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Principally 1-Absorbing Right Primary Ideals

Hatice Çay[1]

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Abstract— This paper first defines the 1-absorbing version of principally right primary ideals (P1ARP ideals), generalizing prime ideals, for noncommutative rings. It then investigates various properties of this ideal structure in different ring settings. It obtains some essential results in ring extensions, such as homomorphic images, product rings, local rings, and idealization. While this study enables the obtaining of original results due to structural differences between commutative and noncommutative rings, it shows that some properties valid in commutative rings are preserved. Finally, the paper concludes by discussing two open problems that could guide future studies.

Keywords — 1-*Absorbing primary ideals, noncommutative rings, prime ideals, pseudo-radicals*

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1. Introduction

Working with non-commutative rings is generally more challenging than working with commutative rings. Consequently, numerous results proven for commutative rings still remain unresolved in the setting of non-commutative rings. In this study, we aim to introduce a suitable version of 1-absorbing primary ideals–originally defined for commutative rings–for non-commutative rings. As a starting point, we conduct a literature review and investigate similar definitions and results in the context of commutative rings. This will serve to illustrate the background and motivation for our work.

The study of prime ideal structures and their various generalizations in commutative rings has been a significant area of research in ring theory, as these concepts contribute significantly to understanding the properties and classification of rings. One of these generalizations is the concept of 1-absorbing primary ideals, introduced by Badawi and Çelikel [1]. The authors proved that a ring containing a 1-absorbing primary ideal that is not primary must be a quasi-local ring. Additionally, they explored the relationship between these ideals and the connection between Noetherian domains and Dedekind domains, presenting several significant results. Subsequently, Nikandish et al. [2] investigated various properties of these ideals and introduced a more general version known as the weakly 1-absorbing primary ideals, integrating the characteristics of weakly prime ideals and 1-absorbing primary ideals. They studied the key properties of these ideals in the context of polynomial rings, principal ideal domains (PIDs), and idealization structures.

[1]hcay@medipol.edu.tr (Corresponding Author)
[1]Department of Computer Programming, Istanbul Medipol University, İstanbul, Türkiye

The study of primary ideals, initially focused on commutative rings, has been expanded to noncommutative settings to investigate more extensive ideal structures. In particular, Birkenmeier et al. [3] introduced various generalizations of primary ideals to address the complexities of noncommutative rings. These generalizations establish a foundation for exploring ideal structures in noncommutative rings, enabling the extension of classical commutative algebra results to broader algebraic systems. Investigating these properties has contributed to significant advancements in understanding the decomposition of ideals in noncommutative rings and their algebraic representations [4–11].

In 2022, Groenewald [11] defined the concept of a weakly right primary ideal in noncommutative rings. Additionally, in 2021, Groenewald [10] defined p-2-absorbing right primary ideals, which generalize principally right primary ideals. The aforementioned generalizations of prime ideals are as in Figure 1.

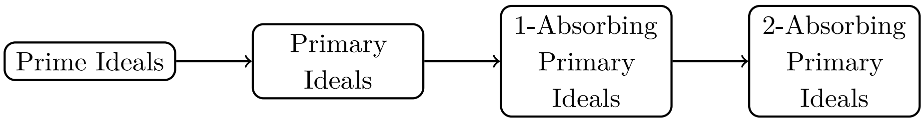

Figure 1. Relations between the aforementioned generalizations of prime ideals

Therefore, this study fills the gap between right principally primary ideals and principally 2-absorbing right primary ideals, which are defined in noncommutative rings, by introducing principally 1-absorbing right primary ideals. This new class of ideals plays a significant role in the characterization of local rings, as demonstrated in Theorem 3.12 and Proposition 3.14. The remainder of this study is organized as follows: Section 2 presents some basic notions to be needed in the following section. Section 3 introduces principally 1-absorbing right primary ideals and explores some of their basic properties. The last section discusses the need for further research.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following section. Throughout this paper, $\mathcal{H}$ denotes a noncommutative ring unless otherwise specified.

Definition 2.1. [3] Let J be a proper ideal of $\mathcal{H}$. Then, the pseudo-radical of J is defined by

$$\sqrt{J} := \sum \left\{ K \lhd \mathcal{H} \mid K^n \subseteq J \text{ for some } n \in \mathbb{Z}^+ \right\}$$

It is clear that $\sqrt{J}$ is an ideal of $\mathcal{H}$. In this context, the term $\mathcal{P}^*(J)$ denotes the prime radical of J, defined as the intersection of all prime ideals of $\mathcal{H}$ containing J. In commutative rings, the definition of the radical of an ideal is the same as this concept. The radical of an ideal in noncommutative rings differs slightly from its definition in commutative rings. It is known that $\sqrt{J} \subseteq \mathcal{P}^*(J)$, and as established in Lemma 2.7 *iv*, $\sqrt{J}$ is strictly contained within $\mathcal{P}^*(J)$. Therefore, this study differs from generalizations related to primary ideals defined in commutative rings. Specifically, the collection of all prime ideals of $\mathcal{H}$, represented by $P(\mathcal{H})$, corresponds to $\mathcal{P}^*(0)$.

Definition 2.2. [1] A proper ideal A of a commutative ring $\mathcal{H}$ is said to be a 1-absorbing primary ideal if, for any nonunit elements $u, v, z \in \mathcal{H}$, the condition $uvz \in A$ implies that either $uv \in A$ or $z \in \sqrt{A}$.

Definition 2.3. [3] An ideal A of $\mathcal{H}$ is defined as a (principally) right primary ideal if, for any (principal) ideals K and L of $\mathcal{H}$ satisfying $KL \subseteq A$, it follows that either $K \subseteq A$ or $L^n \subseteq A$ for some $n \in \mathbb{Z}^+$.

Definition 2.4. [3] An ideal A of $\mathcal{H}$ is referred to as a (principally) semiprimary ideal if, for any (principal) ideals K and L of $\mathcal{H}$ such that $KL \subseteq A$, it holds that either $K^l \subseteq A$ or $L^n \subseteq A$ for some positive integers l and n.

Definition 2.5. [11] An ideal A of $\mathcal{H}$ is defined as a weakly (principally) right primary ideal if, any time K and L are (principal) ideals of $\mathcal{H}$ satisfying $\{0\} \neq KL \subseteq A$, then either $K \subseteq A$ or there is $n \in \mathbb{Z}^+$ such that $L^n \subseteq A$.

Definition 2.6. [10] An $A \lhd \mathcal{H}$ is defined as a p-right 2-absorbing primary ideal if, for any elements $u, v, k \in \mathcal{H}$, the condition $u\mathcal{H}v\mathcal{H}k \subseteq A$ implies that at least one of the following holds: $uv \in A$, $uk \in \sqrt{A}$ or $vk \in \sqrt{A}$.

Lemma 2.7. [3] The following properties hold for some ideals K, L, and J in $\mathcal{H}$:

i. If $K \subseteq L$, then $\sqrt{K} \subseteq \sqrt{L}$.

ii. If $K \subseteq \sqrt{J}$, then $K^n \subseteq J$ for some $n \in \mathbb{Z}^+$ under the condition that K is finitely generated or there exists an $m \in \mathbb{Z}^+$ such that $(\sqrt{J})^m \subseteq J$. In particular, if $\sqrt{J}$ is finitely generated, then there exists an $n \in \mathbb{Z}^+$ such that $(\sqrt{J})^n \subseteq J$.

iii. $\sqrt{KL} = \sqrt{K \cap L} = \sqrt{K} \cap \sqrt{L}$

iv. If $(\sqrt{J})^l \subseteq J$, for an $l \in \mathbb{Z}^+$, then $\sqrt{J} = \mathcal{P}^*(J) = \sqrt{\sqrt{J}}$.

Definition 2.8. [10] An ideal J of $\mathcal{H}$ is called a principally 2-absorbing right primary ideal of $\mathcal{H}$ if, for all $x, y, z \in \mathcal{H}$, whenever $x\mathcal{H}y\mathcal{H}z \subseteq J$, this means that $xy \in J$ or $xz \in \sqrt{J}$ or $yz \in \sqrt{J}$.

Note that the definition of semiprime in noncommutative rings is as follows:

Definition 2.9. An ideal L of $\mathcal{H}$ is said to be semiprime if, for any ideal J of $\mathcal{H}$, whenever a positive power of J, say J^k, is contained in L for some natural number k, then J itself must also be contained in L.

Lemma 2.10. [10] For a ring homomorphism $\varphi : H_1 \to H_2$ and ideals $A_1 \lhd H_1$ and $A_2 \lhd H_2$, the following hold:

i. $\varphi^{-1}(\sqrt{A_2}) = \sqrt{\varphi^{-1}(A_2)}$

ii. If $\mathrm{Ker}(\varphi) \subseteq A_1$, then $\varphi(\sqrt{A_1}) \subseteq \sqrt{\varphi(A_1)}$.

Lemma 2.11. [12] If, for each nonunit $x \in \mathcal{H}$ and each unit $y \in \mathcal{H}$, the sum $x + y$ is a unit, then $\mathcal{H}$ is a local ring.

Remark 2.12. [13] If $x \in \sqrt{J}$, then there exists $n \in \mathbb{Z}^+$ such that $(\langle x \rangle)^n \subseteq J$. Thus, for any ideals I, J of $\mathcal{H}$, if $I \nsubseteq \sqrt{J}$ then $I^n \nsubseteq J$, for each $n \in \mathbb{N}$.

Unless otherwise stated, throughout this paper, all rings are assumed to be noncommutative and possess a nonzero identity element, denoted by $\mathcal{H}$.

3. Main Results

This section defines principally 1-absorbing right primary ideals and investigates some of their basic properties.

Definition 3.1. A proper ideal J of a ring $\mathcal{H}$ is called a principally 1-absorbing right primary ideal if, for all nonunit elements $x, y, z \in \mathcal{H}$, the condition $x\mathcal{H}y\mathcal{H}z \subseteq J$ implies that either $xy \in J$ or $z \in \sqrt{J}$.

For the sake of convenience, we will use the abbreviations P1ARP to refer to "principally 1-absorbing right primary," and PRP refer to "principally right primary" for the remainder of our work.

We should note that a principally 2-absorbing right primary ideal is more general than a P1ARP ideal. Hence, every P1ARP ideal is a subclass of a principally 2-absorbing right primary ideal. To support this claim, we provide a counterexample.

Example 3.2. Let $\mathcal{H} = M_2(\mathbb{Z})$ and $J = M_2(\langle 12 \rangle)$. From [10], J is a principally 2-absorbing right primary ideal. However, J is not a P1ARP ideal since, although

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \subseteq J$$

it can be observed that

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin J \quad \text{and} \quad \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin \sqrt{J}$$

Theorem 3.3. Let J be a P1ARP ideal of $\mathcal{H}$. If $x\mathcal{H}yK \subseteq J$, for all nonunits $x, y \in \mathcal{H}$ and proper ideals K of $\mathcal{H}$, then $xy \in J$ or $K \subseteq \sqrt{J}$.

Proof. Suppose that $x\mathcal{H}yK \subseteq J$, for nonunits $x, y \in \mathcal{H}$ and $K \triangleleft \mathcal{H}$ with $xy \notin J$ and $K \nsubseteq \sqrt{J}$. Then, there exists an $a \in K$ such that $a \notin \sqrt{J}$. Hence, $x\mathcal{H}y\mathcal{H}a \subseteq J$, while neither $xy \in J$ nor $a \in \sqrt{J}$, contradicting the fact that J is a P1ARP. □

Theorem 3.4. Let K be a proper ideal of $\mathcal{H}$. The following statements are equivalent:

i. K is a P1ARP ideal of $\mathcal{H}$.

ii. For some proper ideals J_1, J_2, J_3 of $\mathcal{H}$ with $J_1J_2J_3 \subseteq K$, $J_1J_2 \subseteq K$ or $J_3 \subseteq \sqrt{K}$.

Proof. Assume that K is a proper ideal of $\mathcal{H}$.

$i \Rightarrow ii$: Let K be a P1ARP ideal and $J_1J_2J_3 \subseteq K$ with $J_1J_2 \nsubseteq K$, for some proper ideals J_1, J_2, J_3 of $\mathcal{H}$. Then, there exist nonunits $x \in J_1$ and $y \in J_2$ such that $xy \notin K$. Since $x\mathcal{H}yJ_3 \nsubseteq K$ and $xy \notin K$, then $J_3 \subseteq \sqrt{K}$ by Theorem 3.3.

$ii \Rightarrow i$: Let x, y, z be some nonunits of $\mathcal{H}$ such that $x\mathcal{H}y\mathcal{H}z \subseteq K$ with $xy \notin K$. Assume that $J_1 = \mathcal{H}x\mathcal{H}$, $J_2 = \mathcal{H}y\mathcal{H}$, and $J_3 = \mathcal{H}z\mathcal{H}$. Then, $J_1J_2J_3 \subseteq K$ and $J_1J_2 \subseteq K$. Hence, $J_3 \subseteq \sqrt{K}$, and thus $z \in \sqrt{K}$. □

Proposition 3.5. Assume that $\mathcal{H}$ has a nonzero identity. For $K \triangleleft H$, if $\sqrt{K}$ is a PRP ideal, then K is a P1ARP1.

Proof. Suppose that $uRvRz \subseteq K$, for some nonunits $u, v, z \in \mathcal{H}$ and $z \notin \sqrt{K}$. Consider that $uvRz \subseteq uRvRz \subseteq K \subseteq \sqrt{K}$. Since $\sqrt{K}$ is a PRP ideal, then $uv \in K$. Therefore, K is a P1ARP. □

Theorem 3.6. If J is a semiprime ideal, then the following condition holds:

$$J \text{ is a prime ideal} \Leftrightarrow J \text{ is a P1ARP ideal}$$

Proof. Let J be a semiprime ideal.

($\Rightarrow$) The proof follows directly from the definitions.

($\Leftarrow$) Assume that $\langle x\rangle\langle y\rangle\langle z\rangle \subseteq J$ and $\langle x\rangle\langle y\rangle \nsubseteq J$, for nonunits $x, y, z \in \mathcal{H}$. Since $xy\mathcal{H}z \subseteq x\mathcal{H}y\mathcal{H}z \subseteq \langle x\rangle\langle y\rangle\langle z\rangle \subseteq J$ and J is a P1ARP ideal, then $\langle z\rangle^n \subseteq J$ by Remark 2.12, and since J is a semiprime ideal, then $\langle z\rangle \subseteq J$. Therefore, $xy \notin J$ and $z \in J$ and hence J is a prime ideal. □

Proposition 3.7. If the radical of J is a prime ideal in $\mathcal{H}$, then J is a P1ARP ideal.

PROOF. Let $x\mathcal{H}y\mathcal{H}z \subseteq J$ with $xy \notin J$, for nonunits $x, y, z \in \mathcal{H}$. Then, $xy\mathcal{H}z \subseteq x\mathcal{H}y\mathcal{H}z \subseteq J \subseteq \sqrt{J}$ since $\sqrt{J}$ is prime and $xy \notin J$, i.e., $xy \notin \sqrt{J}$, then $z \in \sqrt{J}$. Hence, J is a P1ARP ideal. □

Theorem 3.8. Let A be a P1ARP ideal of $\mathcal{H}$. If $k \in \mathcal{H} \setminus A$ is a nonunit element, then $(A : \langle k\rangle) = \{t \in \mathcal{H} : \langle k\rangle t \subseteq A\}$ is a PRP ideal of $\mathcal{H}$.

PROOF. Let A be a P1ARP ideal of $\mathcal{H}$ and $k \in \mathcal{H} \setminus A$ be a nonunit element. For some nonunits $u, v \in \mathcal{H}$, assume that $u\mathcal{H}v \subseteq (A : \langle k\rangle)$. If $u \notin (A : \langle k\rangle)$, then $\langle k\rangle u \nsubseteq A$. Since $\langle k\rangle uHv \subseteq A$ and $\langle k\rangle u \nsubseteq A$, then $kHuHv \subseteq A$ and $ku \notin A$. Therefore, $v \in \sqrt{A} \subseteq \sqrt{(A : \langle k\rangle)}$ since A is a P1ARP ideal. Hence, $(A : \langle k\rangle)$ is a PRP ideal of $\mathcal{H}$. □

3.1. Homomorphic Images

The subsection investigates the relations between ring homomorphisms and P1ARP ideals.

Theorem 3.9. The following conditions hold under the surjective ring homomorphism $\varphi : H_1 \to H_2$.

i. If A_2 is a P1ARP ideal of H_2, then $\varphi^{-1}(A_2)$ is a P1ARP ideal of H_1.

ii. If A_1 is a P1ARP ideal with $\mathrm{Ker}(\varphi) \subseteq A_1$, then $\varphi(A_1)$ is a P1ARP ideal of H_2.

PROOF. Let $\varphi : H_1 \to H_2$ be a surjective ring homomorphism.

i. For some nonunits $u, v, z \in H_1$, suppose that $uH_1vH_1z \subseteq \varphi^{-1}(A_2)$. Then, $\varphi(uH_1vH_1z) \subseteq \varphi(u)H_2\varphi(v)H_2\varphi(z) \subseteq A_2$. Thus, $\varphi(u)\varphi(v) \in A_2$ or $\varphi(z) \in \sqrt{A_2}$, i.e., $uv \in \varphi^{-1}(A_2)$ or $z \in \varphi^{-1}(\sqrt{A_2}) = \sqrt{\varphi^{-1}(A_2)}$. Hence, $\varphi^{-1}(A_2)$ is a P1ARP ideal.

ii. For some nonunits $u, v, z \in H_2$, assume that $uH_2vH_2z \subseteq \varphi(A_1)$. Hence, there exist $\varphi(k) = u$, $\varphi(l) = v$, and $\varphi(m) = z$ such that $\varphi(kH_1lH_1m) = uH_2vH_2z \subseteq \varphi(A_1)$. Since $\mathrm{Ker}(\varphi) \subseteq A_1$, then $kH_1lH_1m \subseteq A_1$. Thus, $kl \in A_1$ or $m \in \sqrt{A_1}$. Therefore, $uv \in \varphi(A_1)$ or $z \in \varphi(\sqrt{A_1}) \subseteq \sqrt{\varphi(A_1)}$. Consequently, $\varphi(A_1)$ is a P1ARP ideal of H_2.

□

Corollary 3.10. Let A_1 and A_2 be proper ideals of $\mathcal{H}$ satisfying the condition $A_1 \subseteq A_2$. Then, A_1 is a P1ARP ideal if and only if A_2/A_1 is a P1ARP ideal of $\mathcal{H}/A_1$.

PROOF. Take $\varphi : \mathcal{H} \to \mathcal{H}/A_1$ with $\varphi(x) = x + A_1$. Assume that A_2 is a P1ARP ideal of $\mathcal{H}$. By Theorem 3.9 *ii*, $\varphi(A_2) = A_2/A_1$ is a P1ARP since $\mathrm{Ker}(\varphi) = A_1 \subseteq A_2$. Conversely, assume that A_2/A_1 is a P1ARP ideal of $\mathcal{H}/A_1$. By Theorem 3.9 *i*, $\varphi^{-1}(A_2/A_1) = A_2$ is a P1ARP. □

3.2. Product Rings

The following identities are well known:

$$\sqrt{J_1 \times \mathcal{H}_2} = \sqrt{J_1} \times \mathcal{H}_2$$

and

$$\sqrt{\mathcal{H}_1 \times J_2} = \mathcal{H}_1 \times \sqrt{J_2}$$

Theorem 3.11. Let $\mathcal{H}$ be the product of two unital rings $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. Then, the following properties are valid.

i. If J_1 is a P1ARP ideal of $\mathcal{H}_1$, then $J_1 \times \mathcal{H}_2$ is a P1ARP ideal of $\mathcal{H}$.

ii. If J_2 is a P1ARP ideal of $\mathcal{H}_2$, then $\mathcal{H}_1 \times J_2$ is a P1ARP ideal of $\mathcal{H}$.

Proof. Let $\mathcal{H}$ be the product of two unital rings $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$.

i. Assume that $(x, y)\mathcal{H}(z, d)\mathcal{H}(k, l) = x\mathcal{H}_1 z\mathcal{H}_1 k \times y\mathcal{H}_2 d\mathcal{H}_2 l \subseteq J_1 \times \mathcal{H}_2$ for nonunits $(x, y), (z, d), (k, l) \in \mathcal{H}$. Since J_1 is a P1ARP ideal, $xz \in J_1$ or $k \in \sqrt{J_1}$. Thus, $(x, y)(z, d) \in J_1 \times \mathcal{H}_2$ or $(k, l) \in \sqrt{J_1} \times \mathcal{H}_2 = \sqrt{J_1 \times \mathcal{H}_2}$. Therefore, $J_1 \times \mathcal{H}_2$ is a P1ARP ideal of $\mathcal{H}$.

ii. The proof is similar to the proof of *i.*

□

3.3. Results in Local Rings

This subsection presents the following useful results on local rings.

Theorem 3.12. If $\mathcal{H}$ has a P1ARP ideal that is not PRP, then $\mathcal{H}$ is local.

Proof. If J is a P1ARP ideal of $\mathcal{H}$ that is not PRP, then there exist nonunits $x, y \in \mathcal{H}$ such that $x\mathcal{H}y \subseteq J$, but $x \notin J$ and $\langle y\rangle^n \nsubseteq J$. Assume that k is a nonunit and l is a unit element of $\mathcal{H}$. Suppose that $k+l$ is a nonunit. Since J is a P1ARP ideal and $k\mathcal{H}x\mathcal{H}y \subseteq J$, then $kx \in J$. Thus, $(k+l)\mathcal{H}x\mathcal{H}y \subseteq J$ and hence $(k+l)x \in J$, i.e., $lx \in J$. But since l is a unit, then $x \in J$, which contradicts the assumption $x \notin J$. Therefore, $\mathcal{H}$ is local by Lemma 2.11. □

Theorem 3.13. Let $\mathcal{H}$ be a local ring with the unique maximal ideal M. Thus, the following are identical:

i. J is a P1ARP ideal.

ii. J is a PRP ideal or $M^2 \subseteq J \subseteq M$.

Proof. Assume that $\mathcal{H}$ is a local ring with the unique maximal ideal M.

$i \Rightarrow ii$: Suppose that J is a P1ARP ideal that is not prime. Then, $J \subsetneq M$, and there exist $x, y \in M \setminus J$ with $x\mathcal{H}y \subseteq J$. Let $k, l \in M$. Then, $(k\mathcal{H}l\mathcal{H})\mathcal{H}x\mathcal{H}y \subseteq J$. Consider that M is the unique maximal ideal of $\mathcal{H}$, $(k\mathcal{H}l) \subseteq M$, $x, y \in M$, and $y \notin J \subseteq \sqrt{J}$. Noting that J is a P1ARP ideal, then $(k\mathcal{H}l)x \subseteq J$. Moreover, since $k, l, x \in M$ and $x \notin J \subseteq \sqrt{J}$, then $kl \in J$.

$ii \Rightarrow i$: It can be observed that if J is a PRP ideal, then it is a P1ARP ideal. Assume that $M^2 \subseteq J \subseteq M$. Hence, J is a proper ideal. Suppose that $x\mathcal{H}y\mathcal{H}z \subseteq J$, for $x, y, z \in M$. Thus, $xy \in M^2 \subseteq J$. Therefore, J is a P1ARP ideal. □

Proposition 3.14. *A* is not a PRP ideal of $\mathcal{H}$ that is a P1ARP ideal if and only if $\mathcal{H}$ is a local ring whose maximal ideal M fulfills $M^2 \neq M$.

Proof. (⇒) Assume that A is not a PRP ideal which is a P1ARP. Then, by Theorem 3.13, $\mathcal{H}$ is a local ring and its maximal ideal M satisfies $M^2 \subseteq A \subset M$ and thus $M^2 \neq M$.

(⇐) Assume that M is a maximal ideal of a local ring $\mathcal{H}$ and $M^2 \neq M$. Thus, $M^2 \subset M$. Suppose that $x, y \in M \setminus M^2$. Thus, $\langle x\rangle\langle y\rangle \subseteq M^2$, but neither $\langle x\rangle \subseteq M^2$ nor $\langle y\rangle^n \subset \langle y\rangle \subseteq M^2$. Thus, M^2 is not a P1ARP ideal. If $x, y, z \in \mathcal{H}$ are nonunits with $x\mathcal{H}y\mathcal{H}z \subseteq M^2$, then M^2 is a P1ARP ideal which is not PRP since $xy \in M^2$. □

Proposition 3.15. Let K and L be P1ARP ideals of $\mathcal{H}$ which are not PRP. Then, $K + M$ and $K \cap L$ (or KL) are P1ARP ideals.

Proof. By Theorem 3.12, $\mathcal{H}$ is a local ring, and from Theorem 3.13, $M^2 \subseteq K \cap L$. Suppose that $uRvRk \subseteq K \cap L$, where $u, v, k \in H$ are nonunits and $k \notin \sqrt{K \cap L} = \sqrt{KL}$. Since $u, v \in M$, then $uv \in M^2 \subseteq K \cap L \subseteq KL$. Therefore, $K \cap L$ (or KL) is a P1ARP ideal. With a similar way, it can be observed that $K + L$ is a P1ARP ideal. □

Proposition 3.16. Let H_1 and H_2 be unital rings, and define $\mathcal{H}$ as their direct product, i.e., $\mathcal{H} = H_1 \times H_2$. If $A_1 \times A_2$ is a P1ARP ideal of $\mathcal{H}$, where A_1 and A_2 are ideals of H_1 and H_2, respectively, then A_1 and A_2 are P1ARP ideals of H_1 and H_2, respectively.

Proof. For some nonunits $u, v, z \in H_1$, assume that $uH_1vH_1z \subseteq A_1$. Then, for $a \in H_2$,

$$(uH_1vH_2z, aH_2aH_2a) \subseteq A_1 \times A_2$$

Since $A_1 \times A_2$ is a P1ARP ideal, then $(u, a)(v, a) \in A_1 \times A_2$ or $(z, a) \in \sqrt{A_1 \times A_2} = \sqrt{A_1} \times \sqrt{A_2}$. Hence, $uv \in A_1$ or $z \in \sqrt{A_1}$ and thus A_1 is a P1ARP ideal of H_1. In similar manner, it can be observed that A_2 is a P1ARP ideal of H_2. □

3.4. Idealization

This section explores certain properties of P1ARP ideals within the idealization of a ring. Recall that the structure $\mathcal{H} \boxplus M$ is referred to as the idealization, where $\mathcal{H}$ is a ring and M is an $\mathcal{H}$-$\mathcal{H}$-bimodule. The multiplication in this ring is defined as follows: Let $k, l \in H$ and $z, t \in M$, then $(k, z)(l, t) = (kl, kt + zl)$. In Remark 3.1 of [12], it is established that an element $(u, v) \in \mathcal{H} \boxplus M$ is a nonunit if and only if u is a nonunit in $\mathcal{H}$.

Theorem 3.17. Let $\mathcal{H}$ be a unital ring and M be a $\mathcal{H}$-$\mathcal{H}$-bimodule. For $K \lhd \mathcal{H}$, $K \boxplus M$ is a P1ARP ideal of $\mathcal{H} \boxplus M$ if and only if K is a P1ARP ideal of $\mathcal{H}$.

Proof. Suppose that $K \boxplus M$ is a P1ARP ideal of $\mathcal{H} \boxplus M$ and that $uRvRz \subseteq K$, where u, v, and z are nonunits in $\mathcal{H}$. Then, for nonunits $(u, 0), (v, 0), (z, 0) \in \mathcal{H} \boxplus M$,

$$(u, 0)(\mathcal{H} \boxplus M)(v, 0)(\mathcal{H} \boxplus M)(z, 0) \subseteq K \boxplus M$$

Since $K \boxplus M$ is a P1ARP ideal of $\mathcal{H} \boxplus M$, then $(u, 0)(v, 0) \in K \boxplus M$ or $(z, 0) \in \sqrt{K \boxplus M} = \sqrt{K} \boxplus M$. Hence, $uv \in K$ or $z \in \sqrt{K}$. Then, K is a P1ARP ideal of $\mathcal{H}$. □

Remark 3.18. For an $\mathcal{H}$-$\mathcal{H}$-bisubmodule N of M and $A \lhd \mathcal{H}$, $A \boxplus N$ is an ideal of $\mathcal{H} \boxplus M$ if and only if $AM + MA \subseteq N$.

Theorem 3.19. Let $\sqrt{A}$ be a PRP ideal and N be an $\mathcal{H}$-$\mathcal{H}$-bisubmodule with $AM + MA \subseteq N$. Then, the ideal $A \boxplus N$ is a P1ARP ideal of $\mathcal{H} \boxplus M$.

Proof. From [14], $\sqrt{A \boxplus N} = \sqrt{A} \boxplus N$ is a PRP ideal of $\mathcal{H} \boxplus M$ since $\sqrt{A}$ is a PRP ideal of $\mathcal{H}$. Thus, by Proposition 3.5, $A \boxplus N$ is a P1ARP ideal of $\mathcal{H} \boxplus M$. □

4. Conclusion

In this study, we have investigated the structural properties of P1ARP ideals, their behaviour under homomorphisms, and the conditions they satisfy in ring extensions such as idealization, leading to significant results. On the other hand, certain localized studies in commutative ring theory—such as the relationship between PIDs and Dedekind domains, as well as ring localizations defined via multiplicatively closed subsets, i.e., S, leading to $S^{-1}H$, remain open problems for the context of noncommutative rings. Since ring localizations defined through structures like Ore extensions differ significantly from those in commutative settings, it is natural to ask whether similar ideal-theoretic properties can be established in these cases.

As a result, we pose the following two open problems for further research:

i. What are the structural properties of P1ARP ideals in noncommutative domains? In such settings, can a meaningful relationship be established between prime ideals and PRP ideals?

ii. What structural properties do P1ARP ideals exhibit in ring localizations defined via Ore or Dorroh extensions? Under what conditions—analogous to those in commutative rings—can similar behaviors be observed in these noncommutative settings?

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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