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On Mappings and Topological Structure of Intuitionistic Fuzzy Parameter Soft Sets

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ABSTRACT

In this study, some algebraic properties of intuitionistic fuzzy parameter soft (IFPS) which combining the strengths of intuitionistic fuzzy sets and parameterized soft set theory, are given with examples. A topological structure is constructed on IFPS sets, providing a foundation for exploring fundamental topological notions such as open and closed sets, interior, closure and neighbourhood systems within this new context. Additionally, the concept of IFPS function is defined and its algebraic properties are investigated. Finally, mappings between IFPS topological spaces are examined and the notion of IFPS continuity is introduced and its properties are analysed.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy parametrized soft sets, Topology

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Introduction

Intuitionistic fuzzy (IF) sets, an important tool in uncertainty modelling, were introduced by Atasanov in 1986 [1]. In this theory, in addition to the membership function in fuzzy sets [2], there is a non-membership function that characterizes the non-membership of an element in the universe. Unlike traditional fuzzy sets that assign a single membership degree to each element in an intuitionistic fuzzy set, the non-membership function includes a degree of hesitation and provides a more comprehensive representation of uncertainty and imprecision. Therefore, for suitable problems, IF sets can be used as a more effective tool than fuzzy sets [3-8].

Another theory used in uncertainty modeling is soft set theory [9]. This theory was introduced by Molodtsov in 1999 as a powerful mathematical tool for handling imprecise, uncertain and incomplete information. Unlike traditional mathematical models, soft sets do not rely on rigid membership functions, but instead use a flexible approach based on a set of linguistic parameters, making them highly adaptable to real-world problems. This advantage of the theory quickly made it popular in the scientific community and has been successfully applied to many fields [10-14].

The increasing complexity of real-world problems has led to the development of hybrid mathematical models that combine the strengths of different uncertainty theories. Among these, hybrid theories based on soft sets [9] have attracted significant attention due to their ability to integrate with other mathematical approaches such as fuzzy sets [7], rough sets [15], and IF sets [1]. By combining the flexibility of soft sets with the improved uncertainty representation of other theories, these hybrid models are providing more robust decision-making tools, especially in areas such as artificial intelligence, pattern recognition, multi-criteria decision analysis, and mathematics [16, 17, 18, 19, 20]. One of these theories is intuitionistic fuzzy parameter soft (IFPS) sets [21], which are based on the principle that parameter sets are IF sets. In addition, topology is one of the important areas of mathematics. Therefore, the topological structures of these hybrid theories have been studied by many researchers, [22- 25].

In this study, Intuitionistic Fuzzy Parameter Soft (IFPS) sets are examined. Following this, the topological structure defined on these sets is presented and an introduction to the basic topological concepts it provides is given. In addition, the IFPS function is defined and its algebraic properties are investigated. Finally, the properties of a function between IFPS topological spaces are defined and the concept of continuity is introduced.

Definition 1 [1]Let X be a non-empty set and $\mu_A: X \to [0,1]$, $\nu_A: X \to [0,1]$ functions and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Then the set $\tilde{A} = \{(x, \mu_A(x), \nu_A(x)): x \in X\}$ is called an intuitionistic fuzzy set. Here the functions μ_A and ν_A are called membership and non-membership functions of x, respectively.

Definition 2 [1]Let X be a non-empty set and let $\tilde{A}$ be an intuitionistic fuzzy set on X. Then $supp_\mu \tilde{A} = \{x \in X: \mu_A(x) \neq 0\}$ and $supp_\nu \tilde{A} = \{x \in X: \nu_A(x) \neq 0\}$ sets are called membership and non-membership support sets, respectively. Also, the union of $supp_\mu \tilde{A}$ and $supp_\nu \tilde{A}$ is called $supp\tilde{A}$.

Definition 3 [1] Let X be a non-empty set, $\tilde{A} = \{(x, \mu_A(x), \nu_A(x)): x \in X\}$, $\tilde{B} = \{(x, \mu_B(x), \nu_B(x)): x \in X\}$ and $\{\tilde{A}_i\}_{i \in I}$,for all $i \in I$, are intuitionistic fuzzy sets on X, where I arbitrary index sets. The basic operations of intuitionistic fuzzy sets are defined as follows;

1) $\tilde{A} \leq \tilde{B} \Leftrightarrow \forall x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$
2) $\tilde{A} \wedge \tilde{B} = \{(x, \mu_{A \wedge B}(x), \nu_{A \wedge B}(x)): \mu_{A \wedge B}(x) = \min\{\mu_A(x), \mu_B(x)\}$
$\nu_{A \wedge B}(x) = \max\{\nu_A(x), \nu_B(x)\}, x \in X\}$
3) $\tilde{A} \vee \tilde{B} = \{(x, \mu_{A \vee B}(x), \nu_{A \vee B}(x)): \mu_{A \vee B}(x) = \max\{\mu_A(x), \mu_B(x)\},$
$\nu_{A \vee B}(x) = \min\{\nu_A(x), \nu_B(x)\}, x \in X\}$
4) $(\tilde{A})^c = \{(x, \nu_A(x), \mu_A(x))\}$
5) $\tilde{1}_X = \{(x, 1,0): x \in X\}$ and $\tilde{0}_X = \{(x, 0,1): x \in X\}$
6) $\underset{i \in I}{\wedge} \tilde{A}_i = \{(x, \mu_{\wedge A_i}(x), \nu_{\wedge A_i}(x)): \mu_{\wedge A_i}(x) = \inf\{\mu_{A_i}(x)\},$
$\nu_{\wedge A_i}(x) = \sup\{\mu_{\wedge A_i}(x)\}, x \in X\}$
7) $\underset{i \in I}{\vee} \tilde{A}_i = \{(x, \mu_{\vee A_i}(x), \nu_{\vee A_i}(x)): \mu_{\vee A_i}(x) = \sup\{\mu_{A_i}(x)\},$
$\nu_{\vee A_i}(x) = \inf\{\mu_{\wedge A_i}(x)\}, x \in X\}$

Definition 4 [8] Let X and Y be non-empty sets, $f: X \to Y$ be a function and $\tilde{A} = \{(x, \mu_A(x), \nu_A(x)): x \in X\}$, $\tilde{B} = \{(y, \mu_B(y), \nu_B(y)): y \in Y\}$ are intuitionistic fuzzy set on X and Y, respectively.

1) $f(\tilde{A})$ is an intuitionistic fuzzy set in Y. Then the membership and non-membership functions for each $y \in Y$ defined as;

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & ; f^{-1}(y) \neq \varnothing \\ 0 & ; f^{-1}(y) = \varnothing \end{cases}$$
$$\nu_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x) & ; f^{-1}(y) \neq \varnothing \\ 0 & ; f^{-1}(y) = \varnothing \end{cases}$$

2) $f^{-1}(\tilde{B})$ is an intuitionistic fuzzy set in X. Then the membership and non-membership functions for each $x \in X$ defined as;

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$$
$$\nu_{f^{-1}(B)}(x) = \nu_B(f(x)).$$

Definition 5 [21] Let E be a set of parameters, X be a non-empty initial universe, $\mu_A: E \to [0,1]$, $\nu_A: E \to [0,1]$ and $F_A: E \to P(X)$, where $\mu_A(e) + \nu_A(e) \leq 1$ for each $e \in E$. Then the $(F, \tilde{A}) = \{((e, \mu_A(e), \nu_A(e)), F_A(e)): e \in E\}$ set is called an intuitionistic fuzzy parameter soft sets.

From now on, the family of all intuitionistic fuzzy parameterized soft sets with parameter set E and initial universe X will be denoted by $IFPS(X, E)$. For convenience of notation, we will denote $(F, \tilde{A})$ simply by the approximation function $F_{\tilde{A}}$.

Definition 6 [21]Let $F_{\tilde{A}} \in IFPS(X, E)$. If $\mu_A(x) = 0$, $\nu_A(x) = 1$ and $F_A(e) = \varnothing$ for each $e \in E$, then $F_{\tilde{A}}$ is called the null intuitionistic fuzzy parameter soft set and is denoted by X_{01}.

Definition 7 [21]Let $F_{\tilde{A}} \in IFPS(X, E)$. If $\mu_A(x) = 1$, $\nu_A(x) = 0$ and $F_A(e) = X$ for each $e \in E$, then $F_{\tilde{A}}$ is called the universal intuitionistic fuzzy parameter soft set and is denoted by X_{10}.

Example 1 [21]Let $E=\{e_1,e_2,e_3\}$ be a set of parameters and $X=\{x_1,x_2,x_3,x_4\}$ initial universe. Then

$$X_{01}=\{((e_1,0,1),\varnothing),((e_2,0,1),\varnothing),((e_3,0,1),\varnothing)\}$$
$$X_{10}=\{((e_1,1,0),X),((e_2,1,0),X),((e_3,1,0),X)\}.$$

Definition 8 [21]Let $F_{\tilde{A}},F_{\tilde{B}}\in IFPS(X,E)$. Then

1) If $\mu_A(e)\leq\mu_B(e),\nu_A(e)\geq\nu_B(e)$ and $F_A(e)\subseteq F_B(e)$ for each $e\in E$, then $F_{\tilde{A}}$ is called the $IFPS$ subset of $F_{\tilde{B}}$ and is denoted by $F_{\tilde{A}}\,\widetilde{\subseteq}\,F_{\tilde{B}}$.

2) The $F_{\tilde{C}}=\{((e,\max\{\mu_A(e),\mu_B(e)\},\min\{\nu_A(e),\nu_B(e)\}),F_{A\cup B}(e)):e\in E\}$ $IFPS$ set is called the $IFPS$ union of $F_{\tilde{A}}$ and $F_{\tilde{B}}$ and is denoted by $F_{\tilde{A}}\,\widetilde{\cup}\,F_{\tilde{B}}$.

3) The $F_{\tilde{D}}=\{((e,\min\{\mu_A(e),\mu_B(e)\},\max\{\nu_A(e),\nu_B(e)\}),F_{A\cap B}(e)):e\in E\}$ $IFPS$ set is called the $IFPS$ intersection of $F_{\tilde{A}}$ and $F_{\tilde{B}}$ and is denoted by $F_{\tilde{A}}\,\widetilde{\cap}\,F_{\tilde{B}}$.

4) The $F_{\tilde{G}}=\{((e,\nu_A(e),\mu_A(e)),F_A^c(e)):e\in E\}$ $IFPS$set is called the $IFPS$ complement of $F_{\tilde{A}}$ and is denoted by $F_{\tilde{A}}^C$.

Example 2 Let $E=\{e_1,e_2,e_3\}$ be a set of parameters, $X=\{x_1,x_2,x_3,x_4\}$ initial universe and $F_{\tilde{A}}=\{((e_1,0.3,0.7),\{x_1,x_3\}),((e_2,0,0.6),X),((e_3,0.7,0.3),\{x_1,x_2,x_3\})\}$ and $F_{\tilde{B}}=\{((e_1,0.2,0.5),\{x_1,x_2\}),((e_2,0.6,0.3),\{x_3,x_4\}),(e_3,0.6,0.4),X)\}$ are IFPS set on $\dot{X}$. Then;

$$F_{\tilde{A}}\,\widetilde{\cup}\,F_{\tilde{B}}=\{((e_1,0.3,0.5),\{x_1,x_2,x_3\}),((e_2,0.6,0.3),X),((e_3,0.7,0.3),X)\}$$
$$F_{\tilde{A}}\,\widetilde{\cap}\,F_{\tilde{B}}=\{((e_1,0.2,0.7),\{x_1\}),((e_2,0,0.6),\{x_3,x_4\}),((e_3,0.6,0.4),\{x_1,x_2,x_3\})\}$$
$$F_{\tilde{A}}^C=\{((e_1,0.7,0.3),\{x_2,x_4\}),((e_2,0.6,0),\varnothing),((e_3,0.3,0.7),\{x_4\})\}.$$

Definition 9 Let $\{F_{\tilde{A}_i}\}_{i\in I}\subseteq IFPS(X,E)$. Then generalized union and intersection are defined as;

$$\widetilde{\bigcup_{i\in I}}F_{\tilde{A}_i}=\{((e,\bigvee_{i\in I}\mu_{A_i}(e),\bigwedge_{i\in I}\nu_{A_i}(e)),\bigcup_{i\in I}F_{A_i}(e)):e\in E\}$$

$$\widetilde{\bigcap_{i\in I}}F_{\tilde{A}_i}=\{((e,\bigwedge_{i\in I}\mu_{A_i}(e),\bigvee_{i\in I}\nu_{A_i}(e)),\bigcap_{i\in I}F_{A_i}(e)):e\in E\}$$

, where I is the arbitrary index set.

Example 3 Let $E=\{e_1,e_2\}$ be a set of parameters, $X=\{x_1,x_2,x_3,x_4\}$ initial universe and $F_{\tilde{A}_n}=\{\left(\left(e_1,\frac{1}{n+2},1-\frac{1}{n+1}\right),\{x_1,x_3\}\right),\left(\left(e_2,\frac{1}{2n+2},\frac{1}{n+2}\right),\{x_1,x_2,x_3\}\right)\}$ for all $n\in\mathbb{N}$. Then;

$$\widetilde{\bigcup_{i\in I}}F_{\tilde{A}_i}=\{((e,\bigvee_{i\in I}\mu_{A_i}(e),\bigwedge_{i\in I}\nu_{A_i}(e)),\bigcup_{i\in I}f_{A_i}(e)):e\in E\}$$
$$=\{((e_1,0.5,0),\{x_1,x_3\}),((e_2,0.5,0),\{x_1,x_2,x_3\})\}$$

$$\widetilde{\bigcap_{i\in I}}F_{\tilde{A}_i}=\{((e,\bigwedge_{i\in I}\mu_{A_i}(e),\bigvee_{i\in I}\nu_{A_i}(e)),\bigcap_{i\in I}f_{A_i}(e)):e\in E\}$$
$$=\{((e_1,0,1),\{x_1,x_3\}),((e_2,0,0.5),\{x_1,x_2,x_3\})\}$$

Definition 10 Let $F_{\tilde{A}}\in IFPS(X,E)$. Then $[F_{\tilde{A}}]=\{((e,\mu_A(e),1-\mu_A(e)),F_A(e)):e\in E)\}$ and $\langle F_{\tilde{A}}\rangle=\{((e,1-\nu_A(e),\nu_A(e)),F_A(e)):e\in E)\}$.

Example 4 Let $E=\{e_1,e_2,e_3\}$ be a set of parameters, $X=\{x_1,x_2,x_3,x_4\}$ initial universe and $F_{\tilde{A}}=\{((e_1,0.1,0.8),\{x_1,x_2\}),((e_2,0.3,0.5),\{x_3,x_4\}),\left((e_3,0.6,0.3),\{x_1,x_2,x_3\}\right)\}$. Then;

$$[F_{\tilde{A}}]=\{((e_1,0.1,0.9),\{x_1,x_2\}),((e_2,0.3,0.7),\{x_3,x_4\}),\left((e_3,0.6,0.4),\{x_1,x_2,x_3\}\right)\}$$

$\langle F_{\tilde{A}}\rangle = \left\{\left((e_1, 0.2,0.8), \{x_1, x_2\}\right), \left((e_2, 0.5,0.5), \{x_3, x_4\}\right), \left((e_3, 0.7,0.3), \{x_1, x_2, x_3\}\right)\right\}$.

Definition 11 Let $IFPS(X,E)$ and $IFPS(Y,K)$ be two families of $IFPS$ sets and $u: X \to Y$ and $p: E \to K$ be two functions. Then u_p is called a $IFPS$ mapping from $IFPS(X,E)$ to $IFPS(Y,K)$ and denoted by $u_p: IFPS(X,E) \to IFPS(Y,K)$.

1) Let $F_{\tilde{A}} \in IFPS(X,E)$ and $\tilde{C} = u(\tilde{A})$, then the image of $F_{\tilde{A}}$ under the $IFPS$ mapping u_p is $F_{\tilde{C}}$ $IFPS$ set in $IFPS(Y,K)$ defined as,

$$F_C(k) = \begin{cases} \bigcup\limits_{e \in p^{-1}(k) \cap supp\tilde{A}} u(f_A(e)) & ; p^{-1}(k) \cap supp\tilde{A} \neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$

for each $k \in K$.

2) Let $F_{\tilde{B}} \in IFPS(Y,K)$ and $\tilde{D} = u^{-1}(\tilde{B})$, then the preimage of $F_{\tilde{B}}$ under the $IFPS$ mapping u_p is $F_{\tilde{D}}$ $IFPS$ set in $IFPS(X,E)$ defined as,

$$F_D(e) = u^{-1}(F_B(p(e)))$$

for each $e \in E$.

Example 5 Let $E = \{e_1, e_2, e_3\}, K = \{k_1, k_2\}$, $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}$ and $F_{\tilde{A}} = \left\{\left((e_1, 0.3,0.5), \{x_2\}\right), \left((e_2, 0.2,0.5), \{x_2, x_3\}\right), \left((e_3, 0.6,0.1), \{x_1, x_2\}\right)\right\}$. For the function $u: X \to Y$ defined as $u(x_1) = y_2, u(x_2) = y_3$, $u(x_3) = y_2$ and the function $p: E \to K$ defined as $p(e_1) = k_3, P(e_2) = k_1, p(e_3) = k_2$, then image of the intuitionistic fuzzy parameterized soft set $F_{\tilde{A}}$ under u_p is an intuitionistic fuzzy parametrezied soft set $F_{\tilde{C}}$ in $IFPS(Y,K)$, where $p(\tilde{A}) = \tilde{C}$. Then for all $k \in K$,

$$\mu_{p(A)}(k_1) = \sup_{e \in p^{-1}(k_1)} \mu_A(e) = \mu_A(e_2) = 0.2 \quad \nu_{p(A)}(k_1) = \inf_{e \in p^{-1}(k_1)} \nu_A(e) = \nu_A(e_2) = 0.5$$
$$\mu_{p(A)}(k_2) = \sup_{e \in p^{-1}(k_2)} \mu_A(e) = \mu_A(e_3) = 0.6 \quad \nu_{p(A)}(k_2) = \inf_{e \in p^{-1}(k_2)} \nu_A(e) = \nu_A(e_3) = 0.1$$
$$\mu_{p(A)}(k_3) = \sup_{e \in p^{-1}(k_3)} \mu_A(e) = \mu_A(e_1) = 0.3 \quad \nu_{p(A)}(k_3) = \inf_{e \in p^{-1}(k_3)} \nu_A(e) = \nu_A(e_1) = 0.5$$

$P(\tilde{A}) = \tilde{C} = \{(k_1, 0.2,0.5\}, (k_2, 0.6,0.1), (k_3, 0.3,0.5\}\}$. Therefore,

$$F_C(k_1) = \bigcup_{e \in p^{-1}(k_1) \cap supp\tilde{A}} u(F_A(e)) = u(F_A(e_2)) = u(\{x_2, x_3\}) = \{y_2, y_3\}$$
$$F_C(k_2) = \bigcup_{e \in p^{-1}(k_2) \cap supp\tilde{A}} u(F_A(e)) = u(F_A(e_3)) = u(\{x_1, x_2\}) = \{y_2, y_3\}$$
$$F_C(k_3) = \bigcup_{e \in p^{-1}(k_3) \cap supp\tilde{A}} u(F_A(e)) = u(F_A(e_1)) = u(\{x_2\}) = \{y_3\}$$

Consequently,
$F_C = \left\{\left((k_1, 0.2,0.5), \{y_2, y_3\}\right), \left((k_2, 0.6,0.1), \{y_2, y_3\}\right), \left((k_3, 0.3,0.5), \{y_3\}\right)\right\}$.

On the other hand, the inverse image of the intuitionistic parameterized soft set $F_{\tilde{B}}$ under u_p is an $IFPS$ set $F_{\tilde{D}}$,where $p^{-1}(\tilde{B}) = \tilde{D}$. Then,

$$\mu_{p^{-1}(B)}(e_1) = \mu_B(p(e_1)) = \mu_B(k_3) = 0.4 \quad \nu_{p^{-1}(B)}(e_1) = \nu_B(p(e_1)) = \nu_B(k_3) = 0.5$$
$$\mu_{p^{-1}(B)}(e_2) = \mu_B(p(e_2)) = \mu_B(k_1) = 0.2 \quad \nu_{p^{-1}(B)}(e_2) = \nu_B(p(e_2)) = \nu_B(k_1) = 0.7,$$
$$\mu_{p^{-1}(B)}(e_3) = \mu_B(p(e_3)) = \mu_B(k_2) = 0.2 \quad \nu_{p^{-1}(B)}(e_3) = \nu_B(p(e_3)) = \nu_B(k_2) = 0.8$$

and so $p^{-1}(\tilde{B}) = \tilde{D} = \{(e_1, 0.4,0.5\}, (e_2, 0.2,0.7), (e_3, 0.2,0.8\}\}$. Therefore,

$$F_D(e_1) = u^{-1}\left(g_B\left(p(e_1)\right)\right) = u^{-1}\left(g_B(k_3)\right) = u^{-1}\left(\{y_1, y_3\}\right) = \{x_2\}$$

$F_D(e_2) = u^{-1}(g_B(p(e_2))) = u^{-1}(g_B(k_1)) = u^{-1}(\{y_1, y_2\}) = \{x_1, x_3\}$
$F_D(e_3) = u^{-1}(g_B(p(e_3))) = u^{-1}(g_B(k_2)) = u^{-1}(\{y_2\}) = \{x_1\},$

. Consequently, $F_{\tilde{D}} = \{((e_1, 0.4,0.5), \{x_2\}), ((e_2, 0.2,0.7), \{x_1, x_3\}), ((e_3, 0.2,0.8), \{x_1\})\}$.

Theorem 1 Let J is set of arbitrary index set, $F_{\tilde{A}}$, $F_{\tilde{A}_i} \in IFPS(X,E)$, $F_{\tilde{B}}$, $F_{\tilde{B}_i} \in IFPS(Y,K)$ and $u_p: IFPS(X,E) \to IFPS(Y,K)$ is a IFPS function. The followings are true:

(1) If $F_{\tilde{A}_1} \tilde{\subseteq} F_{\tilde{A}_2}$ then $u_p(F_{\tilde{A}_1}) \tilde{\subseteq} u_p(F_{\tilde{A}_2})$
(2) If $F_{\tilde{B}_1} \tilde{\subseteq} F_{\tilde{B}_2}$ then $u_p^{-1}(F_{\tilde{B}_1}) \tilde{\subseteq} u_p^{-1}(F_{\tilde{B}_2})$
(3) $F_{\tilde{A}} \tilde{\subseteq} u_p^{-1}(u_p(F_{\tilde{A}}))$
(4) $u_p(u_p^{-1}(F_{\tilde{B}})) \tilde{\subseteq} F_{\tilde{B}}$
(5) $u_p(\tilde{\cup}_{i\in J} F_{\tilde{A}_i}) = \tilde{\cup}_{i\in J} u_p(F_{\tilde{A}_i})$
(6) $u_p(\tilde{\cap}_{i\in J} F_{\tilde{A}_i}) \tilde{\subseteq} \tilde{\cap}_{i\in J} u_p(F_{\tilde{A}_i})$,
(7) $u_p^{-1}(\tilde{\cup}_{i\in J} F_{\tilde{B}_i}) = \tilde{\cup}_{i\in J} u_p^{-1}(F_{\tilde{B}_i})$
(8) $u_p^{-1}(\tilde{\cap}_{i\in J} F_{\tilde{B}_i}) = \tilde{\cap}_{i\in J} u_p^{-1}(F_{\tilde{B}_i})$
(9) $(u_p^{-1}(F_{\tilde{B}}))^c = u_p^{-1}(F_{\tilde{B}}^c)$
(10) $u_p^{-1}(Y_{10}) = X_{10}$
(11) $u_p^{-1}(\mathrm{Y}_{01}) = \mathrm{X}_{01}$
(12) $u_p(X_{10}) \tilde{\subseteq} Y_{10}$
(13) $u_p(\mathrm{X}_{01}) = \mathrm{Y}_{01}$.

Proof. (1) Let $F_{\tilde{B}_1} = u_p(F_{\tilde{A}_1})$ and $F_{\tilde{B}_2} = u_p(F_{\tilde{A}_2})$. Since $\tilde{B}_1 = p(\tilde{A}_1)$, $\tilde{B}_2 = p(\tilde{A}_2)$ and $\tilde{B}_1 \leq \tilde{B}_2$, it is sufficient to prove that $F_{B_1}(k) \subset F_{B_2}(k)$ for each $k \in K$.

$$F_{B_1}(k) = \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp\tilde{A}_1} u(F_{A_1}(e)) & ; p^{-1}(k) \cap supp\tilde{A}_1 \neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$\subset \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp\tilde{A}_2} u(F_{A_2}(e)) & ; p^{-1}(k) \cap supp\tilde{A}_2 \neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= F_{B_2}(k)$$

(2) Let $u_p^{-1}(F_{\tilde{B}_1}) = F_{\tilde{A}_1}$ and $u_p^{-1}(F_{\tilde{B}_2}) = F_{\tilde{A}_2}$. Since $\tilde{B}_1 = p(\tilde{A}_1)$, $\tilde{B}_2 = p(\tilde{A}_2)$ and $\tilde{B}_1 \leq \tilde{B}_2$, it is sufficient to prove that $F_{A_1}(e) \subset F_{A_2}(e)$ for each $e \in E$.

$$F_{A_1}(e) = u^{-1}(F_{B_1}(p(e)))$$
$$\subset u^{-1}(F_{B_2}(p(e)))$$
$$= F_{A_2}(e)$$

(3) Let $F_{\tilde{B}} = u_p(F_{\tilde{A}})$ and $F_{\tilde{C}} = u_p^{-1}(G_{\tilde{B}})$. Since $\tilde{A} \leq p^{-1}(p(\tilde{A})) = p^{-1}(\tilde{B}) = \tilde{C}$, it is sufficient to prove that $F_A(e) \subseteq F_C(e)$ for each $e \in E$.

$$F_C(e) = u^{-1}(F_B(p(e)))$$
$$= u^{-1}(\bigcup_{e\in p^{-1}(p(e))\cap supp\tilde{A}} u(F_A(e)))$$
$$= \bigcup_{e\in p^{-1}(p(e))\cap supp\tilde{A}} u^{-1}(u(F_A(e)))$$
$$\supseteq F_A(e)$$

This completes the proof.

(4) Let $u_p^{-1}(G_{\tilde{B}}) = F_{\tilde{A}}$ and $F_{\tilde{C}} = u_p(F_{\tilde{A}})$. Since $\tilde{B} \geq p(p^{-1}(\tilde{B})) = p(\tilde{A}) = \tilde{C}$ it is sufficient to prove that $F_C(k) \subseteq F_B(k)$ for each $k \in K$.

$$F_C(k) = \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp\tilde{A}} u(F_A(e)) & ; p^{-1}(k)\cap supp\tilde{A}\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \begin{cases} \bigcup_{e\in p^{-1}(k)\cap suppp^{-1}(\tilde{B})} u(u^{-1}(F_B(p(e)))) & ; p^{-1}(k)\cap supp(p^{-1}(\tilde{B}))\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$\subseteq \begin{cases} \bigcup_{e\in p^{-1}(k)\cap p^{-1}(supp\tilde{B})} F_B(p(e)) & ; p^{-1}(k)\cap p^{-1}(supp\tilde{B})\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \begin{cases} \bigcup_{e\in p^{-1}(k)\cap p^{-1}(supp\tilde{B})} F_B(p(e)) & ; p(e)=k\in supp\tilde{B} \\ \varnothing & ; \text{other} \end{cases}$$
$$= F_B(k)$$

(5) Let $F_{\tilde{B}_i} = u_p(F_{\tilde{A}_i})$ and $F_{\tilde{B}} = u_p(\tilde{\cup}_{i\in J}(F_{\tilde{A}_i}))$.Then, $\tilde{B} = p(\vee \tilde{A}_i) = \vee p(\tilde{A}_i) = \vee \tilde{B}_i$ and for each $k\in K$,

$$F_B(k) = \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp \bigvee_{i\in J}\tilde{A}_i} u(\bigcup_{i\in J} F_{A_i}(e)) & ; p^{-1}(k)\cap supp \bigvee_{i\in J}\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp \bigvee_{i\in J}\tilde{A}_i \bigcup_{i\in J}} u(F_{A_i}(e)) & ; p^{-1}(k)\cap supp \bigvee_{i\in J}\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \begin{cases} \bigcup_{i\in J}\left(\bigcup_{e\in p^{-1}(k)\cap supp\tilde{A}_i} u\left(F_{A_i}(e)\right)\right) & ; p^{-1}(k)\cap supp\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \bigcup_{i\in J} \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp\tilde{A}_i} u(F_{A_i}(e)) & ; p^{-1}(k)\cap supp\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \cup_{i\in J} F_{B_i}(k)$$

This completes the proof.

(6) Let $F_{\tilde{B}_i} = u_p(F_{\tilde{A}_i})$ and $F_{\tilde{B}} = u_p(\tilde{\cap}_{i\in J}(F_{\tilde{A}_i}))$. Then $\tilde{B} = p(\wedge \tilde{A}_i) \leq \wedge p(\tilde{A}_i) = \wedge \tilde{B}_i$ and for each $k\in K$,

$$F_B(k) = \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp \bigwedge_{i\in J}\tilde{A}_i} u(\bigcap_{i\in J} F_{A_i}(e)) & ; p^{-1}(k)\cap supp \bigwedge_{i\in J}\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$\subseteq \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp \bigwedge_{i\in J}\tilde{A}_i}\left(\bigcap_{i\in J} u\left(F_{A_i}(e)\right)\right) & ; p^{-1}(k)\cap supp \bigwedge_{i\in J}\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \begin{cases} \bigcap_{i\in J}\left(\bigcup_{e\in p^{-1}(k)\cap\tilde{A}_i} u\left(F_{A_i}(e)\right)\right) & ; p^{-1}(k)\cap supp\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \bigcap_{i\in J} \begin{cases} \bigcup_{e\in p^{-1}(k)\cap supp\tilde{A}_i} u(F_{A_i}(e)) & ; p^{-1}(k)\cap supp\tilde{A}_i\neq \varnothing \\ \varnothing & ; \text{other} \end{cases}$$
$$= \cap_{i\in J} F_{B_i}(k)$$

This completes the proof.

(7) Let $F_{\tilde{A}_i} = u_p^{-1}(F_{\tilde{B}_i})$ and $F_{\tilde{A}} = u_p^{-1}(\tilde{\cup}_{i\in J} F_{\tilde{B}_i})$.Then, $\tilde{A} = p^{-1}(\vee \tilde{B}_i) = \vee p^{-1}(\tilde{B}_i) = \vee \tilde{A}_i$ and for each $e\in E$,

$$F_A(e) = u^{-1}(\bigcup_{i\in J} F_{B_i}(p(e)))$$
$$= \bigcup_{i\in J} u^{-1}(F_{B_i}(p(e)))$$
$$= \bigcup_{i\in J} F_{A_i}(e)$$

This completes the proof.

(8) Let $F_{\tilde{A}_i} = u_p^{-1}(F_{\tilde{B}_i})$ and $F_{\tilde{A}} = u_p^{-1}(\tilde{\cap}_{i\in J} F_{\tilde{B}_i})$.Then, $\tilde{A} = p^{-1}(\wedge \tilde{B}_i) = \wedge p^{-1}(\tilde{B}_i) = \wedge \tilde{A}_i$ and for each $e\in E$,

$$\begin{aligned} F_A(e) &= u^{-1}(\underset{i\in J}{\cap} F_{B_i}(p(e))) \\ &= \underset{i\in J}{\cap} u^{-1}(F_{B_i}(p(e))) \\ &= \underset{i\in J}{\cap} F_{A_i}(e) \end{aligned}$$

This completes the proof.

(9) Let $u_p^{-1}(F_{\tilde{B}}) = F_{\tilde{A}}$ and $u_p^{-1}(F_{\tilde{B}}^c) = H_{\tilde{C}}$. Then, $\tilde{A} = p^{-1}(\tilde{B})$ and $\tilde{C} = p^{-1}(\tilde{B}^c)$ are fuzzy sets in E and for each $e \in E$, $F_C(e) = u^{-1}(F_B^c(p(e)) = u^{-1}(Y \backslash F_B(p(e)) = X \backslash u^{-1}(F_B(p(e)) = X \backslash F_A(e) = F_{A^c}(e)$.This shows that the approximation functions of $F_{\tilde{C}}$ and $F_{\tilde{A}}^c$ are the same. Thus the proof is completed.

(10) Let $F_{\tilde{A}} = u_p^{-1}(Y_{10})$. Then $f_A(e) = u^{-1}(Y_{10}(p(e))) = u^{-1}(Y) = X$ for each E. This shows that $F_{\tilde{A}} = X_{10}$.

(11) Since $p^{-1}(K)$ is a fuzzy empty set, then the proof is clear.

(12) Let $G_{\tilde{B}} = u_p(X_{10})$. In this situation $\tilde{B} = p(\tilde{E}) \leq \tilde{K}$. Moreover, for each $k \in K$,

$$\begin{aligned} F_B(k) &= \begin{cases} \underset{e\in p^{-1}(k)\cap supp 1_E}{\cup} u(X_{10}(e)) & ; p^{-1}(k) \cap supp\tilde{1}_E \neq \varnothing \\ \varnothing & ; \text{other} \end{cases} \\ &= \begin{cases} \underset{e\in p^{-1}(k)}{\cup} u(X) & ; p^{-1}(k) \neq \varnothing \\ \varnothing & ; \text{other} \end{cases} \\ &\subset Y_{10}(k) \end{aligned}$$

(13) Since $p(\tilde{0}_E) = \tilde{0}_K$, the proof is clear.

Definition 12 Let E be a set of parameters, X be the initial universe and $\tau \subseteq IFPS(X, E)$. If τ satisfies the following conditions, then τ is called an $IFPS$ topology.

1) $X_{01}, X_{10} \in \tau$

2) If $F_{\tilde{A}}, F_{\tilde{B}} \in \tau$, then $F_{\tilde{A}} \tilde{\cap} F_{\tilde{B}} \in \tau$.

3) If $\{F_{\tilde{A}_i}\}_{i\in I} \in \tau$, $\underset{i\in I}{\tilde{\cup}} F_{\tilde{A}_i} \in \tau$.

Here, the triple (X, τ, E) is called an intuitionistic fuzzy parameterized soft topological space. Each element of τ is called an $IFPS$ open set ($IFPOS$). The sets whose complement is $IFPOS$ are called $IFPS$ closed sets ($IFPCS$). The family of all $IFPS$ closed sets in the space (X, τ, E) is denoted by τ^k.

Example 7 *Let* $E = \{e_1, e_2, e_3\}, X = \{x_1, x_2, x_3\}$. *Then*

$F_{\tilde{A}} = \{((e_1, 0.2, 0.5), \{x_1, x_2\}), ((e_2, 0.3, 0.6), X)\}$

$F_{\tilde{B}} = \{((e_1, 0.4, 0.6), \{x_1\}), ((e_2, 0.1, 0.5), \{x_2, x_3\}), ((e_3, 0.3, 0.5), \{x_1)\}$

$F_{\tilde{C}} = \{((e_1, 0.2, 0.6), \{x_1\}), ((e_2, 0.1, 0.6), \{x_2, x_3\})\}$

$F_{\tilde{D}} = \{((e_1, 0.4, 0.5), \{x_1, x_2\}), ((e_2, 0.3, 0.5), X), ((e_3, 0.3, 0.5), \{x_1\})\}$

. The family $\tau = \{X_{01}, X_{10}, F_{\tilde{A}}, F_{\tilde{B}}, F_{\tilde{C}}, F_{\tilde{D}}\}$ is an $IFPS$ topology on $IFPS(X, E)$.

Theorem 2 Let E be a set of parameters, X be the initial universe, and (X, τ, E) be an $IFPS$ topological space. Then the followings are true.

1) $X_{01}, X_{10} \in \tau^k$

2) If $F_{\tilde{A}}, F_{\tilde{B}} \in \tau^k$, then $F_{\tilde{A}} \tilde{\cup} F_{\tilde{B}} \in \tau^k$.

3) If $\{F_{\tilde{A}_i}\}_{i\in I} \in \tau^k$, then $\underset{i\in I}{\tilde{\cap}} F_{\tilde{A}_i} F_{A_i} \in \tau^k$.

Proof. The proof can be done easily.

Example 8 Let us consider the topology τ in the Example 7 . Then;

$F_{\tilde{A}}^c = \{((e_1, 0.5, 0.2), \{x_3\}), ((e_3, 1, 0), X)\}$

$F_{\tilde{B}}^c = \{((e_1, 0.6, 0.4), \{x_2, x_3\}), ((e_2, 0.5, 0.1), \{x_1\}), ((e_3, 0.5, 0.3), \{x_2, x_3\})\}$

$F_{\tilde{C}}^c = \{((e_1, 0.6, 0.2), \{x_2, x_3\}), ((e_2, 0.6, 0.1), \{x_1\}), ((e_3, 1, 0), X)\}$

$F_{\tilde{D}}^{c} = \{((e_1, 0.5,0.4), \{x_3\}), ((e_3, 0,0.3), \{x_2, x_3\})\}$.
Thus $\tau^k = \{X_{01}, X_{10}, F_{\tilde{A}}^{c}, F_{\tilde{B}}^{c}, F_{\tilde{C}}^{c}, F_{\tilde{D}}^{c}\}$.

Definition 13 Let E be a set of parameters, X be an initial universe and (X, τ_1, E), (X, τ_2, E) be two $IFPS$ topological space. In this situation, If $\tau_1 \subseteq \tau_2$, τ_1 is said to be coarser than τ_2 or τ_2 is said to be finer than τ_1.

Example 9 Let $E = \{e_1, e_2, e_3\}, X = \{x_1, x_2, x_3\}$. Then,
$F_{\tilde{A}} = \{((e_1, 0.3,0.4), \{x_1, x_2\}), ((e_2, 0.5,0.2), \{x_2\})\}$
$F_{\tilde{B}} = \{((e_1, 0.4,0.6), \{x_2\}), ((e_2, 0.1,0.5), \{x_1, x_3\}), ((e_3, 0.3,0.5), \{x_1, x_2\})\}$
$F_{\tilde{C}} = \{((e_1, 0.3,0.4), X), ((e_2, 0.5,0.3), X), ((e_3, 0.3,0), \{x_1, x_2\})\}$
$F_{\tilde{D}} = \{((e_1, 0.4,0.6), X), ((e_2, 0.5,0.4), X), ((e_3, 0.4,0.5), X)\}$
. Thus $\tau_1 = \{X_{01}, X_{10}, F_{\tilde{A}}, F_{\tilde{B}}, F_{\tilde{C}}\}$ and $\tau_2 = \{X_{01}, X_{10}, F_{\tilde{A}}, F_{\tilde{B}}, F_{\tilde{C}}, F_{\tilde{D}}\}$ families are topologies. Moreover, $\tau_1 \subseteq \tau_2$. So τ_2 is coarser than τ_1 or τ_2 is finer than τ_1.

Proposition 1 Let E be a set of parameters, X be the initial universe, (X, τ, E) be an $IFPS$ topological space. Then

$$[\tau] = \{[F_{\tilde{A}}]: F_{\tilde{A}} \in \tau\}$$
$$\langle\tau\rangle = \{\langle F_{\tilde{A}}\rangle: F_{\tilde{A}} \in \tau\}$$

families are $IFPS$ topologies.

Proof. Let's show that $[\tau]$ is a $IFPS$ topology. Similarly, it can be shown that $\langle\tau\rangle$ is also a topology.
$IFPST$ 1) Since $[X_{01}] = X_{01}$, $[X_{10}] = X_{10}$, $X_{01}, X_{10} \in \tau]$.
$IFPST$ 2) Let $[F_{\tilde{A}}], [F_{\tilde{B}}] \in [\tau]$ Then there are $IFPS$ sets such that $F_{\tilde{A}}, F_{\tilde{B}} \in \tau$. Therefore $F_{\tilde{A}} \,\tilde{\cap}\, F_{\tilde{B}} \in \tau$. On the other hand;

$$\begin{aligned}
[F_{\tilde{A}}] \,\tilde{\cap}\, [F_{\tilde{B}}] &= \{(e, \mu_A(e), 1-\mu_A(e), F_A(e)): e \in E)\} \,\tilde{\cap}\, \\
&\quad \{(e, \mu_B(e), 1-\mu_B(e), F_B(e)): e \in E)\} \\
&= \left\{\begin{pmatrix} e, \min\{\mu_A(e), \mu_B(e)\}, \max\{1-\mu_A(e), 1-\mu_B(e)\}, \\ F_A(e) \cap F_B(e)): e \in E \end{pmatrix}\right\} \\
&= \left\{\begin{pmatrix} e, \min\{\mu_A(e), \mu_B(e)\}, 1-\min\{\mu_A(e), \mu_B(e)\}, \\ F_A(e) \cap F_B(e)): e \in E \end{pmatrix}\right\} \\
&= [F_{\tilde{A}} \,\tilde{\cap}\, F_{\tilde{B}}]
\end{aligned}$$

Consequently, $[F_{\tilde{A}}] \,\tilde{\cap}\, [F_{\tilde{B}}] \in [\tau]$
$IFPST$ 3) Let $\left[F_{\tilde{A}_i}\right] \in [\tau]$ for all $i \in I$. In this situation, there are $IFPS$ sets such that $F_{\tilde{A}_i} \in \tau$. In this situation $\underset{i\in I}{\cup} F_{\tilde{A}_i} \in \tau$. On the other hand;

$$\underset{i\in I}{\tilde{\cup}} \left[F_{\tilde{A}_i}\right] = \left\{\left(e, \max\{\mu_{A_i}(e)\}, \min\{1-\mu_{A_i}(e)\}, \underset{i\in I}{\cup} F_{A_i}\right): e \in E\right)\right\} = \left[\underset{i\in I}{\tilde{\cup}} F_{\tilde{A}_i}\right].$$

Consequently, $\underset{i\in I}{\tilde{\cup}} \left[F_{\tilde{A}_i}\right] \in [\tau]$.

Definition 14 Let E be a set of parameters, X be an initial universe, (X, τ, E) $IFPS$ topological space and $F_A \in IFPS(X, E)$. Then;

$$\overline{F_{\tilde{A}}} = \tilde{\cap}\{F_{\tilde{B}}: F_{\tilde{B}} \in \tau^k \text{ ve } F_{\tilde{A}} \,\tilde{\subseteq}\, F_{\tilde{B}}\}$$
$$F_{\tilde{A}}^{\circ} = \tilde{\cup}\{F_{\tilde{B}}: F_{\tilde{B}} \in \tau \text{ ve } F_{\tilde{B}} \,\tilde{\subseteq}\, F_{\tilde{A}}\}$$

These sets are called $IFPS$ closure and $IFPS$ interior of $F_{\tilde{A}}$, respectively.

Proposition 2 Let E be set of parameters, X be an initial universe, (X, τ, E) be $IFPS$ topological space. Then;
1) $F_{\tilde{A}} \in \tau^k \Leftrightarrow F_{\tilde{A}} = \overline{F_{\tilde{A}}}$
2) $F_{\tilde{A}} \in \tau \Leftrightarrow F_{\tilde{A}} = F_{\tilde{A}}^{\circ}$

Proof. 1) ($:\Rightarrow$) Let $F_{\tilde{A}} \in \tau^k$ Here, $\overline{F_{\tilde{A}}}$ is the intersection of all $IFPS$ closed sets covering $F_{\tilde{A}}$ and since $F_{\tilde{A}} \in \tau^k$, $F_{\tilde{A}} = \overline{F_{\tilde{A}}}$.

(:⇐) Let $F_{\tilde{A}} = \overline{F_{\tilde{A}}}$. $\overline{F_{\tilde{A}}}$ is $IFPS$ closed because it is the intersection of $IFPS$ closed sets

2) (⇒:) Let $F_{\tilde{A}} \in \tau$. Since $F_{\tilde{A}}^{\circ}$ is the union of all $IFPS$ open subsets $F_{\tilde{A}}$, $F_{\tilde{A}} = F_{\tilde{A}}^{\circ}$.

(:⇐) Let $F_{\tilde{A}} = F_{\tilde{A}}^{\circ}$. Then $F_{\tilde{A}} \in \tau$.

Example 10 Let's consider the topology τ in Example 7. For

$$F_{\tilde{S}} = \left\{\left((e_1, 0.3,0.5), \{x_1, x_2\}\right), \left((e_2, 0.3,0.6), X\right), \left((e_3, 0.2,0.7), \{x_2, x_3\}\right)\right\}$$

$IFPS$ set in $IFPS(X,E)$. Then,

$F_{\tilde{S}}^{\circ} = \tilde{\cup}\{F_{\tilde{T}} : F_{\tilde{T}} \in \tau \text{ and } F_{\tilde{T}} \tilde{\subseteq} F_{\tilde{S}}\} = F_{\tilde{A}} \tilde{\cup} X_{01} = F_{\tilde{A}}$

$\overline{F_{\tilde{S}}} = \tilde{\cap}\{F_{\tilde{P}} : F_{\tilde{P}} \in \tau^k \text{ and } F_{\tilde{S}} \tilde{\subseteq} F_{\tilde{P}}\} = X_{10}$.

Theorem 3 Let E be a set of parameters, X be an initial universe, (X,τ,E) $IFPS$ topological space and $F_A \in IFPS(X,E)$. Then,

1) $(\overline{F_{\tilde{A}}})^c = (F_{\tilde{A}}^c)^{\circ}$.

2) $(F_{\tilde{A}}^{\circ})^c = \overline{(F_{\tilde{A}})}$.

Proof. 1)

$$\begin{aligned}(\overline{F_{\tilde{A}}}) &= (\tilde{\cap}\{F_{\tilde{B}} : F_{\tilde{B}} \in \tau^k \text{and } F_{\tilde{A}} \tilde{\subseteq} F_{\tilde{B}}\})^c \\ &= \tilde{\cup}\{F_{\tilde{B}}^c : F_{\tilde{B}} \in \tau^k \text{ and } F_{\tilde{A}} \tilde{\subseteq} F_{\tilde{B}}\} \\ &= \tilde{\cup}\{F_{\tilde{B}}^c : F_{\tilde{B}} \in \tau^k \text{and } F_{\tilde{B}}^c \tilde{\subseteq} F_{\tilde{A}}^c\} \\ &= \tilde{\cup}\{F_{\tilde{C}} : F_{\tilde{C}} \in \tau \text{ and } F_{\tilde{C}} \tilde{\subseteq} F_{\tilde{A}}^c\} \\ &= (F_{\tilde{A}}^c)^{\circ}\end{aligned}$$

2) The proof can be done similarly to 1).

Theorem 4 Let E be set of parameters, X be an initial universe, (X,τ,E) be $IFPS$ topological space. Then,

1) $\overline{X_{10}} = X_{10}$ and $\overline{X_{01}} = X_{01}$

2) If $F_{\tilde{A}} \in (X,\tau,E)$, then $F_{\tilde{A}} \subseteq \overline{F_{\tilde{A}}}$.

3) If $F_{\tilde{A}}, F_{\tilde{B}} \in (X,\tau,E)$ and $F_{\tilde{A}} \subseteq F_{\tilde{B}}$, then $\overline{F_{\tilde{A}}} \subseteq \overline{F_{\tilde{B}}}$

Proof. It can be easily shown from the closure definition.

Proposition 4 Let (X,τ,E) be an $IFPS$ topological space, $F_{\tilde{A}}$ and $F_{\tilde{B}}$ $IFPS$ sets in X. Then,

a) $F_{\tilde{A}}^{\circ} \tilde{\subseteq} F_{\tilde{A}}$ and $F_{\tilde{A}} \tilde{\subseteq} \overline{F_{\tilde{A}}}$.

b) $F_{\tilde{A}} \tilde{\subseteq} F_{\tilde{B}}$ $F_{\tilde{A}}^{\circ} \tilde{\subseteq} F_{\tilde{B}}^{\circ}$ and $\overline{F_{\tilde{A}}} \tilde{\subseteq} \overline{F_{\tilde{B}}}$.

c) $\left(F_{\tilde{A}}^{\circ}\right)^{\circ} = F_{\tilde{A}}^{\circ}$ and $\overline{\overline{F_{\tilde{A}}}} = \overline{F_{\tilde{A}}}$.

d) $(F_{\tilde{A}} \tilde{\cap} F_{\tilde{B}})^{\circ} = F_{\tilde{A}}^{\circ} \tilde{\cap} F_{\tilde{B}}^{\circ}$ and $\overline{(F_{\tilde{A}} \tilde{\cup} F_{\tilde{B}})} = \overline{F_{\tilde{A}}} \tilde{\cup} \overline{F_{\tilde{B}}}$.

Proof. a) It can be easily shown from the interior and closure definitions.

b) Let $F_{\tilde{A}} \tilde{\subseteq} F_{\tilde{B}}$. Then ;

$$\begin{aligned}F_{\tilde{A}}^{\circ} &= \tilde{\cup}\{F_{\tilde{S}} : F_{\tilde{S}} \in \tau \text{ and } F_{\tilde{S}} \tilde{\subseteq} F_{\tilde{A}}\} \\ &\tilde{\subseteq} \tilde{\cup}\{F_{\tilde{S}} : F_{\tilde{S}} \in \tau \text{ and } F_{\tilde{S}} \tilde{\subseteq} F_{\tilde{B}}\} \\ &= F_{\tilde{B}}^{\circ}\end{aligned}$$

Similarly, it can be shown that $\overline{F_{\tilde{A}}} \tilde{\subseteq} \overline{F_{\tilde{B}}}$.

c) Since the proposition is from 2 and $\overline{F_{\tilde{A}}}$ is closed, then $\overline{\overline{F_{\tilde{A}}}} = \overline{F_{\tilde{A}}}$. Similarly, it can be shown that $\left(F_{\tilde{A}}^{\circ}\right)^{\circ} = F_{\tilde{A}}^{\circ}$.

d) Since $(F_{\tilde{A}} \tilde{\cap} F_{\tilde{B}})^{\circ} \subseteq F_{\tilde{A}}^{\circ}$ and $(F_{\tilde{A}} \tilde{\cap} F_{\tilde{B}})^{\circ} \subseteq F_{\tilde{B}}^{\circ}$, then $(F_{\tilde{A}} \tilde{\cap} F_{\tilde{B}})^{\circ} \subseteq F_{\tilde{A}}^{\circ} \tilde{\cap} F_{\tilde{B}}^{\circ}$. On the otherhand, Since $F_{\tilde{A}}^{\circ} \subseteq F_{\tilde{A}}$ and $F_{\tilde{B}}^{\circ} \subseteq F_{\tilde{B}}$, then $F_{\tilde{A}}^{\circ} \tilde{\cap} F_{\tilde{B}}^{\circ} \subseteq F_{\tilde{A}} \cap F_{\tilde{B}}$. Moreover, since $F_A^{\circ} \cap G_B^{\circ} \in \tau$, then $F_{\tilde{A}}^{\circ} \tilde{\cap} F_{\tilde{B}}^{\circ} \subseteq (F_{\tilde{A}} \tilde{\cap} F_{\tilde{B}})^{\circ}$.This completes the proof.

Definition 15 Let (X,τ,E) and (Y,σ,K) $IFPS$ topological spaces, $u_p : IFPS(X,E) \to IFPS(Y,K)$ $IFPS$ function. If $u_p^{-1}(F_{\tilde{B}}) \in \tau$ for all $F_{\tilde{B}} \in \sigma$, then u_p is said to be $IFPS$ continuous.

Example 11 Let us consider u_p $IFPS$ function in Example 5 and $\tau = \{X_{01}, X_{10}, F_{\tilde{B}}\}$ and $\sigma = \{Y_{01}, Y_{10}, F_{\tilde{B}}\}$ be two topological spaces. Then u_p is continuous.

Theorem 5 Let $(X,\tau),(Y,\sigma)$ be two $IFPS$ and u_p: $(X,\tau) \to (Y,\sigma)$be a intuitionistic fuzzy parameterized function. Then the following are equivalent:

(1) u_p is $IFPS$ continuous;

(2) $u_p^{-1}(F_{\tilde{B}})$ $IFPS$ is closed, for all $F_{\tilde{B}} \in \sigma^k$;

(3) $u_p(\overline{F_{\tilde{A}}}) \cong \overline{u_p(F_{\tilde{A}})}$, for all $F_{\tilde{A}} \in IFPS(X,E)$;

(4) $\overline{u_p^{-1}(F_{\tilde{B}})} \cong u_p^{-1}(\overline{F_{\tilde{B}}})$ for all $F_{\tilde{B}} \in IFPS(Y,K)$;

(5) $u_p^{-1}(F_{\tilde{B}}^{\circ}) \cong (u_p^{-1}(F_{\tilde{B}}))^{\circ}$ for all $F_{\tilde{B}} \in IFPS(Y,K)$;

Proof. (1)⇒(2) Let u_p be $IFPS$ continuous and $F_{\tilde{K}} \in \sigma^k$. Then $F_{\tilde{K}}^c \in \sigma$. $u_p^{-1}(F_{\tilde{K}}^c) \in \tau$ since $IFPS$ is continuous. From here $(u_p^{-1}(F_{\tilde{K}}))^c \in \tau$. This gives $u_p^{-1}(F_{\tilde{K}}) \in \tau^k$.

(2)⇒(3) Let $F_{\tilde{A}} \in IFPS(X,E)$. From the proposition 1 3) $F_{\tilde{A}} \cong u_p^{-1}(u_p(F_{\tilde{A}}))$. On the otherhand, since $u_p(F_{\tilde{A}}) \cong \overline{u_p(F_{\tilde{A}})}$ and from Proposition 1 (2), $F_{\tilde{A}} \cong u_p^{-1}(u_p(F_{\tilde{A}})) \cong u_p^{-1}(\overline{u_p(F_{\tilde{A}})}) \in \tau^k$. From proposition 1 (1), It becomes $u_p(\overline{F_{\tilde{A}}}) \cong u_p(u_p^{-1}(\overline{u_p(F_{\tilde{A}})}) \cong \overline{u_p(F_{\tilde{A}})}$, this means it is $u_p(\overline{F_{\tilde{A}}}) \cong \overline{u_p(F_{\tilde{A}})}$.

(3)⇒(4) Let $F_{\tilde{B}} \in IFPS(Y,K)$. If $u_p^{-1}(F_{\tilde{B}})$ is taken instead of $F_{\tilde{A}}$ in (3), $u_p(\overline{u_p^{-1}(F_{\tilde{B}})}) \cong \overline{u_p(u_p^{-1}(F_{\tilde{B}}))} \cong \overline{F_{\tilde{B}}}$ is obtained. This shows that $\overline{u_p^{-1}(F_{\tilde{B}})} \cong u_p^{-1}(\overline{F_{\tilde{B}}})$.

(4)⇔(5) It is clear from proposition 1 (9) and Theorem 3.

(5)⇒(1) Let $F_{\tilde{B}} \in \sigma$. Then $u_p^{-1}(F_{\tilde{B}}) = u_p^{-1}(F_{\tilde{B}}^{\circ}) \cong (u_p^{-1}(F_{\tilde{B}}))^{\circ} \cong u_p^{-1}(F_{\tilde{B}})$. Since $F_{\tilde{B}}$ is a $IFPS$ open set, from (5). This gives that $u_p^{-1}(F_{\tilde{B}})$ is an $IFPS$ open set. Consequently, the function u_p is $IFPS$ continuous.

Conflicts of Interest

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