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MODIFIED Z-ITERATIVE METHOD FOR SIGNAL ENHANCEMENT AND IMAGE DEBLURRING

SİNYAL GELİŞTİRME VE GÖRÜNTÜ BULANIKLIĞINI GİDERME İÇİN DEĞİŞTİRİLMİŞ Z-İTERATİF YÖNTEMİ

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Abstract

This study introduces a novel iterative method for nonexpansive mappings in uniform convex Banach spaces. Weak and strong convergence theorems are derived under specific assumptions. To substantiate the theoretical findings, numerical examples are presented, and the fixed point of the nonexpansive mapping is approximated computationally using Matlab R2016a. In addition, the paper discusses the applications of the proposed method in the fields of image deblurring and signal enhancement. Image deblurring techniques aim to reduce the blurriness in an image and restore it to a sharper and more distinct form, whereas signal enhancement involves reducing noise to improve the clarity of a signal, thereby enhancing the signal-to-noise ratio. The main contribution of this work lies in the development of a novel algorithm for both image deblurring and signal enhancement, providing an effective approach to address these challenges.

Keywords: Fixed points, iterative method, image deblurring, signal enhancement.

Öz

Bu çalışma, düzgün dışbükey Banach uzaylarında genişlemeyen eşlemeler için yeni bir yinelemeli yöntem sunmaktadır. Zayıf ve güçlü yakınsama teoremleri belirli varsayımlar altında türetilmiştir. Teorik bulguları desteklemek için sayısal örnekler sunulmuş ve genişlemeyen eşlemenin sabit noktası Matlab R2016a kullanılarak hesaplamalı olarak yaklaşık olarak hesaplanmıştır. Ek olarak, makalede önerilen yöntemin görüntü bulanıklığını giderme ve sinyal iyileştirme alanlarındaki uygulamaları tartışılmaktadır. Görüntü bulanıklığını giderme teknikleri, bir görüntüdeki bulanıklığı azaltmayı ve daha keskin ve daha belirgin bir forma geri getirmeyi amaçlarken, sinyal iyileştirme, bir sinyalin netliğini iyileştirmek için gürültüyü azaltmayı ve böylece sinyal-gürültü oranını iyileştirmeyi içerir. Bu çalışmanın temel katkısı, hem görüntü bulanıklığını giderme hem de sinyal iyileştirme için yeni bir algoritmanın geliştirilmesinde yatmaktadır ve bu zorlukların ele alınması için etkili bir yaklaşım sağlamaktadır.

Anahtar Kelimeler: Sabit noktalar, yinelemeli yöntem, görüntü bulanıklığı giderme, sinyal iyileştirme.

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1. INTRODUCTION

Let $\mathbb{Q}$ be a nonvoid subset of a Banach space $\wp$. A mapping $\Upsilon: \mathbb{Q} \to \mathbb{Q}$ is said to be nonexpansive if $\|\Upsilon x - \Upsilon y\| \leq \|x - y\|$ for every $x, y \in \mathbb{Q}$. A point $\sigma^* \in \mathbb{Q}$ is a fixed point of Υ iff $\Upsilon\sigma^* = \sigma^*$. Let $F_\Upsilon := \{\sigma^* \in \mathbb{Q}: \Upsilon\sigma^* = \sigma^*\}$. In the last twenty-five years, the theory of fixed points for nonexpansive mappings has attracted considerable scholarly attention, owing to its vast range of applications in fields such as integral equations, differential equations, convex optimization, signal processing, and various other fields.

Browder pioneered the study of fixed point existence for nonexpansive mappings in Hilbert spaces (Browder, 1965) Subsequently, Göhde extended this result to uniformly convex Banach spaces (Göhde, 1965), while Goebel and Kirk further broadened Browder's findings, applying them to reflexive Banach spaces (Goebel and Kirk, 1990). Since then, numerous studies have examined various extensions and generalizations of the nonexpansive mappings.

Noteworthy iterative techniques are widely applied to approximate the fixed points of specific mappings. The Picard method (Picard, 1890), proposed by Picard, is regarded as the most fundamental and widely used approach in fixed point theory and is identified by $x_0 \in \mathbb{Q}$

$$x_n = \Upsilon x_{n-1} = \Upsilon^n x_0, \;\; n \geq 1. \tag{1}$$

Picard's method is effective for obtaining fixed points of contraction-type mappings; however, it becomes impractical when applied to nonexpansive mappings. Therefore, for nonexpansive operators, the Mann iteration method (Mann, 1953) is often employed, as it provides a more general framework compared to Picard's method (1). This method was characterized by an initial point $x_0 \in \mathbb{Q}$ and was further defined by

$$x_{n+1} = (1 - \varrho_n)x_n + \varrho_n \Upsilon x_n \,, \; n \geq 1, \tag{2}$$

where $\{\varrho_n\} \subseteq (0,1)$. The convergence rates of both schemes (1) and (2) are generally slow, and it is typically challenging to determine a common fixed point. Due to the inadequacy of the iteration scheme given by equation (2) for Lipschitzian and pseudocontractive mappings, the Ishikawa iteration method (Ishikawa, 1974) was identified by an initial point $x_0 \in \mathbb{Q}$ as follows

$$\begin{cases} x_{n+1} = (1 - \varrho_n)x_n + \varrho_n \Upsilon y_n \,, \\ y_n = (1 - \zeta_n)x_n + \zeta_n \Upsilon x_n, \;\; , \; n \geq 1, \end{cases} \tag{3}$$

where $\{\varrho_n\}, \{\zeta_n\} \subseteq (0,1)$.
Researchers developed the following S −iteration method with the initial point $x_0 \in \mathbb{Q}$ as follows:

$$\begin{cases} x_{n+1} = (1 - \varrho_n)\Upsilon x_n + \varrho_n \Upsilon y_n \,, \\ y_n = (1 - \zeta_n)x_n + \zeta_n \Upsilon x_n, \;\; n \geq 1, \end{cases} \tag{4}$$

where $\{\varrho_n\},\{\zeta_n\} \subseteq (0,1)$ (Agarwal et al., 2007). Authors established that this iteration for contraction mappings converges at the same rate as the method (1), yet achieves faster convergence than the method (2).

Authors presented a new iteration scheme with the initial point $x_0 \in \mathbb{Q}$ as follows:

$$\begin{cases} x_{n+1} = (1-\varrho_n)\Upsilon y_n + \varrho_n \Upsilon z_n\,, \\ y_n = (1-\zeta_n)\Upsilon x_n + \zeta_n \Upsilon z_n\,, \\ z_n = (1-\vartheta_n)x_n + \vartheta_n \Upsilon x_n, \quad n \geq 1, \end{cases} \tag{5}$$

where $\{\varrho_n\},\{\zeta_n\},\{\vartheta_n\} \subseteq (0,1)$ (Abbas & Nazir, 2014). The iteration method defined by (5) is faster than Picard, Mann and $S-$iteration.

Researchers presented an innovative iterative algorithm designed to approximate the fixed point of Suzuki's generalized nonexpansive mappings as follows (Thakur et al., a2016):

$$\begin{cases} x_{n+1} = \Upsilon y_n\,, \\ y_n = \Upsilon[(1-\varrho_n)x_n + \varrho_n z_n], \\ z_n = (1-\zeta_n)x_n + \zeta_n \Upsilon x_n\,, \quad n \geq 1, \end{cases} \tag{6}$$

where $\{\varrho_n\},\{\zeta_n\} \subseteq (0,1)$. Authors established that the proposed iteration (6) not only converges at a faster rate than the previously known iterations but also exhibits robustness and stability with regard to the parameter variations.

Authors introduce a novel three-step iterative process, referred to as the $M-$iteration process, for the approximation of fixed points as follows:

$$\begin{cases} x_{n+1} = \Upsilon y_n\,, \\ y_n = \Upsilon z_n, \\ z_n = (1-\varrho_n)x_n + \varrho_n \Upsilon x_n\,, \quad n \geq 1, \end{cases} \tag{7}$$

where $\{\varrho_n\} \subseteq (0,1)$ (Ullah and Arshad, 2018). The authors conducted a numerical comparison to assess the convergence speed of the $M-$iteration process in relation to established iteration methods.

Researchers presented a novel iterative algorithm, termed $Z-$ iteration process, as follows:

$$\begin{cases} c_n = \Upsilon k_n, \\ k_n = (1-\varrho_n)x_n + \varrho_n \Upsilon x_n, \\ b_n = \Upsilon f_n, \\ f_n = (1-\zeta_n)c_n + \zeta_n \Upsilon c_n\,, \\ x_{n+1} = \Upsilon b_n, \quad n \geq 1 \end{cases} \tag{8}$$

where $\{\varrho_n\},\{\zeta_n\} \subseteq (0,1)$ (Srivastava et al., 2024). The authors illustrated that the proposed iteration scheme exhibited faster convergence compared to the current

modern iteration schemes. The authors proposed a new iterative algorithm based on the Chatterjea-Suzuki-C condition and gave convergence theorems.
Building on the above work, we devised a modified $Z-$iteration process containing a nonexpansive mapping, by generating the sequence $\{x_n\}$ as follows:

$$\begin{cases} c_n = \Upsilon k_n, \\ k_n = (1-\varrho_n)x_n + \varrho_n \Upsilon x_n, \\ b_n = \Upsilon[(1-\theta_n)x_n + \theta_n f_n], \\ f_n = (1-\zeta_n)c_n + \zeta_n \Upsilon c_n\,, \\ x_{n+1} = \Upsilon[(1-\gamma_n)x_n + \gamma_n b_n],\ \ n \geq 1 \end{cases} \tag{9}$$

where $\{\varrho_n\}, \{\zeta_n\}, \{\theta_n\}, \{\gamma_n\} \subseteq (0,1)$.

If $\theta_n = \gamma_n \equiv 1$ for $n \geq 1$, then (9) reduces to the $Z-$ iteration process (8).

The structure of this paper is outlined as follows: Section 2 provides the necessary preliminary definitions and lemmas. Section 3 offers a comprehensive convergence analysis of (9) in the context of a nonexpansive mapping within Banach spaces. Sections 4 is focused on the practical applications of the proposed methodology, specifically addressing signal recovery and image deblurring. Section 5, the Conclusion is presented.

In order to examine the theoretical foundation and practical relevance of our approach, we begin with some essential definitions and lemmas in the next part.

2. PRELIMINARIES

In this section, we gather essential concepts and important conclusions that will be regularly applied.

From this point on, $\rightarrow$ ($\rightharpoonup$) will represent the strong (weak) convergence.

Definition 2.1. (Senter, 1974) The mapping $\Upsilon: \mathbb{Q} \rightarrow \mathbb{Q}$ with $F_\Upsilon \neq \emptyset$ is said to satisfy *condition* (I) if there exists a nondecreasing function $f: [0,\infty) \rightarrow [0,\infty)$ with $f(r) > 0$, $f(0) = 0$, for all $r > 0$ such that $\|x - \Upsilon x\| \geq f\big(d(x, F_\Upsilon)\big)$ for all $x \in \mathbb{Q}$, where $d(x, F_\Upsilon) = \inf_{y \in F_\Upsilon} d(x, y)$.

Definition 2.2. (Opial, 1967) A Banach space $\wp$ said to satisfy *Opial's property* if for any sequence $\{x_n\}$ in $\wp$, $x_n \rightharpoonup x$ that $liminf_{n\to\infty}\|x_n - x\| < liminf_{n\to\infty}\|x_n - y\|$ for all $y \in \wp$ and $y \neq x$.

Definition 2.3. (Agarwal et al., 2009) Let $\wp$ be real Banach space, $\Upsilon: \mathbb{Q} \rightarrow \mathbb{Q}$ be a mapping with $F_\Upsilon \neq \emptyset$. Then Υ is said to be *demiclosed* at $q^* \in \wp$ if, for all $\{t_n\} \subseteq \mathbb{Q}$ such that $\{t_n\} \rightharpoonup p^* \in Q$, $\{\Upsilon t_n\} \rightarrow q^*$ imply $\Upsilon p^* = q^*$.

Lemma 2.4. (Tan & Xu, 1993) Let $\{e_n\}$, $\{v_n\}$ and $\{z_n\}$ be sequences of nonnegative real numbers such that

$$e_{n+1} \leq (1 + z_n)e_n + v_n$$

for every $n \geq 1$, and $\sum_{n=1}^{\infty} \upsilon_n < \infty$ and $\sum_{n=1}^{\infty} z_n < \infty$, then $lim_{n\to\infty} e_n$ exists. Additionally, if $\{e_n\}$ contains a subsequence $\{e_{n_k}\} \to 0$, then $lim_{n\to\infty} e_n = 0$.

Lemma 2.5. (Chidume et al., 2003) Let $\wp$ be a uniformly convex Banach space and $\mathbb{Q}$ be a nonvoid, closed, convex subset of a Banach space $\wp$. If $\Upsilon : \mathbb{Q} \to \wp$ is an asymptotically nonexpansive mapping, $(I - \Upsilon)$ is demiclosed at zero.

Lemma 2.6. (Cho et al., 2004.) A Banach space $\wp$ is uniformly convex iff if there is a continuous, strictly increasing and convex function g: $[0, \infty) \to [0, \infty)$, $g(0) = 0$ such that

$$\|\varrho\mathfrak{u} + \zeta\mathfrak{v} + \vartheta\mathfrak{w}\|^2 \leq \varrho\|\mathfrak{u}\|^2 + \zeta\|\mathfrak{v}\|^2 + \vartheta\|\mathfrak{w}\|^2 - \varrho\zeta g(\|\mathfrak{u} - \mathfrak{v}\|),$$

for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in B_r$ and $\varrho, \zeta, \vartheta \subset [0,1]$ with $\varrho + \zeta + \vartheta = 1$, where $B_r = \{x \in \wp: \|x\| \leq r\}$.

The mathematical formulation of the proposed method has been thoroughly discussed; however, providing a brief overview of the underlying principles of signal processing and image deblurring is essential to offer context and bridge the gap between theory and application.

Image deblurring is a classical inverse problem in signal and image processing, aiming to recover a sharp image from a blurred observation caused by various factors (Wang & Tao, 2014). It is typically modeled as a convolution with a blur kernel plus noise (Ji & Wang, 2011). Deblurring methods range from classical Wiener filtering to modern variational and sparse representation techniques (Wiener,1949).

Signal recovery from degraded or incomplete data is a fundamental topic in signal processing (Vaseghi, 2008). Methods including Fourier and wavelet transforms, regularization, and compressed sensing have substantially advanced reconstruction and denoising, underpinning deblurring methodologies (Levy, 1946; Grossmann & Morlet, 1984).

Drawing from these fields, the proposed method utilizes mathematical models to improve deblurring quality, emphasizing the integration of theoretical and practical considerations.

3. MAIN RESULTS

This part is devoted to establishing the main convergence results of the proposed iterative method for nonexpansive mappings. We begin with key lemma and continue with the main convergence theorems under appropriate assumptions.

Lemma 3.1. Let $\mathbb{Q}$ be a nonvoid closed convex subset of a real normed space $\wp$. Let $\Upsilon: \mathbb{Q} \to \mathbb{Q}$ be a nonexpansive mapping. Assume that $\{x_n\}$ given by (9), where $\{\varrho_n\}$, $\{\zeta_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$ be in $(0,1)$. Suppose $F_\Upsilon \neq \emptyset$ and

$$\begin{cases} liminf_{n\to\infty}\zeta_n > 0, \\ liminf_{n\to\infty}\theta_n > 0, \\ liminf_{n\to\infty}\gamma_n > 0, \\ 0 < liminf_{n\to\infty}\varrho_n < limsup_{n\to\infty}\varrho_n < 1. \end{cases} \quad (10)$$

Then,

(i) $lim_{n\to\infty}\|x_n - \sigma^+\|$ exists for all $\sigma^+ \in F_\Upsilon$;
(ii) $lim_{n\to\infty}\|x_n - \Upsilon x_n\| = 0$.

Proof. (i) We can choose a $\sigma^+ \in F_\Upsilon$. Using (9), we attain

$$\|k_n - \sigma^+\| \le \|(1-\varrho_n)x_n + \varrho_n \Upsilon x_n - \sigma^+\| \le \|x_n - \sigma^+\|, \tag{11}$$

$$\|c_n - \sigma^+\| \le \|\Upsilon k_n - \sigma^+\| \le \|k_n - \sigma^+\| \le \|x_n - \sigma^+\|, \text{ (by (11))}. \tag{12}$$

It follows from (9) and (12) that

$$\begin{aligned}
&\|b_n - \sigma^+\| \le \|\Upsilon[(1-\theta_n)x_n + \theta_n f_n] - \sigma^+\| \\
&\le (1-\theta_n)\|x_n - \sigma^+\| + \theta_n\|f_n - \sigma^+\| \\
&\le (1-\theta_n)\|x_n - \sigma^+\| \\
&+\theta_n\|[(1-\zeta_n)c_n + \zeta_n \Upsilon c_n] - \sigma^+\| \\
&\le (1-\theta_n)\|x_n - \sigma^+\| + \theta_n\|c_n - \sigma^+\| \\
&\le (1-\theta_n)\|x_n - \sigma^+\| + \theta_n\|x_n - \sigma^+\| \; (by\ (12)) \\
&\le \|x_n - \sigma^+\|.
\end{aligned} \tag{13}$$

From (9) and (13), we own

$$\begin{aligned}
\|x_{n+1} - \sigma^+\| &\le \|\Upsilon[(1-\gamma_n)x_n + \gamma_n b_n] - \sigma^+\| \\
&\le (1-\gamma_n)\|x_n - \sigma^+\| + \gamma_n\|b_n - \sigma^+\| \\
&\le \|x_n - \sigma^+\| \; \big(\text{by (13)}\big).
\end{aligned} \tag{14}$$

We can infer that $\{\|x_n - \sigma^+\|\}$ is nonincreasing and bounded. This means that $lim_{n\to\infty}\|x_n - \sigma^+\|$ exists.

(ii) It follows from (9) and Lemma 2.6 that

$$\begin{aligned}
&\|c_n - \sigma^+\|^2 \le \|\Upsilon k_n - \sigma^+\|^2 \\
&\le \|k_n - \wp\|^2 \\
&\le (1-\varrho_n)\|x_n - \wp\|^2 + \varrho_n\|\Upsilon x_n - \wp\|^2 \\
&-\varrho_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n - x_n\|) \\
&\le (1-\varrho_n)\|x_n - \sigma^+\|^2 - \varrho_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n - x_n\|),
\end{aligned} \tag{15}$$

and by (15),

$$\begin{aligned}
\|f_n - \sigma^+\|^2 &\le (1-\zeta_n)\|c_n - \sigma^+\|^2 + \zeta_n\|\Upsilon c_n - \sigma^+\|^2 \\
&-\zeta_n(1-\zeta_n)\mathscr{g}_2(\|\Upsilon c_n - c_n\|) \\
&\le \|c_n - \sigma^+\|^2 \\
&\le (1-\varrho_n)\|x_n - \sigma^+\|^2 - \varrho_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n - x_n\|).
\end{aligned} \tag{16}$$

Thus, owing to (9), (16) and Lemma 2.6 we have

$$\begin{aligned}
&\|b_n-\sigma^+\|^2 \\
&\le \|\Upsilon[(1-\theta_n)x_n+\theta_n f_n]-\sigma^+\|^2 \\
&\le \|[(1-\theta_n)x_n+\theta_n f_n]-\sigma^+\|^2 \\
&\le (1-\theta_n)\|x_n-\sigma^+\|^2+\theta_n\|f_n-\sigma^+\|^2 \\
&-\theta_n(1-\theta_n)\mathscr{g}_3(\|f_n-x_n\|) \\
&\le (1-\theta_n)\|x_n-\sigma^+\|^2+\theta_n\|f_n-\sigma^+\|^2 \\
&\le (1-\theta_n)\|x_n-\sigma^+\|^2 \\
&+\theta_n[(1-\varrho_n)\|x_n-\sigma^+\|^2-\varrho_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n-x_n\|)](by\ (16)) \\
&\le \|x_n-\sigma^+\|^2-\varrho_n\theta_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n-x_n\|),
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\|x_{n+1}-\sigma^+\|^2 \\
&\le \|\Upsilon[(1-\gamma_n)x_n+\gamma_n b_n]-\sigma^+\|^2 \\
&\le \|[(1-\gamma_n)x_n+\gamma_n b_n]-\sigma^+\|^2 \\
&\le (1-\gamma_n)\|x_n-\sigma^+\|^2+\gamma_n\|b_n-\sigma^+\|^2 \\
&-\gamma_n(1-\gamma_n)\mathscr{g}_4(\|x_n-b_n\|) \\
&\le (1-\gamma_n)\|x_n-\sigma^+\|^2+\gamma_n\|b_n-\sigma^+\|^2 \\
&\le (1-\gamma_n)\|x_n-\sigma^+\|^2 \\
&+\gamma_n[\|x_n-\sigma^+\|^2-\varrho_n\theta_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n-x_n\|)](by\ (17)) \\
&\le \|x_n-\sigma^+\|^2-\varrho_n\theta_n\gamma_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n-x_n\|).
\end{aligned} \tag{18}$$

When the necessary adjustments are made in (18), we obtain the following expression

$$\varrho_n\theta_n\gamma_n(1-\varrho_n)\mathscr{g}_1(\|\Upsilon x_n-x_n\|)\le \|x_n-\sigma^+\|^2-\|x_{n+1}-\sigma^+\|^2.$$

By the assumption in (10), there exists a positive integer n_0 and $\aleph,\aleph^*\in(0,1)$ such that

$$0<\aleph<\zeta_n,\ 0<\aleph<\theta_n,\ 0<\aleph<\gamma_n \text{ and } \varrho_n<\aleph^*<1\ ,\ n_0\le n.$$

Then,

$$\aleph^3(1-\aleph)\mathscr{g}_1(\|\Upsilon x_n-x_n\|)\le \|x_n-\sigma^+\|^2-\|x_{n+1}-\sigma^+\|^2,\ n_0\le n. \tag{19}$$

It follows from (19) that for $n_0\le\omega$

$$\begin{aligned}
\sum_{n=n_0}^{\omega}\mathscr{g}_1(\|\Upsilon x_n-x_n\|) &\le \frac{1}{\aleph^3(1-\aleph)}\left\{\sum_{n=n_0}^{\omega}(\|x_n-\sigma^+\|^2-\|x_{n+1}-\sigma^+\|^2)\right\} \\
&\le \frac{1}{\aleph^3(1-\aleph)}\left\|x_{n_0}-\sigma^+\right\|^2.
\end{aligned}$$

Then $\sum_{n=n_0}^{\omega}\mathscr{g}_1(\|\Upsilon x_n-x_n\|)<\infty$, thus $lim_{n\to\infty}\mathscr{g}_1(\|\Upsilon x_n-x_n\|)=0$. Since $\mathscr{g}$ is a continuous and strictly increasing with $\mathscr{g}(0)=0$, we have $lim_{n\to\infty}\|x_n-\Upsilon x_n\|=0$.

Theorem 3.2. Let $\mathbb{Q}$ be a nonvoid closed convex subset of a real uniformly convex Banach space $\wp$. Let $\Upsilon: \mathbb{Q} \to \mathbb{Q}$ be a nonexpansive mapping. Assume that $\{x_n\}$ given by (9). If $\wp$ is equipped with *Opial's property* and $F_\Upsilon \neq \emptyset$, then $x_n \rightharpoonup \sigma^+$.

Proof. Due to $F_\Upsilon \neq \emptyset$, we can conclude using Lemma 3.1 that $\{x_n\}$ is bounded and $lim_{n\to\infty}\|x_n - \Upsilon x_n\| = 0$. As $\wp$ is a real uniformly convex Banach space, we may assume that x_n converges weakly σ^+ as $n \to \infty$. By Lemma 3.1, we get $\sigma^+ \in F_\Upsilon$. Let $\left\{x_{n_g}\right\}$ and $\left\{x_{m_g}\right\}$ be subseqeunces of $\{x_n\}$, such that x_{n_g} and x_{m_g} converge weakly σ^+ and σ^* respectively as $g \to \infty$. We assume $\sigma^+ \neq \sigma^*$, from Lemma 2.5 and 3.1, we get $\sigma^+, \sigma^* \in F_\Upsilon$. Then, by Opial's property, we obtain $lim_{n\to\infty}\|x_n - \sigma^*\| = lim_{g\to\infty}\left\|x_{n_g} - \sigma^*\right\| < lim_{g\to\infty}\left\|x_{n_g} - \sigma^+\right\| = lim_{g\to\infty}\left\|x_{m_g} - \sigma^+\right\| < lim_{g\to\infty}\left\|x_{m_g} - \sigma^*\right\| = lim_{n\to\infty}\|x_n - \sigma^*\|$. Since this is a contradiction, we attain $\sigma^+ = \sigma^*$. Therefore, $x_n \rightharpoonup \sigma^+$.

Theorem 3.3. Let $\mathbb{Q}$ be a nonvoid closed convex subset of a real uniformly convex Banach space $\wp$. Let $\Upsilon: \mathbb{Q} \to \mathbb{Q}$ be a nonexpansive mapping. Assume that $\{x_n\}$ given by (9). If Υ satisfies *Condition (I)* and $F_\Upsilon \neq \emptyset$, then $x_n \to \sigma^+$.

Proof. Owing to Lemma 3.1 (i), we notice that $lim_{n\to\infty}\|x_n - \sigma^+\|$ exists for all $\sigma^+ \in F_\Upsilon$, therefore $lim_{n\to\infty}d(x_n, F_\Upsilon)$ exists. Suppose that $lim_{n\to\infty}\|x_n - \sigma^+\| = \Lambda$ for some $\Lambda \geq 0$. For $\Lambda = 0$, the result is obviously true. Next, if $\Lambda > 0$, then by the assumption and *Condition (I)*, we get

$$\|x_n - \Upsilon x_n\| \geq f\big(d(x_n, F_\Upsilon)\big). \tag{20}$$

From Lemma 3.1 (ii), we own

$$lim_{n\to\infty}\|x_n - \Upsilon x_n\| = 0. \tag{21}$$

Since f is nondecreasing function, so by using (21) with (20), we have $lim_{n\to\infty}f\big(d(x_n, F_\Upsilon)\big) = 0$.

From above, we have two subsequences $\left\{x_{n_g}\right\}$ of $\{x_n\}$ and $\{t_g\} \subset F_\Upsilon$ such that $\left\|x_{n_g} - t_g\right\| \leq 2^{-g}$ for all $g \in N$. Using Lemma 3.1 (i), we obtain

$$\left\|x_{n_{g+1}} - t_g\right\| \leq \left\|x_{n_g} - t_g\right\| \leq 2^{-g}.$$

Hence

$$\|t_{g+1} - t_g\| \leq \left\|t_{g+1} - x_{n_{g+1}}\right\| + \left\|x_{n_{g+1}} - t_g\right\| \leq 2^{-g-1} + 2^{-g} \to 0 \; as \; n \to \infty.$$

Which implies that $\{t_g\}$ is a Cauchy sequence in F_Υ, so it converges to same σ^+. As F_Υ is closed, so $\sigma^+ \in F_\Upsilon$ and then $x_{n_g} \to \sigma^+$. Since Lemma 3.1 (i), $lim_{n\to\infty}\|x_n - \sigma^+\|$ exists, we have $x_n \to \sigma^+ \in F_\Upsilon$.

Theorem 3.4. Let $\mathbb{Q}$ be a nonvoid closed convex subset of a real uniformly convex Banach space $\wp$. Let $\Upsilon: \mathbb{Q} \to \mathbb{Q}$ be a nonexpansive mapping. Assume that $\{x_n\}$ given by (9). Then $liminf_{n\to\infty} d(x_n, F_\Upsilon) = 0$ iff $\{x_n\}$ converges to a point of F_Υ, where $d(x_n, F_\Upsilon) = inf\{\|x_n - \sigma^*\|: \sigma^* \in F_\Upsilon\}$.

Proof. If the sequence $\{x_n\}$ converges to a point $\sigma^* \in F_\Upsilon$, then it is obvious that $liminf_{n\to\infty} d(x_n, F_\Upsilon) = 0$.

Next, for the first part taking $liminf_{n\to\infty} d(x_n, F_\Upsilon) = 0$ for any fixed point $\sigma^* \in F_\Upsilon$. From Lemma 3.1 (i), $lim_{n\to\infty}\|x_n - \sigma^*\|$ exists for any $\sigma^* \in F_\Upsilon$; thus, $lim_{n\to\infty} d(x_n, F_\Upsilon) = 0$.

Now, by assertion is that $\{x_n\}$ is a Cauchy sequence in $\wp$. As $lim_{n\to\infty} d(x_n, F_\Upsilon) = 0$, and for a given $\varepsilon > 0$, there exists $\sigma_0^* \in Z_0^+$ such that for any $n \geq \sigma_0^*$, $d(x_n, F_\Upsilon) < \frac{\varepsilon}{2}$, and so

$$inf\{\|x_n - \sigma^*\|: \sigma^* \in F_\Upsilon\} < \frac{\varepsilon}{2}.$$

Herefrom,

$$inf\left\{\left\|x_{\sigma_0^*} - \sigma^*\right\|: \sigma^* \in F_\Upsilon\right\} < \frac{\varepsilon}{2}.$$

Therefore, there exists $\sigma^* \in F_\Upsilon$ such that $\left\|x_{\sigma_0^*} - \sigma^*\right\| < \frac{\varepsilon}{2}$.

Now, for $m, n \geq \sigma_0^*$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - \sigma^*\| + \|x_n - \sigma^*\| \leq 2\left\|x_{\sigma_0^*} - \sigma^*\right\| < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in $\wp$. Since $\wp$ is closed, $lim_{n\to\infty} x_n = \sigma^+$ for some $\sigma^+ \in \wp$. Next, $lim_{n\to\infty} d(x_n, F_\Upsilon) = 0$ implies $d(\sigma^+, F_\Upsilon) = 0$; hence, we have $\sigma^+ \in F_\Upsilon$.

Example 3.5. Let $\wp$ be the real line with the usual norm $|.|$, $\mathbb{Q} = (-1,1)$. Assume that $\Upsilon x = -x$ for $x \in \mathbb{Q}$. Then Υ is a nonexpansive mapping. Clearly, $F_\Upsilon = \{0\}$. Set

$$\varrho_n = \zeta_n = \theta_n = \gamma_n = \frac{21}{66}.$$

In order to easily calculate, we modifed our iteration scheme for $n = 1$. This scheme (9) is defined as follows: $x_0 \in \mathbb{Q}$

$$\begin{cases} c_0 = \Upsilon k_0, \\ k_0 = \left(1 - \frac{21}{66}\right) x_0 + \frac{21}{66} \Upsilon x_0, \\ b_0 = \Upsilon\left[\left(1 - \frac{21}{66}\right) x_0 + \frac{21}{66} f_0\right], \\ f_0 = \left(1 - \frac{21}{66}\right) c_0 + \frac{21}{66} \Upsilon c_0 \,, \\ x_1 = \Upsilon\left[\left(1 - \frac{21}{66}\right) x_0 + \frac{21}{66} b_0\right]. \end{cases}$$

Analogically, we can calculate $x_2, x_3, \cdots, x_n, \cdots$. All numerical computations were carried out using Matlab R2016a. We give the first twenty values of $\{x_n\}$ as in the Table 1 below for the initial term $x_0 = -0{,}40000$ and $x_0 = 0{,}70000$, resp. With help of the Table 1, we obtain that $x_n \to 0{,}00000$.

Table 1. Numeric Outcomes of the Proposed Itarative Method with Initial Guesses

Item	$x_0 = -0{,}40000$	$x_0 = 0{,}70000$
1	$0{,}1913$	$-0{,}3348$
2	$-0{,}0915$	$0{,}1601$
3	$0{,}0438$	$-0{,}0766$
4	$-0{,}0209$	$0{,}0366$
5	$0{,}0100$	$-0{,}0175$
6	$-0{,}0048$	$0{,}0084$
7	$0{,}0023$	$-0{,}0040$
8	$-0{,}0011$	$0{,}0019$
9	$5{,}2367e-04$	$-9{,}1643e-04$
10	$-2{,}5045e-04$	$4.3830e-04$
11	$1{,}1978e-04$	$-2{,}0962e-04$
12	$1{,}1978e-04$	$1{,}0025e-04$
13	$-5{,}7288e-05$	$-4.7948e-05$
14	$2{,}7399e-05$	$2{,}2932e-05$
15	$-1{,}3104e-05$	$-1{,}0967e-05$
16	$6{,}2670e-06$	$5{,}2453e-06$
17	$-2{,}9973e-06$	$-2{,}5086e-06$
18	$1{,}4335e-06$	$1{,}1998e-06$
19	$-6{,}8559e-07$	$-5{,}7381e-07$
20	$3{,}2789e-07$	$2{,}7443e-07$

Example 3.6. (Thakur et al., b2016) Let $\wp$ be the real line with the usual norm $|.|$, $\mathbb{Q} = [5{,}37]$. Assume that $\Upsilon x = \sqrt{54 - 9x + x^2}$ for $x \in \mathbb{Q}$. Then Υ is a map. Clearly, $F_\Upsilon = \{6\}$. Set

$$\varrho_n = \zeta_n = \theta_n = \gamma_n = \frac{8}{14}.$$

In the Table 2, we give the first ten values of the array for the initial term $x_0 = 9{,}1567, x_0 = 27{,}9813, x_0 = 35{,}0471$.

Table 2. Numerical Results of the Proposed Iterative Method with Initial Estimates

Item	$x_0 = 9{,}1567$	$x_0 = 27{,}9813$	$x_0 = 35{,}0471$
1	6,6338	27,9813	35,0471
2	6,0878	19,6952	26,2692
3	6,0111	12,5933	18,1535
4	6,0014	7,9532	11,4201
5	6,0002	6,3321	7,4246
6	6,0000	6,0437	6,2238
7	6,0000	6,0055	6,0289
8	6,0000	6,0007	6,0036
9	6,0000	6,0001	6,0004
10	6,0000	6,0000	6,0001

By means of the Table 2, we get $x_n \to 6$. This means that the Theorem 3.4 is applicable.

Example 3.7. Let $\wp$ be the real line with the usual norm $|.|$, $\mathbb{Q} = [0, \infty)$. Assume that $\Upsilon x = sinx$ for $x \in \mathbb{Q}$. Then Υ is a map. Clearly, $F_\Upsilon = \{0\}$. Set

$$\varrho_n = \zeta_n = \theta_n = \gamma_n = \frac{8}{14}.$$

In the Table 3 presents a comparison of the convergence rates to the fixed point using the Mann iteration, S-iteration, Abbas & Nazir iteration method, and the modified Z-iteration.

Table 3. Comparison of Convergence Rates

n	Iteration (2)	Iteration (4)	Iteration (5)	Iteration (9)
1	$\pi/3$	$\pi/3$	$\pi/3$	$\pi/3$
2	0,9437	0,8338	0,7801	0,7774
3	0,8671	0,7195	0,6549	0,6452
4	0,8073	0,6439	0,5770	0,5632
5	0,7588	0,5888	0,5223	0,5061
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0,6050	0,4389	0,3801	0,3610
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0,2222	0,1478	0,1243	0,1143
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
300	0,1308	0,0863	0,0723	0,0662
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	0,1018	0,0670	0,0561	0,0513

For the initial term $x_0 = {}^{\pi}/_{3}$, based on the data in the Table 3, it has been observed that the modified Z-iteration is faster than the aforementioned iterations.

4. APPLICATIONS OF THE MODIFIED Z-ITERATIVE METHOD

In this section, we present the applications of the proposed algorithm in the contexts of signal enhancement and image deblurring.

4.1. Application to Signal Enhancement

Signal enhancement is a technique used to minimize noise, thereby improving the clarity of the signal and enhancing the signal-to-noise ratio (Vandeginste et al., 1998). Signal enhancement plays a crucial role in various engineering disciplines, particularly in areas such as image analysis, audio processing, medical imaging, and telecommunications (Vaseghi, 2009). To address the challenges of complex signal enhancement, iterative methods are commonly employed (Byrne, 2004). These methods are essential for optimizing processes and improving the precision of solutions in signal enhancement (Cadzow, 1988). Inspired by these principles, we developed a program that generates a noisy signal based on a sine wave, applies an iterative algorithm to enhance the signal, and visualizes the original, noisy, and enhanced signals using Matlab R2016a. The step-by-step explanation of the code is as follows:

1. Step: Creation of the Original Signal (clean sine wave) and Noisy Signal (sine wave with Gaussian noise added).
2. Step: Application of a New Iterative Signal Enhancement Algorithm.
3. Step: Visualization of the Outputs.

Let's take a sequence α_n which is equal to other sequences ($\{\varrho_n\}$, $\{\zeta_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$).

In this context, x_n refers to the initial signal, and α_n represents the learning rate that regulates the balance between the contribution of the previous iteration and the new signal. This rate controls the interaction between the current and previous signals. If the learning rate is too low, the noise reduction process may slow down or fail to produce the desired results (see Figure 1).

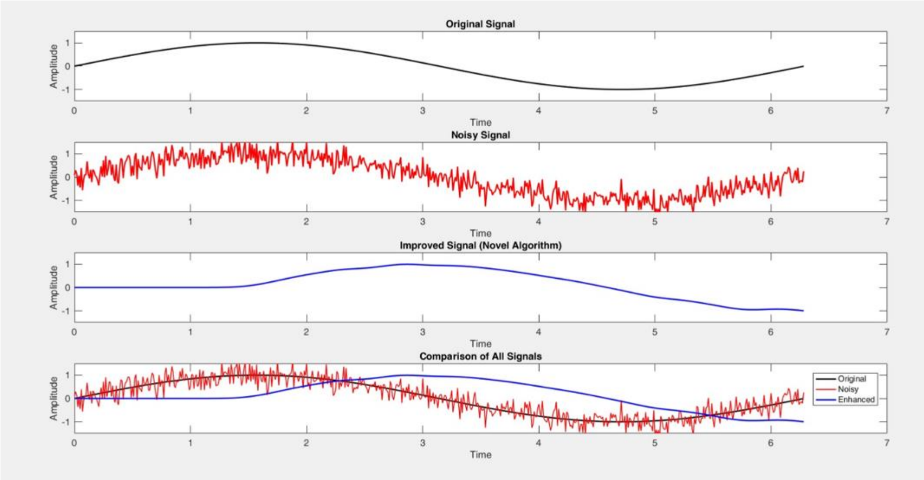

Figure 1. When $\alpha_n = 0{,}3$ we observe the Original/Noisy/Enhanced Signals.

Iterative methods update the signal in a balanced manner by assigning equal importance to the previous and current signals, resulting in a smoother and more gradual enhancement (see Figure 2).

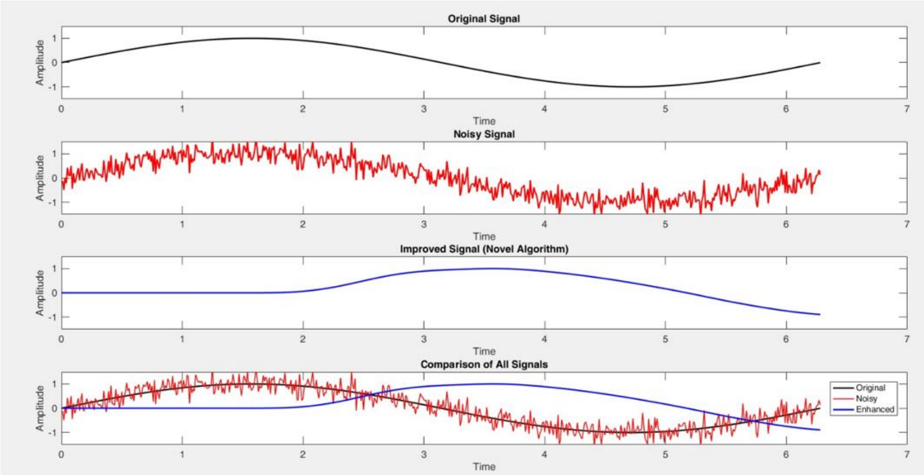

Figure 2. When $\alpha_n = 0{,}5$ we observe the Original/Noisy/Enhanced Signals.

When the learning rate is too high, the algorithm may overcorrect and miss the optimal solution (see Figure 3).

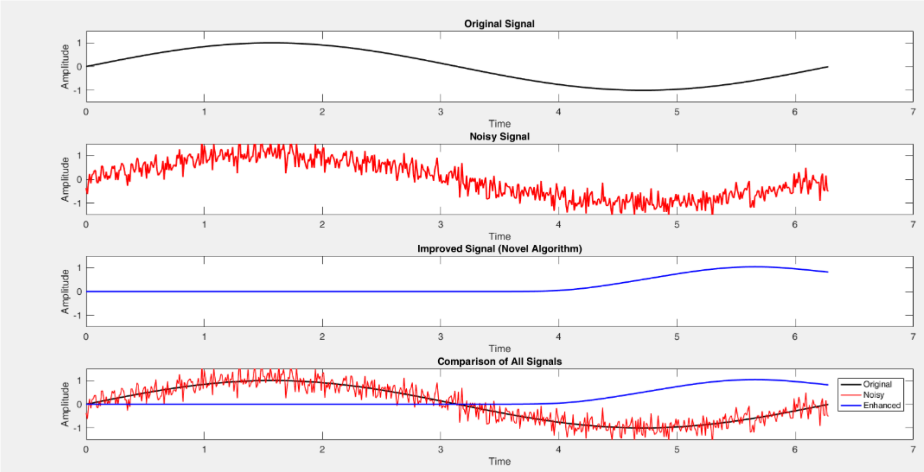

Figure 3. When $\alpha_n = 0{,}9$, we observe the Original/Noisy/Enhanced Signals.

The gradual enhancement of the signal demonstrates the algorithm's effective and stable noise reduction process. As the signal improves, it becomes progressively clearer and eventually stabilizes, becoming close to the original. This indicates that the algorithm is functioning correctly, removing noise without over-smoothing the signal (Proakis & Manolakis, 2007; Haykin, 2002).

On the other hand, critical points such as how the method performs under high noise levels and the computational complexity of the algorithm are not discussed here. Under high noise conditions, the signal conditioning capacity of the algorithm may decrease or longer iterations may be required. In addition, due to the filtering and iterative conditioning processes, the processing power, time, and resources (CPU, memory, etc.) required for the algorithm to run increases, meaning that the algorithm performs more computations. This can be a disadvantage in real-time applications or large datasets. Explicitly stating and working on these limitations will allow us to better understand the practical application areas and performance limits of the method.

4.2. Application to Image Deblurring

The goal of deblurring an image is to obtain a sharp image from a blurry one; blur is often caused by factors such as the rapid movement of the object (or camera shake) or poor focusing (Lee, 1990; Navarro et al., 2011) The primary objective of these algorithms is to progressively enhance the image in each iteration, reduce the blur, and gradually approach the original sharp version (Peters, 1995; Biemond et al., 1990).

We developed a code in Matlab R2016a that applies an iterative deblurring algorithm to a motion-blurred image. This algorithm aims to recover the original image from a blurry one and improves its prediction with each iteration. The reconstructed images, using the proposed algorithm, is shown after 50th iterations (Unblurred Image with New Iteration) (see Figure 4).

In conclusion, these algorithms can be applied in various fields; however, more complex types of blur may require additional techniques and parameter adjustments to achieve effective results. Careful selection of parameters and updating of the code will enhance the accuracy of the results Fergus et al., 2006).

Figure 4. Visualization of the Result After the 50th Iteration

5. CONCLUSION

This study presents a novel iterative technique for nonexpansive mappings in uniformly convex Banach spaces. Weak and strong convergence theorems are established under certain conditions. To support the theoretical results, numerical examples are provided, and the fixed point of the nonexpansive mapping is approximated computationally using Matlab R2016a.

Signal enhancement and iterative algorithms, including techniques such as Bayesian enhancement, adaptive methods, and filtering, are widely used in modern engineering applications to improve signal quality. These methods are particularly crucial in reducing noise and distortions and enhancing the accuracy of solutions. On the other hand, iterative methods form the foundation of the image deblurring process. These methods offer an effective way to reduce blur and obtain a result closer to the original by gradually improving the image. Starting from the initial blurred image, they work to enhance and sharpen the output with each iteration. Both signal enhancement and image restoration techniques will contribute to progress in a wide range of applications, including image processing, space imaging, medical imaging, and computer vision.

As future work, the method could be extended to more general settings, such as nonlinear mappings or other classes of Banach spaces beyond uniform convexity. Additionally, incorporating blur kernel estimation and adaptive parameter optimization directly into the algorithm would greatly improve its practical usability in real-world imaging applications. Such developments would open new avenues in both the theoretical study of iterative methods and their application to increasingly complex signal enhancement problems.

Authors Contribution

Hüseyin Gül contributed for Methodology, Investigation, Writing – original draft, Reviewing, Esra Yolaçan contributed for Methodology, Software, Investigation, Writing – original draft, Reviewing.

Competing Interest
The authors declare that they have no competing interests related to the content of this manuscript.

Ethical and Informed Consent for Data Used
Ethical approval and informed consent for data use were not required for this research, as the study did not involve human subjects, personal data, or sensitive information. The data utilized were obtained from publicly available and anonymized sources, and all aspects of the research adhered to ethical standards and legal requirements.

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