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Research Article

Bayesian analysis of the beta regression model subject to linear inequality restrictions with application

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Abstract

Recent studies in machine learning are based on models in which parameters or state variables are restricted by a restricted boundedness. These restrictions are based on prior information to ensure the validity of scientific theories or structural consistency based on physical phenomena. The valuable information contained in the restrictions must be considered during the estimation process to improve the accuracy of the estimation. Many researchers have focused on linear regression models subject to linear inequality restrictions, but generalized linear models have received little attention. In this paper, the parameters of beta Bayesian regression models subjected to linear inequality restrictions are estimated. The proposed Bayesian restricted estimator, which is demonstrated by simulated studies, outperforms ordinary estimators. Even in the presence of multicollinearity, it outperforms the ridge estimator in terms of the standard deviation and the mean squared error. The results confirm that the proposed Bayesian restricted estimator makes sparsity in parameter estimating without using the regularization penalty. Finally, a real data set is analyzed by the new proposed Bayesian estimation method.

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1. Introduction

Beta regression models (BRMs), proposed by [14], have become a popular choice for analyzing response variables that fall within the range $(0, 1)$. Various applications of BRMs have been studied for instance the percentages of body fat [13], the proportion of crude oil after distillation and fractionation [32], the color characteristics of hazelnuts [19]. Typically, the maximum likelihood estimator is used to estimate the model parameters. For recent developments in parameter estimation methods for beta regression models (BRMs), see [2, 4, 5, 25, 37]. However, in certain applications, incorporating prior information about

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the parameters into the model can be advantageous. This prior information can enhance the accuracy of the estimation process. It is typically introduced into the model in the form of constraints, linear or non-linear, and may involve equality or inequality conditions.

In BRMs, linear equality restrictions have been studied by [6,34], who aimed to improve the Beta Liu Estimator (BLE) and the Beta maximum likelihood estimator (BMLE), respectively, by using shrinkage methods such as James-Stein, positive James-Stein, and preliminary test approaches. Compared to equality restrictions, there are situations where it becomes necessary to impose linear inequality constraints on regression parameters. These constraints help maintain structural consistency with physical phenomena or ensure the validity of scientific theories. For example, in applied econometrics, certain regression coefficients may be required to be non-negative or non-positive [7, 31]. In hyperspectral imaging, physical considerations often demand that coefficients be non-negative [26]. Similar applications of inequality restrictions can be found in astronomy and zoology [41], as well as in geodesy [42].

In the classical framework, linear regression models subject to linear inequality restrictions have been extensively studied (see [11,12,18,23,29]). More recently, in the era of big data, it has been shown that incorporating non-negativity constraints can induce sparsity in linear regression models without the need for regularization [27, 38]. Similar findings have also been reported for generalized linear models (GLMs) [20]. However, Bayesian models provide a straightforward approach for incorporating linear inequality restrictions into the estimation process. Several studies have investigated Bayesian inference in linear regression models subject to such restrictions. Notable contributions include [15, 16], [10], [28], and more recently [40] and [35]. Most of these studies focus on multiple linear regression models; however, similar restrictions can also arise in applications where GLMs are applicable. [17] introduced an algorithm for Bayesian estimation under linear inequality restrictions in GLMs, but it relies on certain conditions that may not always be satisfied. For example, these conditions do not hold in the case of BRMs with a logit link function. More recently, [36] proposed an algorithm within the GLM framework that accommodates linear inequality restrictions using any link function, specifically in Gamma regression models. Therefore, a practical method that enables Bayesian inference in BRMs subject to linear inequality restrictions is needed. The aim of this paper is to address this gap by focusing on Bayesian inference in BRMs under such constraints.

We organize the article as follows: The BRMs and the maximum likelihood estimator of the regression parameters are presented in Section 2. Section 3 introduces our Bayesian estimation method for BRMs subject to linear inequality constraints. In Section 4, we compare the performance of the proposed Bayesian estimator with existing methods through two simulation studies. The results indicate that our estimator outperforms alternatives such as the Maximum Likelihood Estimator (MLE) and the Ridge estimators in terms of standard deviation, mean squared error, and relative efficiency. Section 5 presents an analysis of a real-life dataset, which confirms the simulation results. Section 6 concludes the paper.

2. Beta Regression Model and Estimation

The beta regression model for a response variable confined to the interval $(0, 1)$ was first proposed by [14], who introduced a monotonic differentiable function, known as the link function, to relate the mean of the response variable to a set of independent variables. Assume y_i is a continuous random variable following the beta probability density function

$$f(y_i) = \frac{\Gamma(\gamma)}{\Gamma(\mu_i\gamma)\Gamma\left((1-\mu_i)\gamma\right)} y_i^{\mu_i\gamma-1}(1-y_i)^{(1-\mu_i)\gamma-1}, \qquad i : 1, 2, \ldots, n, \tag{2.1}$$

where $0 < y_i, \mu_i < 1$, and $\gamma > 0$. Here, $\Gamma(.)$ denotes the gamma function and γ is referred to as the precision parameter, which is assumed to be known and constant throughout this paper. The model allows the mean of the response variable to depend on a linear predictor through a link function $g(.)$, defined as

$$g(\mu_i) = \log(\frac{\mu_i}{1-\mu_i}) = \boldsymbol{X}_i^\top \boldsymbol{\beta} = \eta_i; \qquad i: 1, 2, \ldots, n$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^\top$ is a vector of p unknown parameters, $X_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^\top$ is vector of covariates for the ith observation and η_i is the linear predictor. To find the estimation of $\boldsymbol{\beta}$, the log-likelihood function is required. For BRM, the log-likelihood function is defined as

$$\ell(y|\boldsymbol{\beta}, \gamma, \boldsymbol{X}) = n \log\left(\Gamma(\gamma)\right) + \sum_{i=1}^{n} \Big\{ (\gamma - 1)\log(1-y_i) - \log(y_i) \Big\}$$
$$+ \sum_{i=1}^{n} \Big\{ \gamma\mu_i \log(\frac{y_i}{1-y_i}) - \log\left(\Gamma(\gamma\mu_i)\right) - \log\left(\Gamma(\gamma(1-\mu_i))\right) \Big\} \tag{2.2}$$

The beta maximum likelihood estimator (BMLE) of the vector $\boldsymbol{\beta}$ is obtained using the iterative re-weighted least squares (IRLS) method as

$$\hat{\boldsymbol{\beta}}_{BMLE} = (\boldsymbol{X}^\top \hat{\boldsymbol{C}} \boldsymbol{X})^\top \boldsymbol{X}^\top \hat{\boldsymbol{C}} \boldsymbol{U}, \tag{2.3}$$

where

$$\hat{\boldsymbol{C}} = diag(C_1, C_2, \ldots, C_n),$$
$$C_i = \gamma \Big\{ \Psi'(\hat{\mu}_i\gamma) + \Psi'((1-\hat{\mu}_i)\gamma) \Big\} \frac{1}{\{g'(\hat{\mu}_i)\}^2},$$
$$\boldsymbol{U} = \hat{\boldsymbol{\eta}} + \hat{\boldsymbol{C}}^{-1} \hat{T}(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{\mu}}),$$
$$\hat{\boldsymbol{T}} = diag(\frac{1}{g'(\hat{\mu}_1)}, \frac{1}{g'(\hat{\mu}_2)}, \ldots, \frac{1}{g'(\hat{\mu}_n)}),$$
$$\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n)^\top; \qquad \tilde{\mu}_i = \Psi(\hat{\mu}_i\gamma) - \Psi((1-\hat{\mu}_i)\gamma),$$
$$\tilde{\boldsymbol{y}} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)^\top; \qquad \tilde{y}_i = \log(\frac{y_i}{1-y_i}),$$

and $\Psi(.)$ denotes digamma function.

3. Bayesian Inference in Restricted Beta Regression

In this section, we first begin with the model under the assumption that $\boldsymbol{X}^\top \hat{\boldsymbol{C}} \boldsymbol{X}$ is non-singular. Then, in the following section, through a simulation study, we demonstrate that when $\boldsymbol{X}^\top \hat{\boldsymbol{C}} \boldsymbol{X}$ becomes singular, the beta Bayesian estimator based on linear inequality restrictions naturally serves as a penalty in the estimation process. The restrictions imposed on the model parameters are defined as follows:

$$\boldsymbol{H}\boldsymbol{\beta} \leq \boldsymbol{G}, \tag{3.1}$$

where $\boldsymbol{H}$ is a pre-specified $q \times p$ matrix, and $\boldsymbol{G}$ is a vector of length q. Here, the number of restrictions q may exceed the number of parameters p, and it is assumed that the restricted subspace defined by (3.1) is non-empty. In traditional Bayesian inference, when no restrictions are imposed on the model parameters, the multivariate normal distribution is typically used as the prior distribution for the parameter vector $\boldsymbol{\beta}$

$$\boldsymbol{\beta} \sim N_p(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta), \tag{3.2}$$

However, in our case, there is a strong belief that the model parameters satisfy the restriction in (3.1). Therefore, using (3.2) as an unrestricted prior may lead to inefficient Bayesian estimates. To illustrate this point, consider the case where $Y_i \sim N(\mu, \sigma^2)$, with known σ^2. The unrestricted estimator of μ is the sample mean ($\bar{Y} = \sum_{i=1}^{n} y_i/n$). If we know that $\mu \leq b$, the sample mean $\bar{Y}$ might violate this constraint. In such cases, a restricted estimator can be defined as:

$$\hat{\mu} = \bar{Y}I(\bar{Y} \leq b) + bI(\bar{Y} > b)$$

where $I(.)$ is the indicator function. Consequently, we incorporate the prior information from (3.1) into the prior distribution by considering the truncated multivariate normal distribution

$$\boldsymbol{\beta} \sim TN_p(\boldsymbol{\mu}_{\beta}, \boldsymbol{\Sigma}_{\beta}, \boldsymbol{H}, \boldsymbol{G}), \tag{3.3}$$

where $TN_p(\boldsymbol{\mu}_{\beta}, \boldsymbol{\Sigma}_{\beta}, \boldsymbol{H}, \boldsymbol{G})$ denotes the truncated multivariate normal distribution, whose probability density function is given by

$$\pi(\boldsymbol{\beta}) = \frac{\exp\left\{(\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})^{\top}\boldsymbol{\Sigma}_{\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})\right\}}{\int_{\boldsymbol{H}\beta \leq \boldsymbol{G}} \exp\left\{(\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})^{\top}\boldsymbol{\Sigma}_{\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})\right\} d\boldsymbol{\beta}} I(\boldsymbol{H}\boldsymbol{\beta} \leq \boldsymbol{G}). \tag{3.4}$$

Using the prior distribution above, the posterior distribution of $\boldsymbol{\beta}$ is given by

$$\begin{aligned} \pi(\boldsymbol{\beta}|Y, X, \gamma) &= \ell(y|\boldsymbol{\beta}, \gamma, X)\pi(\boldsymbol{\beta}) \\ &\propto \left[\sum_{i=1}^{n}\left\{\gamma\mu_i \log(\frac{y_i}{1-y_i}) - \left\{\log(\Gamma(\gamma\mu_i)) + \log\left(\Gamma(\gamma(1-\mu_i))\right)\right\}\right\}\right] \\ &\quad \times \exp\left\{(\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})^{\top}\boldsymbol{\Sigma}_{\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_{\beta})\right\} I(\boldsymbol{H}\boldsymbol{\beta} \leq \boldsymbol{G}). \end{aligned} \tag{3.5}$$

Obviously, this posterior distribution does not have a closed-form expression. Therefore, to obtain the estimator, one must generate random samples from the distribution in (3.5). [17] showed that if the likelihood function has the following form

$$\ell(Y|\boldsymbol{\beta}, \boldsymbol{X}) \propto \exp\{Y^{\top}\boldsymbol{X}\boldsymbol{\beta} - \sum_{i=1}^{n}\Upsilon(\boldsymbol{X}_i^{\top}\boldsymbol{\beta})\} \tag{3.6}$$

where $\Upsilon(.)$ is a positive, convex, and invertible function such that $\Upsilon''(z) > 0 \quad \forall z$, then the product slice sampling method can be efficiently used to draw samples from the posterior. However, when comparing the likelihood function of the Beta regression model in (2.2) with the form in (3.6), it becomes evident that the conditions required to apply the algorithm of [17] are not satisfied in the case of Beta regression models.

We use the Metropolis-Hastings algorithm to derive the beta Bayesian Linear Inequality Restricted Estimator (BBIRE). The proposal distribution in the algorithm is specified as

$$\boldsymbol{\beta} \sim TN_p(\boldsymbol{\beta}^{(t-1)}, \boldsymbol{\Sigma}_{pro}, \boldsymbol{H}, \boldsymbol{G}). \tag{3.7}$$

where $\boldsymbol{\Sigma}_{pro}$ is a $p \times p$ positive definite matrix.

Several algorithms have been proposed for generating samples from a truncated multivariate normal distribution subject to linear inequality constraints (see, for example, [8,15,16,21,30,33]). However, most of these methods are only practical when the number of restrictions $q < p$. In this study, we employ the sampling method introduced by [22], which uses a sequence of Gibbs sampling cycles to generate samples from the truncated multivariate normal distribution. In each cycle, sampling from truncated univariate normal distributions is performed using efficient, customized rejection sampling techniques tailored to the type of restriction.

In the following section, we compute the BBIRE using the samples obtained from the Metropolis-Hastings algorithm described above. Our analysis demonstrates that the proposed estimator outperforms existing methods, even in the presence of multicollinearity within the dataset.

4. Simulation Study

In this section, the performance of the proposed estimator is illustrated using two simulated data scenarios. In **Scenario A**, the covariates are independent or exhibit weak intercorrelation. In contrast, **Scenario B** assumes a high degree of intercorrelation among the covariates.

4.1. Random Data Generation

In both scenarios, the predictor function and the mean are considered as follows:

$$\eta_i = X_{i1}\beta_1 + X_{i2}\beta_2 + X_{i3}\beta_3 + X_{i4}\beta_4, \tag{4.1}$$

$$\mu_i = \frac{\exp\{\eta_i\}}{1+\exp\{\eta_i\}}; \qquad i : 1, 2, \ldots, n \tag{4.2}$$

The covariates are generated from a multivariate normal distribution with a mean vector $\mathbf{0} = (0, 0, 0, 0)^\top$ and a covariance matrix $\mathbb{C}$, where $\mathbb{C}_{ij} = \rho^{|i-j|}$; $\quad i, j = 1, 2, \ldots, 4$. The parameter ρ controls the degree of intercorrelation among the covariates. The true values of the regression coefficients are set as $\boldsymbol{\beta} = (1, 1, 1, 1)^\top$. We also investigated the effect of sample size on the performance of the BBIRE estimator compared to other estimators, using sample sizes of $n = 20$ and $n = 50$. Different values of the precision parameter γ are considered, specifically $\gamma = 5$ and $\gamma = 10$. Finally, the observations of the response variable are generated from $Beta(\mu_i\gamma, (1-\mu_i)\gamma)$.

It is possible that values of zero and one may appear in the generated data. To address this issue, we follow the recommendation of [39] by rescaling the values of the dependent variable using the following transformation

$$\tilde{Y}_i = \frac{Y_i(n-1)+0.5}{n}. \tag{4.3}$$

Based on the parameter values, the inequality restrictions of interest are specified as follows:

$$\begin{aligned} \beta_1 &\leq 1.5, \\ \beta_1 - \beta_2 + \beta_3 &\leq 1.5, \\ \beta_3 &\leq 1.5. \end{aligned}$$

4.2. Specializing Hyperparameters

The hyperparameters in (3.3) are set as

$$\boldsymbol{\mu}_{\beta} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\Sigma}_{\beta} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \tag{4.4}$$

For the proposal distribution, the covariance matrix is set as the inverse of the Fisher information matrix

$$\boldsymbol{\Sigma}_{pro} = I^{-1}(\boldsymbol{\beta}) = \frac{1}{\gamma}(\boldsymbol{X}^\top \hat{\boldsymbol{C}} \boldsymbol{X})^{-1} \tag{4.5}$$

where $\hat{\boldsymbol{C}}$ is calculated using the BMLE. In addition to BBIRE and the ordinary estimators used in both scenarios, the Beta Bayesian unrestricted estimator (BBUNE) is also obtained by employing the multivariate normal distribution with the hyperparameters specified in (4.4) as the prior distribution for $\boldsymbol{\beta}$.

4.3. Criteria for Evaluating the Estimators

After designing our experiment, the criteria for comparing the estimators are defined. To determine the proposed Bayesian estimator, the mean of the simulated data is used, which corresponds to assuming a squared error loss function. Since this loss function is applied to find the proposed estimator, the mean squared error (MSE) of the estimators obtained from 100 replicated data sets is calculated in both scenarios to evaluate the performance of the proposed Bayesian estimator compared to alternative estimators. The MSE of an estimator of β_j, e.g. $\hat{\beta}_j$, is calculated as follows:

$$MSE(\hat{\beta}_j) = \frac{1}{100}\sum_{k=1}^{100}(\hat{\beta}_{kj} - \beta_j^{true})^2, \tag{4.6}$$

where $\hat{\beta}_{kj}$ is the estimate of β_j in the kth replication. Another criterion, relative efficiency (RE), is defined by setting the proposed Bayesian estimator (BBIRE) as a benchmark and is computed as

$$RE(\hat{\beta}_j) = \frac{MSE(\hat{\beta}_j)}{MSE(\hat{\beta}_{j(BBIRE)})}. \tag{4.7}$$

4.4. Simulation Results

This subsection presents the results of the simulation study. In each scenario, $10,000$ samples are generated using the Metropolis-Hastings algorithm, with the first $1,000$ samples discarded as burn-in to reduce the influence of initial values. The simulation study is implemented in the R programming language, utilizing the **betareg** [9] and **tmvmnorm** [24] packages. It is worth noting that convergence of the Markov chain was assessed using three different initial values in both scenarios, and no convergence issues were detected.

Results of Scenario A. In this scenario, the correlation parameter ρ is set to 0 and 0.5, corresponding to no and weak inter-correlation among the covariates, respectively. The proposed Bayesian estimator is compared with the beta Bayesian unrestricted estimator (BBUNE) and the Beta maximum likelihood estimator (BMLE). Tables 1 and 2 present the estimates, standard deviations (SD), MSE, and RE of each coefficient in the beta regression model, based on 100 replications and varying values of the precision parameter. The results indicate that the proposed Bayesian estimator consistently outperforms the other estimators in terms of SD, MSE, and RE. Furthermore, as the sample size increases, a general decrease in SD, MSE, and often RE is observed, while the superiority of the proposed estimator remains evident.

Results of Scenario B. In this scenario, high inter-correlation among the covariates is introduced by setting ρ to 0.90 and 0.95. The proposed Bayesian estimator is compared with the Beta Ridge Estimator (BRE) and the Beta Bayesian Unrestricted Estimator (BBUNE). To determine the ridge parameter for BRE, several estimators from the literature were considered and the one that produced the lowest MSE in the designed experiment was selected. Let $\boldsymbol{E}$ be the matrix of eigenvectors of $\boldsymbol{X}^\top \hat{\mathbf{C}} \boldsymbol{X}$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)^\top$ the corresponding eigenvalues, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)^\top = \boldsymbol{E}\hat{\boldsymbol{\beta}}_{\text{BMLE}}$. The chosen ridge parameter estimator is then given by:

$$\hat{k} = \frac{\lambda_{max}}{\gamma\alpha^2_{max}}$$

where $\alpha^2_{\text{max}} = \max_j(\alpha_j^2)$ and $\lambda_{\text{max}} = \max_j(\lambda_j)$.

Estimates, standard deviations (SD), MSE and RE of each coefficient in the beta regression model, based on 100 replications, are reported in Table 3 for $\gamma = 5$ and Table 4 for $\gamma = 10$. The results clearly show that the BBIRE yields significantly lower MSE and SD for all coefficients compared to both BRE and BBUNE. As expected, increasing the sample size leads to reductions in SD, MSE, and RE for each coefficient. These results suggest that incorporating linear inequality restrictions into the Bayesian model can enhance the accuracy of parameter estimates, even in the presence of strong multicollinearity among the covariates.

5. Application

The application of the proposed methodology is demonstrated through an empirical study based on Turkey's well-being index for 2015, as documented by Aktas (year). This index comprises several dimensions, including housing, employment, income and wealth, health, education, environmental quality, safety, civic engagement, and access to public and social services. The life satisfaction index, which serves as the response variable, ranges from 0 to 1, with values closer to one indicating a higher quality of life. The data set was retrieved from the official website of the Turkish Statistical Institute.

Of the 41 available indicators, we focus on nine specific indicators, according to the selection made by [1], to serve as covariates in the model. These are: number of rooms per person (X_1), percentage of households declaring to not meet basic needs (X_2), satisfaction rate with public health services(X_3), average point of necessary placement scores of the system for the transition to secondary education from basic education (X_4), satisfaction rate with public education services (X_5), percentage of population receiving waste services (X_6), satisfaction rate with public safety services (X_7), access rate of the population to the sewerage and pipe system (X_8), and finally, we consider the level of happiness as a response variable. To begin with, the degree of multicollinearity among the covariates is examined. Table 5 presents the correlation matrix of the selected variables, revealing strong correlations among several covariates. Consequently, we compute and compare four estimators: the BBIRE, BBUNE, BMLE, and BRE.

Inequality restrictions on regression coefficients are imposed based on findings from previous studies [1,3], and are specified as follows:

$$\beta_2 \leq 0,\ \beta_3 \geq 0,\ \beta_5 \geq 0,\ \beta_6 \leq 0. \tag{5.1}$$

In line with the simulation settings in Section 4, the Metropolis-Hastings algorithm described in Section 3 is used to generate $10,000$ samples, the first $1,000$ discarded as burn-in to mitigate the influence of initial values. The hyperparameters for the prior and proposal distributions are specified in Section 4. For the ridge parameter in the BRE, following the approach in [1], the ridge penalty is defined as:

$$\hat{k} = \frac{\lambda_{min}}{\gamma \alpha^2_{min}}$$

in which $\alpha^2_{min} = \min\limits_{1\leq j\leq p}(\alpha^2_j)$ and $\lambda_{min} = \min\limits_{1\leq j\leq p}(\lambda_j)$. To evaluate the performance of the proposed Bayesian restricted estimator, a bootstrap case resampling method is employed. From the complete data set of 81 observations, bootstrap samples of size 30 are drawn with 100 replications. For each bootstrap sample, the estimators, their standard deviations, and relative efficiencies are calculated. The final estimates and standard deviations are derived as the sample means and sample standard deviations in the 100 bootstrap replications.

Table 1. Results of simulation for Scenario A when $\gamma = 5$.

			$n = 20$				$n = 50$			
ρ	Parameters	Estimators	Estimates	SD	MSE	R.E	Estimates	SD	MSE	R.E
	β_1	BMLE	1.0329	0.2397	0.0580	2.1993	1.0065	0.1552	0.0239	1.8570
		BBUNE	1.0002	0.2195	0.0477	1.8098	0.9909	0.1387	0.0191	1.4880
		BBIRE	0.9341	0.1491	0.0264	-	0.9780	0.1118	0.0129	-
	β_2	BMLE	1.0465	0.2521	0.0651	2.2703	1.0305	0.1395	0.0202	1.5640
		BBUNE	1.0151	0.2245	0.0501	1.7484	1.0138	0.1291	0.0167	1.2923
0		BBIRE	1.0438	0.1644	0.0287	-	1.0194	0.1125	0.0129	-
	β_3	BMLE	1.0708	0.2320	0.0583	2.9931	0.9790	0.1483	0.0222	1.5158
		BBUNE	1.0335	0.2002	0.0408	2.0938	0.9625	0.1405	0.0210	1.4296
		BBIRE	0.9578	0.1337	0.0195	-	0.9589	0.1144	0.0147	-
	β_4	BMLE	1.0141	0.2642	0.0693	2.0318	1.0263	0.1593	0.0258	1.9179
		BBUNE	0.9839	0.2359	0.0554	1.6231	1.0082	0.1429	0.0203	1.5075
		BBIRE	0.9818	0.1847	0.0341	-	1.0080	0.1163	0.0135	-
	β_1	BMLE	0.9497	0.2831	0.0819	1.6374	0.9905	0.1494	0.0222	1.7998
		BBUNE	0.9173	0.2732	0.0807	1.6137	0.9722	0.1332	0.0183	1.4861
		BBIRE	0.8693	0.1824	0.0500	-	0.9621	0.1049	0.0123	-
	β_2	BMLE	0.9559	0.3076	0.0956	2.5631	0.9935	0.1510	0.0226	1.8423
		BBUNE	0.9323	0.2782	0.0812	2.1765	0.9753	0.1363	0.0190	1.5467
0.5		BBIRE	1.0072	0.1940	0.0373	-	0.9948	0.1113	0.0123	-
	β_3	BMLE	1.0107	0.3393	0.1141	2.4199	0.9623	0.1498	0.0236	1.5364
		BBUNE	0.9707	0.2921	0.0853	1.8099	0.9453	0.1385	00.0220	1.4301
		BBIRE	0.8989	0.1932	0.0472	-	0.9346	0.1059	0.0154	-
	β_4	BMLE	0.9605	0.3487	0.1220	2.0149	0.9574	0.1594	0.0270	1.5837
		BBUNE	0.9279	0.2976	0.0929	1.5346	0.9386	0.1447	0.0245	1.4394
		BBIRE	0.9574	0.2435	0.0605	-	0.9482	0.1204	0.0170	-

Table 2. Results of simulation for Scenario A when $\gamma = 10$.

			$n = 20$				$n = 50$			
ρ	Parameters	Estimators	Estimates	SD	MSE	R.E	Estimates	SD	MSE	R.E
0	β_1	BMLE	1.0210	0.1975	0.0391	2.3230	0.9850	0.1248	0.0157	1.6030
		BBUNE	1.0068	0.1733	0.0298	1.7701	0.9862	0.1153	0.0134	1.3676
		BBIRE	0.9681	0.1263	0.0168	-	0.9760	0.0963	0.0098	-
	β_2	BMLE	1.0570	0.2302	0.0557	2.3686	0.9937	0.1192	0.0141	1.5781
		BBUNE	1.0404	0.2074	0.0442	1.8806	0.9946	0.1040	0.0107	1.2020
		BBIRE	1.0450	0.1473	0.0235	-	0.9995	0.0950	0.0089	-
	β_3	BMLE	1.0503	0.2264	0.0533	2.8775	1.0083	0.1152	0.0132	1.5597
		BBUNE	1.0361	0.2072	0.0438	2.3661	1.0099	0.1066	0.0114	1.3401
		BBIRE	0.9844	0.1358	0.0185	-	1.0012	0.0925	0.0085	-
	β_4	BMLE	1.0182	0.2097	0.0439	1.9941	0.9794	0.1301	0.0172	1.4514
		BBUNE	1.0053	0.1898	0.0357	1.6232	0.9807	0.1182	0.0142	1.1997
		BBIRE	0.9978	0.1490	0.0220	-	0.9808	0.1076	0.0118	-
0.5	β_1	BMLE	1.0161	0.2483	0.0613	2.3705	0.9894	0.1335	0.0177	1.6373
		BBUNE	0.9906	0.2297	0.0523	2.0222	0.9854	0.1267	0.0161	1.4849
		BBIRE	0.9480	0.1530	0.0259	-	0.9746	0.1015	0.0108	-
	β_2	BMLE	1.0313	0.2565	0.0661	2.2718	1.0023	0.1361	0.0183	1.6052
		BBUNE	1.0015	0.2412	0.0576	1.9785	0.9962	0.1262	0.0158	1.3803
		BBIRE	1.0370	0.1674	0.0291	-	1.0078	0.1071	0.0114	-
	β_3	BMLE	1.0056	0.2364	0.0554	1.9416	1.0075	0.1313	0.0171	1.4638
		BBUNE	0.9780	0.2304	0.0530	1.8593	1.0039	0.1276	0.0161	1.3786
		BBIRE	0.9282	0.1536	0.0285	-	0.9899	0.1082	0.0117	-
	β_4	BMLE	1.0459	0.2366	0.0575	2.0748	0.9829	0.1397	0.0196	1.6578
		BBUNE	1.0193	0.2048	0.0419	1.5121	0.9768	0.1258	0.0162	1.3691
		BBIRE	1.0185	0.1663	0.0277	-	0.9771	0.1069	0.0118	-

Table 3. Results of simulation for Scenario B when $\gamma = 5$.

			$n = 20$				$n = 50$			
ρ	Parameters	Estimators	Estimates	SD	MSE	R.E	Estimates	SD	MSE	R.E
0.9	β_1	BRE	0.9341	0.3719	0.1412	1.7573	0.8346	0.2583	0.0934	1.4753
		BBUNE	1.0133	0.4611	0.2107	2.6210	0.8656	0.2926	0.1029	1.6244
		BBIRE	0.8663	0.2513	0.0804	-	0.8338	0.1899	0.0633	-
	β_2	BRE	0.8310	0.4477	0.2270	1.8872	0.8897	0.2695	0.0841	2.2456
		BBUNE	0.8792	0.5434	0.3070	2.5520	0.9230	0.2962	0.0928	2.4779
		BBIRE	1.0699	0.3414	0.1203	-	1.0237	0.1930	0.0374	-
	β_3	BRE	0.8569	0.4190	0.1943	1.2818	0.8919	0.2758	0.0870	1.4902
		BBUNE	0.9292	0.5336	0.2869	1.8930	0.9306	0.3126	0.1016	1.7393
		BBIRE	0.7829	0.3248	0.1516	-	0.8721	0.2060	0.0584	-
	β_4	BRE	0.8042	0.4148	0.2087	1.1899	0.9113	0.2490	0.0692	1.2982
		BBUNE	0.8534	0.5265	0.2959	1.6872	0.9542	0.2886	0.0846	1.5853
		BBIRE	0.9483	0.4177	0.1754	-	0.9928	0.2320	0.0533	-
0.95	β_1	BRE	1.1185	0.3966	0.1697	4.1024	0.8015	0.3506	0.1611	1.2388
		BBUNE	1.2658	0.5440	0.3636	8.7876	0.8181	0.4249	0.2118	1.6294
		BBIRE	0.9148	0.1856	0.0414	-	0.7721	0.2808	0.1300	-
	β_2	BRE	0.8060	0.5961	0.3894	1.1995	0.8990	0.3198	0.1114	1.5408
		BBUNE	0.8691	0.7998	0.6504	2.0033	0.9324	0.3970	0.1606	2.2204
		BBIRE	1.2061	0.5339	0.3247	-	1.0523	0.2651	0.0723	-
	β_3	BRE	0.7800	0.5615	0.3606	1.0633	0.9015	0.4007	0.1686	1.5189
		BBUNE	0.7741	0.7847	0.6606	1.9481	0.9408	0.4757	0.2275	2.0492
		BBIRE	0.6047	0.4297	0.3391	-	0.8488	0.2984	0.1110	-
	β_4	BRE	0.7568	0.5782	0.3902	0.8297	0.8865	0.4017	0.1726	1.0589
		BBUNE	0.7276	0.8045	0.7149	1.5202	0.9370	0.4806	0.2326	1.4270
		BBIRE	0.9049	0.6826	00.4703	-	0.9759	0.4051	0.1630	-

Table 4. Results of simulation for Scenario B when $\gamma = 10$.

ρ	Parameters	Estimators	$n = 20$				$n = 50$			
			Estimates	SD	MSE	R.E	Estimates	SD	MSE	R.E
0.90	β_1	BRE	0.9311	0.3373	0.1174	1.5126	0.9757	0.2252	0.0508	2.0275
		BBUNE	0.9664	0.4061	0.1644	2.1192	0.9891	0.2341	0.0544	2.1708
		BBIRE	0.8187	0.2125	0.0776	-	0.9339	0.1445	0.0251	-
	β_2	BRE	0.8678	0.4338	0.2038	1.8568	0.8907	0.2438	0.0708	2.8426
		BBUNE	0.8867	0.5340	0.2951	2.6889	0.8991	0.2571	0.0756	3.0377
		BBIRE	1.1118	0.3134	0.1097	-	1.0066	0.1585	0.0249	-
	β_3	BRE	0.9305	0.4341	0.1914	1.7062	0.9480	0.2630	0.0712	1.7666
		BBUNE	0.9590	0.5072	0.2563	2.2851	0.9589	0.2754	0.0768	1.9054
		BBIRE	0.7894	0.2617	0.1122	-	0.8984	0.1740	0.0403	-
	β_4	BRE	0.9554	0.4040	0.1636	1.0711	0.9351	0.2444	0.0634	1.4894
		BBUNE	0.9901	0.4844	0.2324	1.5216	0.9469	0.2553	0.0674	1.5836
		BBIRE	1.0993	0.3799	0.1527	-	0.9876	0.2069	0.0425	-
0.95	β_1	BRE	1.1144	0.4113	0.1806	3.1281	0.9480	0.3256	0.1077	1.6747
		BBUNE	1.1903	0.5195	0.3034	5.2563	0.9709	0.3587	0.1283	1.9948
		BBIRE	0.9177	0.2269	0.0577	-	0.8743	0.2213	0.0643	-
	β_2	BRE	0.7790	0.5356	0.3329	1.8111	0.9026	0.2972	0.0969	2.2706
		BBUNE	0.7389	0.6536	0.4911	2.6722	0.9234	0.3237	0.1096	2.5666
		BBIRE	1.0844	0.4224	0.1838	-	1.0814	0.1909	0.0427	-
	β_3	BRE	0.8875	0.4474	0.2108	1.1278	0.9398	0.3477	0.1233	1.7289
		BBUNE	0.8935	0.5782	0.3423	1.8312	0.9603	0.3651	0.1336	1.8726
		BBIRE	0.7380	0.3457	0.1869	-	0.8610	0.2292	0.0713	-
	β_4	BRE	0.8909	0.5385	0.2990	0.9037	0.8976	0.3346	0.1213	1.5707
		BBUNE	0.9388	0.6701	0.4483	1.3550	0.9057	0.3547	0.1334	1.7278
		BBIRE	1.0400	0.5767	0.3308	-	0.9809	0.2787	0.0772	-

Table 5. Correlation matrix of covariates of real data set.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	1.000	-0.822	0.572	0.877	0.416	0.154	0.480	0.197
X_2		1.000	-0.585	-0.759	-0.372	-0.182	-0.399	-0.248
X_3			1.000	0.497	0.839	0.039	0.843	0.110
X_4				1.000	0.312	0.174	0.414	0.241
X_5					1.000	-0.173	0.891	-0.112
X_6						1.000	-0.225	0.931
X_7							1.000	-0.159
X_8								1.000

Summary statistics for all parameters are presented in Table 6. The results indicate that the BBIRE exhibits the lowest standard deviation among all estimators. To further evaluate the performance of the proposed Bayesian estimator, Table 7 presents the total simulated relative efficiency (TSRE), which is computed using the following formula:

$$TSRE(\hat{\beta}) = \frac{\sum_{j=0}^{8} MSE(\hat{\beta}_j)}{\sum_{j=0}^{8} MSE(\hat{\beta}_{j(BBIRE)})} \tag{5.2}$$

Since the values of TSRE for all traditional estimators are larger than one, it indicates that BBIRE outperforms the other estimators.

Table 6. Bootstrapped estimates and standard deviation (SD) of model parameters for real data set.

	BMLE		BRE		BBUNE		BBIRE	
	Estimates	SD	Estimates	SD	Estimates	SD	Estimates	SD
intercept	2.7208	2.2456	2.1614	1.9728	2.7598	2.2714	2.9719	1.8160
X_1	-0.0413	0.6825	0.0258	0.6206	-0.0404	0.6885	-0.0545	0.6248
X_2	-2.0861	1.2015	-1.7173	1.1993	-2.1067	1.2161	-2.0456	0.8096
X_3	1.7496	1.9315	1.4798	1.5576	1.7415	1.9315	2.1923	1.1235
X_4	-0.6216	0.4777	-0.5611	0.4502	-0.6290	0.4809	-0.6051	0.4668
X_5	0.9944	1.6243	0.7943	1.3594	1.0076	1.6168	1.6395	0.7606
X_6	-0.8395	1.0613	-0.6617	0.8476	-0.8456	1.0706	-1.2225	0.7236
X_7	-1.2312	2.3902	-0.7414	1.9481	-1.2393	2.4084	-2.4460	1.7759
X_8	0.3845	1.2416	0.2677	1.0157	0.3878	1.2487	0.6641	0.9028

Table 7. TSRE of estimators for real data set.

	BMLE	BRE	BBUNE
TSRE	1.8657	1.4719	1.8813

6. Conclusion

This paper has addressed the problem of Bayesian estimation of parameters, which are restricted by some linear inequality restrictions in Beta regression models. Beta regression models with logistic link functions do not satisfy the conditions of the estimation parameter method mentioned by [17]. Thus, a new method for estimating restricted parameters in the Beta regression model has been presented and is also feasible for any other members of GLM. The simulation results illustrated that the proposed method provides a parameter estimation that outperforms well-known estimators even if the design matrix is ill-conditioned. The real data application also shows the practicality of the proposed method in estimating the parameters.

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