bar{\omega}\tau) = \dfrac{(a\bar{\omega})^2}{b^2} \\ \dfrac{1}{b^2}((\bar{\omega}^2 - c)^2 + (a\bar{\omega})^2) = 1 \end{cases}$$

$t^2 + (a^2 - 2c)t + c^2 - b^2 = 0$ (30)
obtained. then

$$t_{1,2} = \frac{-(a^2-2c) \pm \sqrt{(a^2-2c)^2 - 4(c^2-b^2)}}{2}$$
$$\bar{\omega}_{\pm}^2 = \frac{-(a^2-2c) \pm \sqrt{(a^2-2c)^2 - 4(c^2-b^2)}}{2} \quad (31)$$

(H9) For any value of a, $\bar{\omega}_{\pm}^2$ has two complex conjugate roots.
let's determine the values τ_k as follows. if the equations (28)-(29) are,

$$\cos(\bar{\omega}\tau) = \frac{\bar{\omega}^2 - c}{b}$$
$$sin(\bar{\omega}\tau) = \frac{a\bar{\omega}}{b}$$

If we divide the equations by

$$\tan(\bar{\omega}\tau_k) = \frac{a\bar{\omega}}{\bar{\omega}^2 - c} \quad (32)$$

if tan^{-1} of both sides of equation (15) is taken,

$$\bar{\omega}\tau_k = \tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + 2k\pi, k = 0,1,2, \ldots \quad (33)$$

from equation (33)

$$\tau_k = \frac{1}{\bar{\omega}}\tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + 2k\pi, k = 0,1,2, \ldots \quad (34)$$

$$\tau_k^{\pm} = \frac{1}{\bar{\omega}_{\pm}}\tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + \frac{2k\pi}{\bar{\omega}_{\pm}^2}, k = 0,1,2, \ldots \quad (35)$$

Let's summarize the above with lemma. The proof of this lemma is given.

Lemma 1.

If **(H7)** and **(H8)** are fulfilled, only the imaginary roots of equation (22) with $\tau = \tau_j^+$ are $\pm i\bar{\omega}$
If **(H7)** and **(H8)** hold, only imaginary root pairs of equation (22) with $\tau = \tau_j^+$ become $\pm i\bar{\omega}$
The following theorem is given to determine the stability of the time-delayed equation.
Theorem 1. If the following conditions are fulfilled, the equilibrium point P^* is asymptotic for all $\tau \geq 0$.

- **a.** All real parts of the roots of the equation $\Delta(\lambda, 0) = 0$ are negative.
- **b.** For all real $(\bar{\omega})$ and $\tau \geq 0; \Delta(i\bar{\omega}, \tau) \neq 0, i = \sqrt{-1}$.

In conclusion, let us obtain the following theorem:
Theorem 2. If the conditions hold for $a < 0, d > 0$, then the equilibrium point of equation (22) is asymptotically stable for all $\tau \geq 0$.
For the proof of contradictory conditions

$$\frac{d}{d\tau}Re\left(\lambda_j^+(\tau_j^+)\right) > 0 \text{ ve } \frac{d}{d\tau}Re\left(\lambda_j^-(\tau_j^-)\right) < 0$$

let us take the derivative of both sides of equation (22):

$$\lambda^2 + a\lambda + c + be^{-\lambda\tau} = 0$$
$$2\lambda\frac{d\lambda}{d\tau} + a\frac{d\lambda}{d\tau} + be^{-\lambda\tau}\left[\tau\frac{d\lambda}{d\tau} + \lambda\right] = 0$$
$$\left[2\lambda + a + be^{-\lambda\tau}\tau\right]\frac{d\lambda}{d\tau} + be^{-\lambda\tau} = 0$$

$$\frac{d\lambda}{d\tau} = \frac{-be^{-\lambda\tau}}{2\lambda + a + be^{-\lambda\tau}\tau} \quad (36)$$

For $\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{\frac{d\lambda}{d\tau}}$,

$$\frac{d\tau}{d\lambda}=\frac{2\lambda+a+be^{-\lambda\tau}\tau}{-be^{-\lambda\tau}} \tag{37}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{2\lambda+a+be^{-\lambda\tau}\tau}{-be^{-\lambda\tau}}=\frac{2\lambda+a}{-be^{-\lambda\tau}}-\frac{\tau}{\lambda} \tag{38}$$

$e^{-\lambda\tau}=\frac{-(\lambda^2+a\lambda+c)}{b}$ for equation

$$\lambda^2+a\lambda+c+be^{-\lambda\tau}=0 \tag{39}$$

let us written the one given in equation (39) for the one given in equation (38),

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{2\lambda+a+be^{-\lambda\tau}\tau}{-be^{-\lambda\tau}}=\frac{2\lambda+a}{\lambda(\lambda^2+a\lambda+c)}-\frac{\tau}{\lambda}$$

$$\text{sign}\left\{\frac{d}{d\tau}(Re\lambda)\right\}=\text{sign}\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}$$

$$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}\right\}=\text{sign}\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\lambda=i\bar{\omega}}\right\}$$

hence the expression,

$$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}\right\}=\text{sign}\left\{\left(\frac{2\lambda+a}{\lambda(\lambda^2+a\lambda+c)}-\frac{\tau}{\lambda}\right)\Big|_{\lambda=i\bar{\omega}}\right\} \tag{40}$$

it can be calculated this way. if we use $i^2=-1$, the calculation gives the following result

$$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}\right\}=\text{sign}\left\{Re\left(\left(\frac{2\lambda+a}{\lambda(\lambda^2+a\lambda+c)}\right)\Big|_{\lambda=i\bar{\omega}}\right)-Re\left(\frac{\tau}{\lambda}\right)\Big|_{\lambda=i\bar{\omega}}\right\}$$

Now, let's calculate sign $\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}\right\}$ as follows:

$Re\left(\left(\frac{2\lambda+a}{\lambda(\lambda^2+a\lambda+c)}\right)\Big|_{\lambda=i\bar{\omega}}\right)$ for value

$$Re\left(\frac{2i\bar{\omega}+a}{i\bar{\omega}(i\bar{\omega}^2+ai\bar{\omega}+c)}\right)=Re\left(\frac{a}{i\bar{\omega}(-\bar{\omega}^2+ai\bar{\omega}+c)}\right)+Re\left(\frac{2i\bar{\omega}}{i\bar{\omega}(-\bar{\omega}^2+ai\bar{\omega}+c)}\right)$$

but,

$$Re\left(\frac{2i\bar{\omega}}{i\bar{\omega}(-\bar{\omega}^2+ai\bar{\omega}+c)}\right)=Re\left(\frac{2}{(-\bar{\omega}^2+ai\bar{\omega}+c)}\frac{(-\bar{\omega}^2-ai\bar{\omega}+c)}{(-\bar{\omega}^2-ai\bar{\omega}+c)}\right)=Re\left(\frac{(-2\bar{\omega}^2-2ai\bar{\omega}+2c)}{(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)}\right)$$
$$=\frac{(-2\bar{\omega}^2+2c)}{(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)}$$

$$Re\left(\frac{a}{i\bar{\omega}(-\bar{\omega}^2+ai\bar{\omega}+c)}\right)=Re\left(\frac{a}{(-\bar{\omega}^2+ai\bar{\omega}+c)}\frac{(-\bar{\omega}^2-ai\bar{\omega}+c)}{(-\bar{\omega}^2-ai\bar{\omega}+c)}\right)=Re\left(\frac{(-a\bar{\omega}^2-a^2i\bar{\omega}+ac)}{i\bar{\omega}(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)}\right)$$
$$=\frac{-a^2}{(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)}$$

therefore,

$$Re\left(\left(\frac{2\lambda+a}{\lambda(\lambda^2+a\lambda+c)}\right)\Big|_{\lambda=i\bar{\omega}}\right)=\frac{(-2\bar{\omega}^2+2c)-a^2}{(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)} \tag{41}$$

$Re\left(\frac{\tau}{\lambda}\right)\Big|_{\lambda=i\bar{\omega}}$ for value,

$$Re\left(\frac{\tau}{\lambda}\right)\Big|_{\lambda=i\bar{\omega}}=Re\left(\frac{\tau}{i\bar{\omega}}\frac{-i\bar{\omega}}{-i\bar{\omega}}\right)=Re\left(\frac{-\tau i\bar{\omega}}{\bar{\omega}^2}\right)=0 \tag{42}$$

Substituting equations (41) and (42) in equation (40) gives the following result:

$$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}\right\}=\text{sign}\left\{\frac{(-2\bar{\omega}^2+2c)-a^2}{(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)}\right\} \tag{43}$$

Theorem 3. let $\tau=\tau^{\pm}$ be defined as in (35), and if we continue with the conditions,

$$\lambda^2+a\lambda+c+be^{-\lambda\tau}=0$$

then the equilibrium point P is unstable for a positive constant m.

$\tau\epsilon[0,\tau_0^+]\cup[\tau_0^-,\tau_0^+]\cup\cdots\cup[\tau_{m-1}^-,\tau_0^?]$
$\tau\epsilon[\tau_0^+,\tau_0^-]\cup[\tau_1^+,\tau_1^-]\cup\cdots\cup[\tau_{m-1}^+,\tau_{m-1}^-]$.

Proof. When the following conditions apply to the theorem, only the following contradictory conditions need to be satisfied.

$\frac{d}{d\tau}(Re\lambda)|_{\tau=\tau^+} > 0$ ve $\left(\frac{d(Re)}{d\tau}\right)\Big|_{\tau=i\bar{\omega}^+} > 0$

$\frac{d}{d\tau}(Re\lambda)|_{\tau=\tau^-} > 0$ ve $\left(\frac{d(Re)}{d\tau}\right)\Big|_{\tau=i\bar{\omega}^-} < 0$

$\bar{\omega}_{\pm}^2 = \frac{-(a^2-2c)\pm\sqrt{(a^2-2c)^2-4(c^2-b^2)}}{2}$

and

$$\tau_k^{\pm} = \frac{1}{\bar{\omega}_{\pm}}\tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2-c}\right) + \frac{2k\pi}{\bar{\omega}_{\pm}^2}, k = 0,1,2,\ldots$$

For theorem 3 conditions

$(a^2-2c)^2 - 4(c^2-b^2) > 0$

$\sqrt{(a^2-2c)^2-4(c^2-b^2)} > 0$

$\text{sign}\left\{\sqrt{(a^2-2c)^2-4(c^2-b^2)}\right\} > 0$

Therefore,

$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}, \tau = \tau^+\right\} > 0.$

Now,

$$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}\right\} = \text{sign}\left\{\frac{(-2\bar{\omega}^2+2c)-a^2}{(\bar{\omega}^4+(a^2-2c)\bar{\omega}^2+c^2)}\right\} = \text{sign}\left\{\sqrt{(a^2-2c)^2-4(c^2-b^2)}\right\} > 0.$$

and,

$$\text{sign}\left\{-\sqrt{(a^2-2c)^2-4(c^2-b^2)}\right\} < 0.$$

therefore,

$$\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\Big|_{\lambda=i\bar{\omega}}, \tau = \tau^+\right\} < 0.$$

this is how contradictions are provided and the proof is completed.

Example: (1) Let's use the parameters $\gamma = 0.37141, v = 0.01623, \delta = 0.0068, \lambda = 0.67758, \mu = 0.00764, I = 60, \tau = 0.5$ for the model.

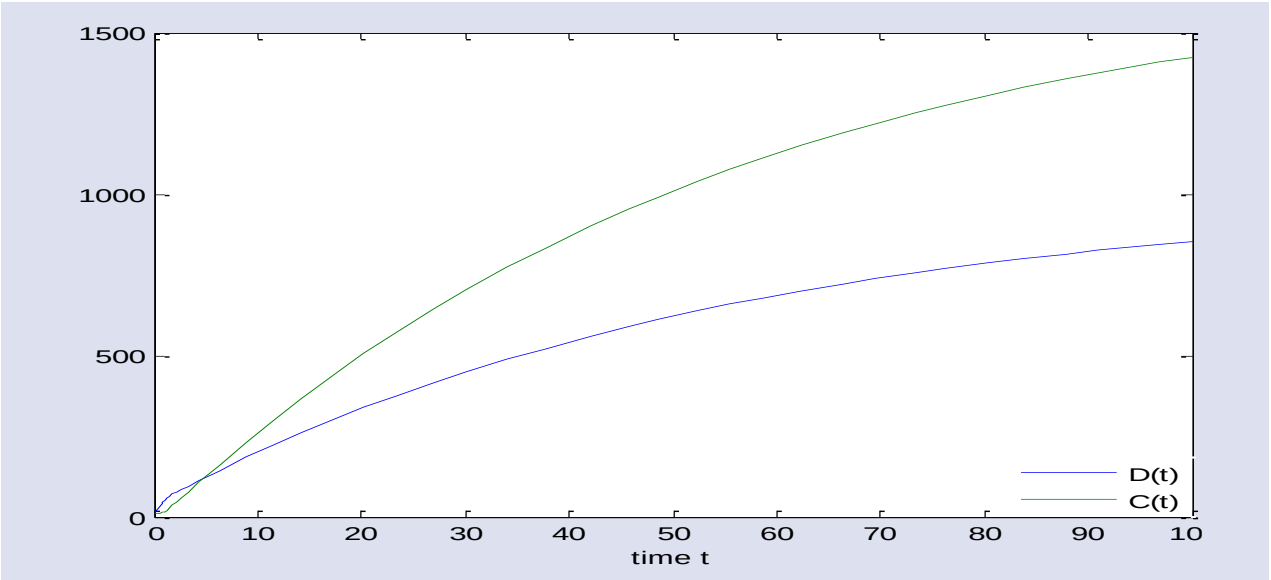

Figure 2. Solution curve of DC model with $\tau = 0.5$

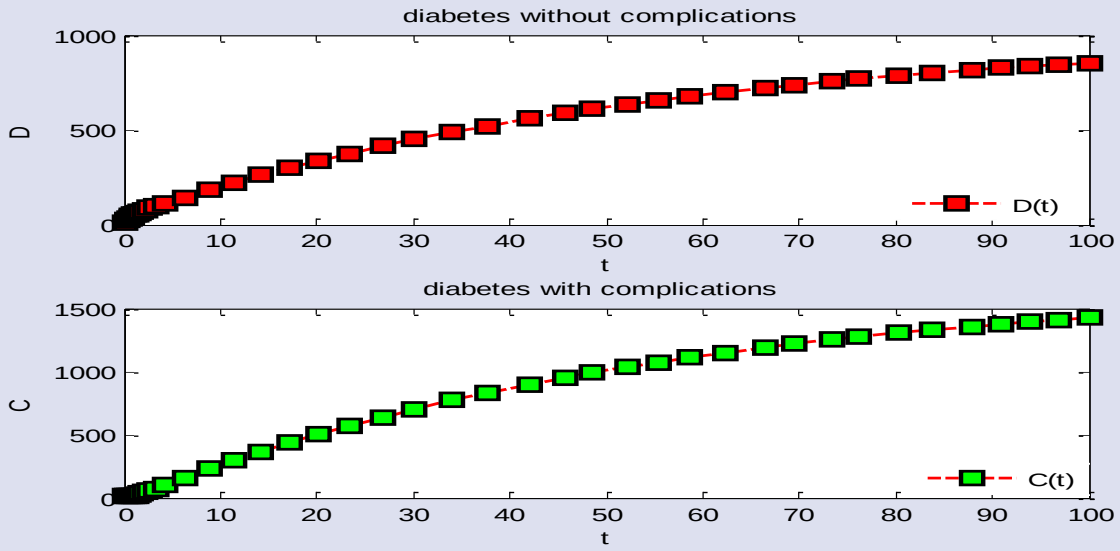

Figure 3. Solution curve of DC model with $\tau = 0.5$

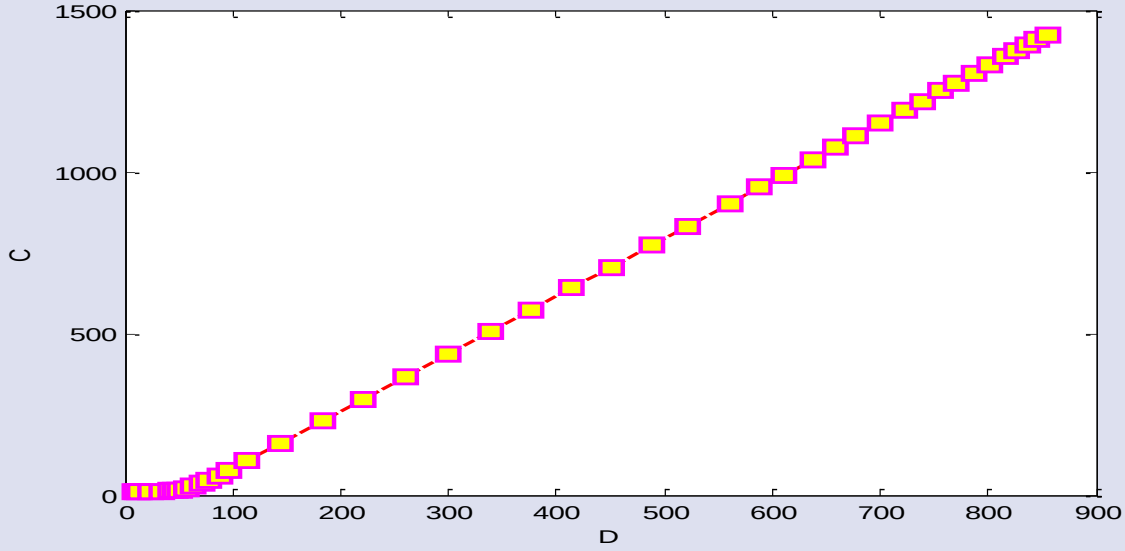

Fig. 4. Phase diagram of diabetes with and without complications of the DC model according to $\tau = 0.5$

The critical value of time delay is $\tau = 0.5$. When $\tau < 0.5$ the equilibrium point $(45.7626,77.1185)$ becomes asymptotically stable; when $\tau < 0.5$, the stability at $(45.7626,77.1185)$ is lost and when $\tau > 0.5$, the equilibrium point $(45.7626,77.1185)$ becomes unstable.

Conclusion

In this work, the diabetes complication (DC) model was analyzed by incorporating the effect of time delays into the system dynamics. The main objective was to investigate how delays influence the stability of equilibrium points and the qualitative behavior of the model.

First, the equilibrium points of the system were determined, and stability analysis was carried out using the characteristic equation. When the delay parameter was set to $\tau = 0$, the positive equilibrium point was obtained as $(D^*, C^*) = (45.7626,77.1185)$). The eigenvalues corresponding to this point were calculated as $(-1.0649,-0.0224)$. Since both eigenvalues have negative real parts, this equilibrium point was identified as locally asymptotically stable. This indicates that, in the absence of delays, the progression of diabetes complications remains under control and the system returns to equilibrium after small perturbations.

Next, the impact of time delay was systematically investigated. For τ = 0.5, the trajectories of the variables D and C were examined numerically and illustrated in Figure 2. The results show that as the delay approaches this critical threshold, the system exhibits oscillatory tendencies, reflecting a transition in its dynamical behavior. To further understand this effect, the range of values around the critical point was studied. As shown in Figure 3, for $\tau < 0.5$ the solution trajectories converge to the positive equilibrium point, confirming asymptotic stability of the system. Conversely, when $\tau > 0.5$, the equilibrium loses stability, and the trajectories diverge as

illustrated in Figure 4. This demonstrates that the system undergoes a delay-induced instability when the critical delay is exceeded.

The overall results reveal that the stability of the DC model strongly depends on the magnitude of the time delay. The existence of a critical delay value ($\tau = 0.5$) acts as a bifurcation threshold: for smaller delays, the equilibrium is stable, while larger delays destabilize the system. This emphasizes that delays in feedback or physiological response can drastically alter the progression of diabetes complications, potentially leading to uncontrolled growth of the complication variables.

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Conflicts of interest

There are no conflicts of interest in this work.

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