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ASYMPTOTIC BEHAVIOR OF THE NON-AUTONOMOUS REACTION-DIFFUSION EQUATION WITH ROBIN BOUNDARY CONDITION

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Abstract. In this paper, we investigate the long-time behavior of the time-dependent reaction-diffusion equation $u_t - \Delta u + a(x)|u|^{\rho}u - b(x)|u|^{\nu}u = h(x,t)$ with Robin boundary condition. We begin this paper with the existence and uniqueness results of the solution to the problem. For the asymptotic behavior, we firstly prove the existence of an absorbing set in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$. The existence of a uniform attractor is obtained in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$.

1. Introduction

We are concerned with the existence of uniform attractors for the process associated with the solutions of the following reaction-diffusion equation:

$$u_t - \Delta u + a(x)|u|^{\rho}u - b(x)|u|^{\nu}u = h(x,t),\ (x,t) \in Q_T,$$

subject to the Robin boundary condition,

$$(\frac{\partial u}{\partial \eta} + k(x')u)|_{\partial\Omega} = 0,\ x' \in \partial\Omega,$$

and the initial condition,

$$u(x,\tau) = u_\tau(x),\ x \in \Omega, \forall \tau \in \mathbb{R}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$; $\rho, \nu > 0$ are given some numbers; T is a positive number; $\tau \in \mathbb{R}$; $Q_T = \Omega \times (\tau, T)$, $\Sigma_T = \partial\Omega \times [\tau, T]$; Δ is the n dimensional Laplace operator; $a : \Omega \to \mathbb{R}^1_+$, $b : \Omega \to \mathbb{R}^1_+$ and $k : \partial\Omega \to \mathbb{R}^1$ are given functions; h is given generalized function. The nonlinearity part and the external force h satisfy some conditions specified later. $\frac{\partial u}{\partial \eta}$ denotes the normal derivative of the function u in direction of the outer normal

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vector η. Here $u(x,t)$ is an unknown function which can represent temperature, population density, or in general the quantity of a substance.

Equation (1.1) generally arises as a mathematical model in various areas such as population dynamics and biological sciences, hydrodynamics and the heat transfer theory. Although reaction-diffusion equations with Dirichlet and Neumann boundary conditions have been extensively studied, very little work has been done for Robin boundary conditions.

The study of uniform attractor for non-autonomous dynamical systems has attracted much attention and has made a lot of progress in recent years(see, [6], [7], [10] and reference therein). But in the last two decades, the dynamical systems have been extensively studied for the autonomous case by using of the concept of global attractors(see, for example [6], [11], [12], [32] and the reference therein). In general a proper extension of the notion of a global attractor for semigroups to the case of process is the so called uniform attractor. Uniform attractors for the non-autonomous systems are the minimal compact sets which uniformly(w.r.t. time symbol) attract every bounded set of the initial data spaces.

The long time behavior of solutions of reaction-diffusion equation with Neumann or Dirichlet conditions has been studied extensively for both autonomous and non-autonomous cases. Moreover, the reaction-diffusion equation with Dirichlet boundary condition, has investigated widely, the existence and uniqueness of solution have been proven in (see [24], [29]), by the Faedo-Galerkin method, and the existence of attractors has been obtained in [2], [3],[12], [14], [15], [23]-[27]. Also, for the reaction-diffusion equations with homogeneous nonlinear boundary condition, the dynamical behavior was considered for both autonomous and non-autonomous cases(see [1], [30], [31]). On the other hand, for the reaction-diffusion equation with Robin boundary condition, the blow-up of solution was discussed in [4], [17], [20]-[22]. In [9], one of the first papers is made to the understanding of this problem with a homogeneous Robin boundary condition in a bounded domain $\Omega \subset R^n, n \leq 3$, it is shown that there exists a compact attractor.

In [19], We showed before the existence and uniqueness of the solution for considered problem as taking initial condition is zero. Moreover in [16], for the autonomous case of this problem, we obtained the existence of global attractor in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$, also proved some asymptotic regularity by using the relative stationary problem. After that in [17], we obtained some conditions for blow-up of solutions of problem (1.1)-(1.3) in finite time.

For the existence of the uniform attractor, we need to show that some kind of compactness of the family of processes. Since here our boundary condition is Robin type (linear boundary condition) and also has some negative coefficient, we come across some additional difficulties in proving the asymptotic compactness in $L_{\rho+2}(\Omega)$. To overcome this, we used some different inequalities such as Young, Hölder and which was given in Lemma 2 as well as Sobolev embedding theorems.

This paper is organized as follows. In the next section, we give some basic definitions and abstract results concerning the uniform attractors for non-autonomous dynamical systems. In section 3, we show that the existence and uniqueness of weak solution and the existence of weak continuity of family of processes associated to the problem. In section 4, we prove the existence of an absorbing set in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$ and a uniform attractor in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$.

2. Preliminaries

We begin with some useful definitions from the theory of uniform attractors for non-autonomous systems which we will use throughout the paper. We refer to [5]-[8] for more details.

Let X, Y be two Banach spaces such that $Y \hookrightarrow X$ continuously, and Σ be a parameter set. $\{U_\sigma(t,\tau), t \geq \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, is said to be a family of processes in X if for any $\sigma \in \Sigma$
$U_\sigma(t,s) o U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau, \ \ \tau \in \mathbb{R}$,
$U_\sigma(\tau,\tau) = Id$ is the identity operator, $\quad \tau \in \mathbb{R}$.
Denote by $\mathcal{B}(X)$, $\mathcal{B}(Y)$ the set of all bounded subsets of X and Y respectively.

Definition 1. *A set $B_0 \in \mathcal{B}(Y)$ is said to be a uniform (w.r.t. $\sigma \in \Sigma$) absorbing set in Y for $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(X)$, there exists $T_0 = T_0(B,\tau)$ such that $\bigcup_{\sigma \in \Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq T_0$.*

Definition 2. *A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is called uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in Y if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(X)$, we have $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is relatively compact in Y, where $\{x_n\} \subset B$, $t_n \subset [\tau, +\infty)$, $t_n \to +\infty$, $\sigma_n \subset \Sigma$ are arbitrary.*

Definition 3. *A subset $\mathcal{A} \subset Y$ is said to be the uniform attractor in Y of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if*

(i) *$\mathcal{A}$ is compact in Y;*
(ii) *for any fixed $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(X)$ we have*

$$\operatorname*{limit}_{t\to\infty}(\sup_{\sigma\in\Sigma}(\mathrm{dist}_Y(U_\sigma(t,\tau)B, \mathcal{A})) = 0$$

where $\mathrm{dist}_Y(A,B) = \sup_{x\in A}\inf_{y\in B}\|x-y\|_Y$ for A, $B \subset Y$;
(iii) *if $\mathcal{A}'$ is closed subset of Y satisfying (ii), then $\mathcal{A} \subset \mathcal{A}'$ (minimality property).*

Definition 4. *The kernel $\mathcal{K}$ of a process $\{U(t,\tau)\}_{\sigma\in\Sigma}$ acting on X consists of all bounded complete trajectories of the process $\{U(t,\tau)\}_{\sigma\in\Sigma}$:*

$$\mathcal{K} = \{u(.) : U(t,\tau)u(\tau) = u(t),\ \mathrm{dist}(u(t),u(0)) \leq C_u,\ \forall t \geq \tau,\ \tau \in \mathbb{R}\}.$$

The set $\mathcal{K}(s) = \{u(s) : u(.) \in \mathcal{K}\}$ is said to be kernel section at time $t = s$, $s \in \mathbb{R}$.

Definition 5. *A function φ is said to be translation bounded in $L_2^{\mathrm{loc}}(\mathbb{R};X)$, if*

$$\|\varphi\|_b^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\varphi(s)\|_X^2 ds < \infty.$$

Denote by $L_2^{\mathrm{b}}(\mathbb{R};X)$ the set of all translation bounded functions in $L_2^{\mathrm{loc}}(\mathbb{R};X)$.

Definition 6. *A function $\varphi \in L_2^{\mathrm{loc}}(\mathbb{R};X)$ is said to be normal if for any $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|\varphi(s)\|_X^2 ds \le \varepsilon.$$

Denote by $L_2^{\mathrm{n}}(\mathbb{R};X)$ the set of all normal functions in $L_2^{\mathrm{loc}}(\mathbb{R};X)$.

Lemma 1. *([31]) If $\varphi_0 \in L_2^{\mathrm{n}}(\mathbb{R};X)$, then for any $\tau \in \mathbb{R}$,*

$$\lim_{\gamma\to\infty} \sup_{t\ge\tau} \int_\tau^t e^{-\gamma(t-s)} \|\varphi(s)\|_X^2 ds = 0.$$

Lemma 2. *([4]) For $1 \le p < \infty$ there exists a positive constant $c_0(\Omega, p)$ such that for every $\varphi \in W_p^1(\Omega)$,*

$$\|\varphi - \frac{1}{\mathrm{meas}(\partial\Omega)} \int_{\partial\Omega} \varphi\|_{L_p(\Omega)} \le c_0(\Omega,p) \|\nabla\varphi\|_{L_p(\Omega)}.$$

3. Existence and Uniqueness Results

We shall assume $h \in L_2^{\mathrm{b}}(\mathbb{R}; L_2(\Omega))$. We will understand the solution of the considered problem in the following sense:

Definition 7. *A function $u(x,t)$, is called the weak solution of problem (1.1)-(1.3) on the interval $[\tau, T]$ if it satisfies the followings;*

$$u \in L_2(\tau,T;W_2^1(\Omega)) \cap L_{\rho+2}(\tau,T;L_{\rho+2}(\Omega)), \quad u_t \in L_2(\tau,T;L_2(\Omega)),$$
$$u(x,\tau) = u_\tau(x) \quad for \quad a.e. \quad x \in \Omega,$$

and

$$\int_\tau^T \int_\Omega u_t \varphi dx dt + \int_\tau^T \int_\Omega Du.D\varphi dx dt + \int_\tau^T \int_\Omega \left(a(x)\,|u|^\rho u - b(x)\,|u|^\nu u\right) \varphi dx dt$$
$$+ \int_\tau^T \int_{\partial\Omega} k(x') u \varphi dx' dt = \int_\tau^T \int_\Omega h \varphi dx dt \tag{3.1}$$

for all $\varphi \in L_2(\tau,T;W_2^1(\Omega)) \cap L_{\rho+2}(\tau,T;L_{\rho+2}(\Omega))$.

Theorem 1. *We assume that the following conditions are satisfied:*

(i) $\rho,\nu>0,\ \nu\le\rho$ *and* $u_\tau\in W_2^1(\Omega)\cap L_{\rho+2}(\Omega)$,

(ii) a *and* b *are positive functions,* $a\in L_\infty(\Omega)$, $b\in\begin{cases}L_{\frac{\rho+2}{\rho-\nu}}(\Omega), & \text{if } \nu<\rho,\\ L_\infty(\Omega), & \text{if } \nu=\rho.\end{cases}$

• *If* $\nu<\rho$ *then there exists a number* $a_0>0$ *such that* $a(x)\ge a_0$ *for a.e.* $x\in\Omega$.

• *If* $\nu=\rho$ *then there exists a number* $b_0>0$ *such that* $a(x)-b(x)\ge b_0$ *for a.e.* $x\in\Omega$.

(iii) $k\in L_{n-1}(\partial\Omega)$ *and there exists a number* $k_0\ge 0$ *such that* $k(x')\ge -k_0$ *for a.e.* $x'\in\partial\Omega$,

$$k_0<\begin{cases}\frac{\min\{a',1\}}{c_3^2}, & \text{if } 0<\nu<\rho,\\[2ex] \frac{\min\{b',1\}}{c_3^2}, & \text{if } \nu=\rho.\end{cases}$$

Then problem (1.1)-(1.3) is solvable for any $\tau\in\mathbb{R}$ *(here* a' *and* b' *are positive numbers such that* $a'<a_0$, $b'<b_0$, c_3 *comes from Sobolev embedding* [1]*).*

Proof. Although the existence of a weak solution was proved in [16], we present another proof with some weaker conditions on relations between coefficient functions.

Consider the approximating solution u_n in the form,

$$u_n(t)=\sum_{j=1}^{n}u_{nj}(t)w_j,$$

where $w_j\subset W_2^1(\Omega)\cap L_{\rho+2}(\Omega)$ is a Hilbert basis of $L_2(\Omega)$ such that $\operatorname{span}\{w_j\}_{j\ge1}$ is dense in $W_2^1(\Omega)\cap L_{\rho+2}(\Omega)$. We get u_n from solving the following problem:

$$\frac{d}{dt}\langle u_n,w_j\rangle+\langle\nabla u_n,\nabla w_j\rangle+\langle a(x)|u_n|^\rho u_n-b(x)|u_n|^\nu u_n,w_j\rangle+\langle k(x')u_n,w_j\rangle_{\partial\Omega}$$
$$=\langle h(x,t),w_j\rangle$$
$$\langle u_n(x,\tau),w_j\rangle=\langle u_\tau,w_j\rangle;\quad j=1,\ \ldots,n$$

In (3.2) replacing w_j by u_n, we get

$$\frac12\frac{d}{dt}(\|u_n\|^2_{L_2(\Omega)})+\|\nabla u_n\|^2_{L_2(\Omega)}+\int\limits_\Omega(a(x)|u_n|^{\rho+2}-b(x)|u_n|^{\nu+2})dx+\int\limits_{\partial\Omega}k(x')u_n^2dx'$$
$$=\int\limits_\Omega h(x,t)u_ndx,$$

$$\frac12\frac{d}{dt}(\|u_n\|^2_{L_2(\Omega)})+\frac12(\min\{1,a_0-\varepsilon\}-k_0c_3^2)(\|u_n\|^2_{W_2^1(\Omega)}+\|u_n\|^{\rho+2}_{L_{\rho+2}(\Omega)}-c')-c(\varepsilon)\|b\|^{\frac{\rho+2}{\rho-\nu}}_{L_{\frac{\rho+2}{\rho-\nu}}(\Omega)}-k_0c_3^2c'$$
$$\le\|h\|_{L_2(\Omega)}\|u_n\|_{L_2(\Omega)},$$

[1] $\|u\|_{L_2(\partial\Omega)}\le c_3\|u\|_{W_2^1(\Omega)}$

$$\frac{1}{2}\frac{d}{dt}(\|u_n\|^2_{L_2(\Omega)}) + (K_1 - \varepsilon_1)\|u_n\|^2_{W^1_2(\Omega)} + K_1\|u_n\|^{\rho+2}_{L_{\rho+2}(\Omega)} \le$$

$$c_2^2 c(\varepsilon_1)\|h\|^2_{L_2(\Omega)} + \varepsilon_1 + K_1 c' + c(\varepsilon)\|b\|^{\frac{\rho+2}{\rho-\nu}}_{L_{\frac{\rho+2}{\rho-\nu}}(\Omega)} + k_0 c_3^2 c',$$

where $K_1 = \frac{1}{2}(\min\{1, a_0 - \varepsilon\} - k_0 c_3^2) > 0$ and $\varepsilon_1 < K_1$. Integrating (3.3) from τ to t, $t \in [\tau, T]$, we have

$$\|u_n(t)\|^2_{L_2(\Omega)} + (K_1 - \varepsilon_1)\int_\tau^t \|u_n(s)\|^2_{W^1_2(\Omega)} ds + K_1 \int_\tau^t \|u_n(s)\|^{\rho+2}_{L_{\rho+2}(\Omega)} ds \le$$

$$\|u_n(\tau)\|^2_{L_2(\Omega)} + c_2^2 c(\varepsilon_1)\int_\tau^t \|h(s)\|^2_{L_2(\Omega)} ds + c(\varepsilon)\int_\tau^t \|b\|^{\frac{\rho+2}{\rho-\nu}}_{L_{\frac{\rho+2}{\rho-\nu}}(\Omega)} ds + (k_0 c_3^2 c' + \varepsilon_1 + K_1 c')(t-\tau).$$

This inequality implies that

$$\{u_n\} \text{ is bounded in } L_\infty(\tau, T; L_2(\Omega)) \cap L_{\rho+2}(\tau, T; L_{\rho+2}(\Omega)) \cap L_2(\tau, T; W^1_2(\Omega)).$$

Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u \text{ weakly star in } L_\infty(\tau, T; L_2(\Omega)),$$

$$\Delta u_n \rightharpoonup \Delta u \text{ weakly in } L_2(\tau, T; (W^1_2(\Omega))^*).$$

On the other hand, replacing w_j by $\partial_t u_n$ in (3.2), we have

$$\|\partial_t u_n\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}(\|\nabla u_n\|^2_{L_2(\Omega)} + \int_{\partial\Omega} k(x') u_n^2 dx' + \frac{2}{\rho+2}\int_\Omega a(x)|u_n|^{\rho+2}dx$$
$$-\frac{2}{\nu+2}\int_\Omega b(x)|u_n|^{\nu+2}dx) = \int_\Omega h(x,t)\partial_t u_n. \tag{3.8}$$

Using the Cauchy inequality, we have

$$\|\partial_t u_n\|^2_{L_2(\Omega)} + \frac{d}{dt}(\|\nabla u_n\|^2_{L_2(\Omega)} + \int_{\partial\Omega} k(x') u_n^2 dx' + \frac{2}{\rho+2}\int_\Omega a(x)|u_n|^{\rho+2}dx$$
$$-\frac{2}{\nu+2}\int_\Omega b(x)|u_n|^{\nu+2}dx) \le \|h\|^2_{L_2(\Omega)}. \tag{3.9}$$

Integrating (3.9) from τ to T and using previous arguments, we obtain

$$\{\partial_t u_n\} \text{ is bounded in } L_2(\tau, T; L_2(\Omega)), \tag{3.10}$$

then we have,

$$\partial_t u_n \rightharpoonup \partial_t u \text{ weakly in } L_2(\tau, T; L_2(\Omega)). \tag{3.11}$$

Let

$$f(x,u) := a(x)\,|u|^\rho\, u - b(x)\,|u|^\nu\, u.$$

Also $f(x,\boldsymbol{.}):\mathbb{R}_1 \longrightarrow \mathbb{R}_1$ is a continuous function and we have

$$\|f\|_{L_{\frac{\rho+2}{\rho+1}}(Q_T)} \leq 2^{\frac{\rho+2}{\rho+1}}\|a\|^{\frac{\rho+2}{\rho+1}}_{L_\infty(Q_T)}\|u\|^{\rho+2}_{L_{\rho+2}(Q_T)} + 2^{\frac{\rho+2}{\rho+1}}\|b\|^{\frac{\rho+2}{\rho+1}}_{L_{\frac{\rho+2}{\rho-\nu}}(Q_T)}\|u\|^{\frac{(\rho+2)(\nu+1)}{\rho+1}}_{L_{\rho+2}(Q_T)}. \tag{3.12}$$

So, f is a bounded mapping from $L_{\rho+2}(Q_T)$ to $L_{\frac{\rho+2}{\rho+1}}(Q_T)$. So we obtain that $\{f(.,u_n)\}$ is bounded in $L_{\frac{\rho+2}{\rho+1}}(Q_T)$, then

$$f(u_n) \rightharpoonup \eta \ \text{ weakly in } \ L_{\frac{\rho+2}{\rho+1}}(Q_T). \tag{3.13}$$

Then we can conclude that $u_n \to u$ strongly in $L_2(\tau,T;L_2(\Omega))$. Hence $u_n \to u$ a.e. in $\Omega\times[\tau,T]$. Since f is continuous, it follows that $f(u_n)\to f(u)$ a.e. in $\Omega\times[\tau,T]$. Then according to Lemma 1.3 (see [13], Chapter 1), we have

$$f(u_n) \rightharpoonup f(u) \ \text{ weakly in } \ L_{\frac{\rho+2}{\rho+1}}(Q_T).$$

Thus $\eta = f(u)$.

Let $g(x',u) := k(x')u$, $g: L_2(\tau,T;W_2^1(\Omega)) \to L_2(\tau,T;W_2^{-\frac{1}{2}}(\partial\Omega))$, g satisfies the following,

$$\|g(u)\|_{L_2(\tau,T;W_2^{-\frac{1}{2}}(\partial\Omega))} \leq c_3^2c_4^2\|k\|_{L_{n-1}(\partial\Omega)}\|u\|_{L_2(\tau,T;W_2^{\frac{1}{2}}(\partial\Omega))}.$$

Thus

$$g(u_n) \rightharpoonup g(u) \ \text{ weakly in } \ L_2(\tau,T;W_2^{-\frac{1}{2}}(\partial\Omega)).$$

Now, combining (3.6), (3.7), (3.11), (3.14) and (3.15) we see that u satisfies (3.1). □

Now we recall the following result for the uniqueness of the solution:

Theorem 2. *([16]) We assume that the conditions of Theorem 1 are satisfied. If there exists a positive number b_1 such that $b(x)\leq b_1$, $b_1 < a_0$ for almost every $x\in\Omega$ when $0<\nu<\rho$, then the solution is unique. Moreover u and v are solutions of problem (1.1)-(1.3), with initial data u_τ and v_τ, respectively, then*

$$\|u(x,t)-v(x,t)\|^2_{L_2(\Omega)} \leq \|u_\tau - v_\tau\|^2_{L_2(\Omega)}e^{2(b_1(\rho+1)+1)t} \quad as \quad \nu<\rho,$$

$$\|u(x,t)-v(x,t)\|^2_{L_2(\Omega)} \leq \|u_\tau - v_\tau\|^2_{L_2(\Omega)}e^{2t} \quad as \quad \nu=\rho.$$

We now define the symbol space Σ for the problem (1.1)-(1.3). Taking a fixed symbol

$$\sigma_0(s) = h_0(s), \quad h_0(s)\in L_2^{\mathrm{b}}(\mathbb{R};L_2(\Omega)).$$

We denote by $L^{\mathrm{loc}}_{2,\mathrm{w}}(\mathbb{R};L_2(\Omega))$ the space $L^{\mathrm{loc}}_2(\mathbb{R};L_2(\Omega))$ endowed with local convergence topology. Set $\Sigma_0=\{h_0(s+h): h\in\mathbb{R}\}$, and let Σ be the closure of Σ_0 in $L^{\mathrm{loc}}_{2,\mathrm{w}}(\mathbb{R};L_2(\Omega))$.

Problem (1.1)-(1.3) can be rewritten in the following form:

$$\partial_t y = A_{\sigma(t)}(y), \quad y|_{t=\tau} = y_\tau$$

where the function $\sigma(t) = h(t)$ is the symbol of the equation. Thanks to these existence and uniqueness theorems, we know that problem (1.1)-(1.3) is well posed, and generates a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ as follows

$$U_\sigma(t,\tau) : W_2^1(\Omega) \cap L_{\rho+2}(\Omega) \to W_2^1(\Omega) \cap L_{\rho+2}(\Omega), \quad U_\sigma(t,\tau)y_\tau = y(.,t),$$

where $U_\sigma(t,\tau)y_\tau$ is the unique weak solution of problem (1.1)-(1.3)(with σ in place of h) at the time t with the initial data y_τ at τ.

We obtain the following corollary immediately by using existence and uniqueness theorems:

Corollary 1. *The family of process $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is $(W_2^1(\Omega) \cap L_{\rho+2}(\Omega)) \times \Sigma$, $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$-weakly continuous, that is for any $u_\tau^n \rightharpoonup u_\tau$ in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$ and $\sigma_n \rightharpoonup \sigma_0$ in Σ, we have*

$$U_{\sigma_n}(t,\tau)u_\tau^n \rightharpoonup U_{\sigma_0}(t,\tau)u_\tau, \quad t \geq \tau.$$

4. Existence of a Uniform Attractor in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$

In this section we will show that the existence of uniform attractor in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$. Now we state our main result obtained in this section.

Theorem 3. *We assume that the conditions of Theorem 2 are satisfied. Suppose that for almost every $x \in \Omega$ and $x' \in \partial\Omega$,*

(i) *if $\nu < \rho$ then there exists a number $a_1 > 0$ such that $a(x) \leq a_1$,*
(ii) *if $0 < \rho \leq 2$ then $k(x') \geq 0$, if $\rho > 2$ then $k(x')$ satisfies the condition (iii) of Theorem 1.*

Then the processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ possesses a uniform attractor $\mathcal{A}$ in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$ for all $h(x,t)$ in $L_\infty(\mathbb{R}; L_2(\Omega))$, $h'(t) \in L_2^{\mathrm{b}}(\mathbb{R}; L_2(\Omega))$.

For the proof of this theorem , we will use Theorem 3.9 (which is in [5]). To see that the conditions of this theorem are satisfied, we give the following lemmas and useful a priori estimate for the uniformly asymptotic compactness and the existence of an absorbing set in corresponding space:

Lemma 3. *Assume that the conditions of Theorem 3 are satisfied. Then the processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ has a bounded uniform absorbing set B in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$ for all $h(t) \in L_2^{\mathrm{b}}(\mathbb{R}; L_2(\Omega))$.*

Proof. Multiplying (1.1) by u, after the integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} = -\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} k(x'^2 dx' - \int_\Omega a(x)|u|^{\rho+2}dx + \int_\Omega b(x)|u|^{\nu+2}dx + \int_\Omega h_0(t)u dx,$$

applying Hölder and Young inequality for the last three terms we deduce that

$$\frac{d}{dt}\|u\|^2_{L_2(\Omega)} - K_1\|u\|^2_{L_2(\Omega)} \leq 2c(\varepsilon_1)\|h_0(t)\|^2_{L_2(\Omega)} + 2c(\varepsilon_2)\mathrm{meas}(\Omega)(b_1)^{\frac{\rho+2}{\rho-\nu}}$$

where $K_1 := \frac{2}{c_2^2}(-\min\{1, a_0-\varepsilon_2\}+k_0c_3^2)<0$, c_2 comes from Sobolev embedding inequality. [2] By Gronwall's lemma we obtain the following inequality :

$$\|u\|^2_{L_2(\Omega)} \leq \|u_\tau\|^2_{L_2(\Omega)}e^{K_1(t-s)}+2c(\varepsilon_1)\int_\tau^t \|h_0(s)\|^2_{L_2(\Omega)}e^{K_1(t-s)}ds + 2c(\varepsilon_2)\text{meas}(\Omega)(b_1)^{\frac{\rho+2}{\rho-\nu}}\frac{e^{-K_1(t-\tau)}}{-K_1},$$

here we have used the fact that

$$\begin{aligned}
&\int_\tau^t \|h_0(s)\|^2_{L_2(\Omega)}e^{K_1(t-s)}ds \\
\leq\ & \int_{t-1}^t e^{K_1(t-s)}\|h_0\|^2_{L_2(\Omega)}ds+\int_{t-2}^{t-1} e^{K_1(t-s)}\|h_0\|^2_{L_2(\Omega)}ds+\int_{t-3}^{t-2} e^{K_1(t-s)}\|h_0\|^2_{L_2(\Omega)}ds+... \\
\leq\ & \int_{t-1}^t \|h_0\|^2_{L_2(\Omega)}ds+\int_{t-2}^{t-1} e^{K_1}\|h_0\|^2_{L_2(\Omega)}ds+\int_{t-3}^{t-2} e^{2K_1}\|h_0\|^2_{L_2(\Omega)}ds+... \\
\leq\ & (1+e^{K_1}+e^{2K_1}+...)\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))} \\
\leq\ & (\frac{1}{1-e^{K_1}})\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))}.
\end{aligned}$$

Thus we get

$$\|u\|^2_{L_2(\Omega)} \leq \|u_\tau\|^2_{L_2(\Omega)}e^{K_1t}+\frac{2c(\varepsilon_1)}{1-e^{K_1}}\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))}+\frac{2c(\varepsilon_2)\text{meas}(\Omega)(b_1)^{\frac{\rho+2}{\rho-\nu}}}{-K_1}e^{K_1\tau}$$

for convenience we denote all terms but except the first term in the right side by r_0

$$\|u(t)\|^2_{L_2(\Omega)} \leq \|u_\tau\|^2_{L_2(\Omega)}e^{K_1t}+r_0.$$

Consequently we can find a T_0 for given $\delta>0$,

$$T_0 := \frac{1}{-K_1}\ln(\frac{\|u_\tau\|^2_{L_2(\Omega)}}{\delta}),$$

such that $\|u\|^2_{L_2(\Omega)} \leq r_1$ for all $t \geq T_0$, $r_1 = r_0+\delta$.

On the other hand, multiplying (1.1) by u_t, after the integration by parts, we have

$$\|\frac{d}{dt}u\|^2_{L_2(\Omega)}+\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L_2(\Omega)}+\frac{d}{dt}\int_\Omega(\frac{a(x)|u|^{\rho+2}}{\rho+2}-\frac{b(x)|u|^{\nu+2}}{\nu+2})dx+\frac{d}{dt}\frac{1}{2}\int_{\partial\Omega}k(x'^2dx'$$

[2] $\|u\|_{L_2(\Omega)} \leq c_2\,\|u\|_{W_2^1(\Omega)}$

$$= \int\limits_{\Omega} h_0 u_t dx,$$

from here

$$\frac{d}{dt}\{\|\nabla u\|^2_{L_2(\Omega)} + \int\limits_{\Omega} 2(\frac{a(x)|u|^{\rho+2}}{\rho+2} - \frac{b(x)|u|^{\nu+2}}{\nu+2})dx + \int\limits_{\partial\Omega} k(x'^2 dx'\} \leq 2\|h_0\|^2_{L_2(\Omega)}. \quad (4.8)$$

We will show that there exists a constant $M_2 = M_2(r_1, a_1) > 0$ such that for all $t \geq T_0 + 1$ the following inequality is satisfied:

$$\int\limits_t^{t+1} \{\|\nabla u(x,s)\|^2_{L_2(\Omega)} + \int\limits_{\Omega} 2(\frac{a(x)|u(x,s)|^{\rho+2}}{\rho+2} - \frac{b(x)|u(x,s)|^{\nu+2}}{\nu+2})dx + \int\limits_{\partial\Omega} k(x'^2(x',s)dx'\}ds \leq M_2. \quad (4.9)$$

For (4.9), if we multiply (1.1) by u, after the integrating by parts, and use the conditions of theorem, we have

$$\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + 2\{\int\limits_{\Omega} |\nabla u|^2 dx + (a_0 - b_1)\int\limits_{\Omega} |u|^{\rho+2} dx + \int\limits_{\partial\Omega} k(x'^2 dx'\} \leq 2b_1 \text{meas}(\Omega) + \int\limits_{\Omega} 2h_0 u dx.$$

Now we can separate the end term of the left side such that $0 < A < 1$, and applying Young-Hölder inequalities, we have

$$\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + 2\{\int\limits_{\Omega} |\nabla u|^2 dx + \frac{(a_0 - b_1)}{a_1}\int\limits_{\Omega} (\frac{a(x)|u|^{\rho+2}}{\rho+2} - \frac{b(x)|u|^{\nu+2}}{\nu+2})dx +$$

$$A\int\limits_{\partial\Omega} k(x)|u|^2 dx'\} \leq 2b_1 \text{meas}(\Omega) + c(\varepsilon_1)\|h_0\|^2_{L_2(\Omega)} + \varepsilon_1\|u\|^2_{L_2(\Omega)} + \varepsilon_1\|u\|^2_{W_2^1(\Omega)} + 4c_4^2 c(\varepsilon_1)(1-A)^2\|k\|^2_{L_{n-1}(\partial\Omega)}.$$

We integrate last inequality from t to $t+1$ where $t \geq T_0 + 1$, we have:

$$\int\limits_t^{t+1} \{(2-\varepsilon_1)\int\limits_{\Omega} |\nabla u|^2 dx + \frac{2(a_0 - b_1)}{a_1}\int\limits_{\Omega} (\frac{a(x)|u|^{\rho+2}}{\rho+2} - \frac{b(x)|u|^{\nu+2}}{\nu+2})dx + 2A\int\limits_{\partial\Omega} k(x'^2 dx'\}d\tau$$

$$\leq \|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))} + 4c_4^2 c(\varepsilon_1)(1-A)^2\|k\|^2_{L_{n-1}(\partial\Omega)} + r_1(1+2\varepsilon_1) + 2b_1 \text{meas}(\Omega).$$

Here $A := \frac{a_0 - b_1}{a_1}$ and $\varepsilon_1 := 2 - A$, then we have

$$\int_t^{t+1} \{\int_\Omega |\nabla u|^2 dx + 2\int_\Omega (\frac{a(x)|u|^{\rho+2}}{\rho+2} - \frac{b(x)|u|^{\nu+2}}{\nu+2})dx + \int_{\partial\Omega} k(x'^2 dx'\} d\tau$$

$$\leq \frac{1}{A}(\|h\|^2_{L_2^b(\mathbb{R};L_2(\Omega))} + 4c_4^2 c(\varepsilon_1)(1-A)^2\|k\|^2_{L_{n-1}(\partial\Omega)} + r_1(1+2\varepsilon_1) + 2b_1 \mathrm{meas}(\Omega)).$$

So we obtain inequality (4.9). Denoting by

$$y = \{\int_\Omega |\nabla u|^2 dx + 2\int_\Omega (\frac{a(x)|u|^{\rho+2}}{\rho+2} - \frac{b(x)|u|^{\nu+2}}{\nu+2})dx + \int_{\partial\Omega} k(x'^2 dx'\}$$

combining with (4.8), (4.9), we have the following inequalities for all $t \geq T_0 + 1$:

$$\frac{d}{dt} y(t) \leq 2\|h_0(t)\|^2_{L_2(\Omega)} \quad \text{and} \quad \int_t^{t+1} y(s)ds \leq M_2.$$

Let $T_0 + 1 \leq t < s \leq t+1$. Then $\frac{d}{ds} y(s) \leq 2\|h_0(s)\|^2_{L_2(\Omega)}$. Integrating in s on $[z; t+1]$, where $z : t < z < t+1$, we obtain $y(t+1) \leq \int_z^{t+1} 2\|h_0\|^2_{L_2(\Omega)} ds + y(z) \leq 2\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))} + y(z)$. Finally if we integrate in z on $[t; t+1]$ we get the wanted estimate,

$$\|\nabla u\|^2_{L_2(\Omega)} + 2\int_\Omega (\frac{a(x)|u|^{\rho+2}}{\rho+2} - \frac{b(x)|u|^{\nu+2}}{\nu+2})dx + \int_{\partial\Omega} k(x'^2 dx' \leq 2\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))} + M_2$$

by using the conditions of Theorem 3 and the last inequality, we have $\forall t \geq T_0 + 1$:

$$\|u\|_{W_2^1(\Omega)} \leq M_3, \quad \|u\|_{L_{\rho+2}(\Omega)} \leq M_3.$$

Thus $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ has a bounded absorbing set in $W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$.

□

Now we give an a priori estimate for the solution of the problem to verifying the uniformly asymptotic compactness in $L_{\rho+2}(\Omega)$.

Lemma 4. *Assume that the conditions of Theorem 3 are satisfied. Then for any $\varepsilon > 0$ and any bounded subset $B \subset W_2^1(\Omega) \cap L_{\rho+2}(\Omega)$ there exist $T_2 = T_2(\varepsilon, B)$ and $\mathcal{M} = \mathcal{M}(\varepsilon, B)$ such that*

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau| \geq \mathcal{M})} |U_\sigma(t,\tau)u_\tau|^{\rho+2} dx < \varepsilon \quad \forall t \geq T_2, \quad \forall u_\tau \in B, \tag{4.11}$$

where $\Omega(|U_\sigma(t,\tau)u_\tau| \geq \mathcal{M}) = \{(x,t) : |U_\sigma(t,\tau)u_\tau| \geq \mathcal{M}\}$ for all normal function h in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$.

Proof. We multiply (1.1) by $(u-\mathcal{M})_+^{\rho+1}$ and integrating on Ω, we obtain

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega(u\geq\mathcal{M})}|(u-\mathcal{M})|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega(u\geq\mathcal{M})}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx$$
$$+\int\limits_{\Omega(u\geq\mathcal{M})}a(x)|u|^{\rho+1}(u-\mathcal{M})^{\rho+1}dx-\int\limits_{\Omega(u\geq\mathcal{M})}b(x)|u|^{\nu+1}(u-\mathcal{M})^{\rho+1}dx$$
$$+\int\limits_{\partial\Omega(u\geq\mathcal{M})}k(x')u(u-\mathcal{M})^{\rho+1}dx'=\int\limits_{\Omega(u\geq\mathcal{M})}h_0(x,t)(u-\mathcal{M})^{\rho+1}dx$$

where $(u-\mathcal{M})_+$ denotes the positive part of $(u-\mathcal{M})$, that is

$$(u-\mathcal{M})_+:=\begin{cases}u-\mathcal{M}, & \text{if } u\geq\mathcal{M},\\ 0, & \text{if } u\leq\mathcal{M}.\end{cases}$$

Set $\Omega_1:=\Omega(u(t)\geq\mathcal{M})$, $\partial\Omega_1:=\partial\Omega(u(t)\geq\mathcal{M})$, we have

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega_1}|(u-\mathcal{M})|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega_1}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx+\int\limits_{\Omega_1}a(x)|u|^{\rho+1}(u-\mathcal{M})^{\rho+1}dx-$$
$$\int\limits_{\Omega_1}b(x)|u|^{\nu+1}(u-\mathcal{M})^{\rho+1}dx+\int\limits_{\partial\Omega_1}k(x')u(u-\mathcal{M})^{\rho+1}dx'=\int\limits_{\Omega_1}h_0(x,t)(u-\mathcal{M})^{\rho+1}dx.$$

Let $0<c<a_0-b_1$ and $\mathcal{M}$ is taken as $\mathcal{M}\geq(\frac{b_1}{a_0-b_1-c})^{\frac{1}{\rho+1}}$, then on Ω_1 we have,

$$a(x)|u|^{\rho+1}-b(x)|u|^{\nu+1}\geq c|u|^{\rho+1}$$

if we use this inequality in (4.13), we have:

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega_1}|(u-\mathcal{M})|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega_1}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx+\int\limits_{\Omega_1}c|u|^{\rho+1}(u-\mathcal{M})^{\rho+1}dx+$$
$$\int\limits_{\partial\Omega_1}k(x')u(u-\mathcal{M})^{\rho+1}dx'\leq\int\limits_{\Omega_1}h_0(x,t)(u-\mathcal{M})^{\rho+1}dx \tag{4.14}$$

by applying Hölder and Young inequality for the last term and using the condition on k, we deduce that

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega_1}|(u-\mathcal{M})|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega_1}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx+\int\limits_{\Omega_1}c|u|^{\rho+1}(u-\mathcal{M})^{\rho+1}dx$$
$$-(k_0+k_0\mathcal{M})\int\limits_{\partial\Omega_1}(u-\mathcal{M})^{\rho+2}dx'-k_0\mathcal{M}\text{meas}(\partial\Omega_1)\leq\int\limits_{\Omega_1}h_0(x,t)(u-\mathcal{M})^{\rho+1}dx$$

by applying Young inequality for the term of right side, we have

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega_1}|(u-M)|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega_1}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx+\int\limits_{\Omega_1}\frac{c}{2}|u|^{\rho+1}(u-\mathcal{M})^{\rho+1}dx$$

$$-(k_0+k_0\mathcal{M})\int\limits_{\partial\Omega_1}(u-\mathcal{M})^{\rho+2}dx'-k_0\mathcal{M}\text{meas}(\partial\Omega_1)\leq\frac{1}{2c}\int\limits_{\Omega_1}h_0^2dx$$

by using $u\geq\mathcal{M}$ and $u-\mathcal{M}\leq u$ for the third term of left side we obtain,

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega_1}|(u-\mathcal{M})|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega_1}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx+\frac{c}{2}\mathcal{M}^\rho\int\limits_{\Omega_1}(u-\mathcal{M})^{\rho+2}dx$$

$$-(k_0+k_0\mathcal{M})\int\limits_{\partial\Omega_1}(u-\mathcal{M})^{\rho+2}dx'-k_0\mathcal{M}\text{meas}(\partial\Omega_1)\leq\frac{1}{2c}\int\limits_{\Omega_1}h_0^2dx$$

by using Lemma 2 and the equation

$$|\nabla((u-\mathcal{M})^{\rho+2}|=2(|(u-\mathcal{M})^{\frac{\rho}{2}+1}\nabla((u-\mathcal{M})^{\frac{\rho+2}{2}})|)$$

we have,

$$\frac{1}{\rho+2}\frac{d}{dt}\int\limits_{\Omega_1}|(u-\mathcal{M})|^{\rho+2}dx+\frac{4(\rho+1)}{(\rho+2)^2}\int\limits_{\Omega_1}(\nabla(u-\mathcal{M})^{\frac{\rho+2}{2}})^2dx+\frac{c}{2}\mathcal{M}^\rho\int\limits_{\Omega_1}(u-\mathcal{M})^{\rho+2}dx-$$

$$(k_0+k_0\mathcal{M})\frac{\text{meas}(\partial\Omega_1)}{\text{meas}(\Omega_1)}(\int\limits_{\Omega_1}(u-\mathcal{M})^{\rho+2}dx+2c_0\int\limits_{\Omega_1}|(u-\mathcal{M})^{\frac{\rho}{2}+1}\nabla((u-\mathcal{M})^{\frac{\rho+2}{2}})|dx)$$

$$\leq\frac{1}{2c}\int\limits_{\Omega_1}h_0^2dx+k_0\mathcal{M}\text{meas}(\partial\Omega_1)$$

and by using Young inequality we have,

$$\frac{d}{dt}\int\limits_{\Omega_1}|(u-\mathcal{M})|^{\rho+2}dx+\mu\int\limits_{\Omega_1}(u-\mathcal{M})^{\rho+2}dx\leq\frac{\rho+2}{2c}\int\limits_{\Omega_1}h_0^2dx+(\rho+2)k_0\mathcal{M}\text{meas}(\partial\Omega_1)$$

where

$$\mu=(\rho+2)[\frac{c}{2}\mathcal{M}^\rho-k_0(\mathcal{M}+1)\frac{\text{meas}(\partial\Omega_1)}{\text{meas}(\Omega_1)}-[c_0(1+\mathcal{M})k_0\frac{\text{meas}(\partial\Omega_1)}{\text{meas}(\Omega_1)}]^2\frac{1}{\varepsilon_2}],$$

and $\varepsilon_2=\frac{\rho+1}{(\rho+2)^2}$. Since $\rho>2$, we can choose $\mathcal{M}$ sufficiently large enough such that $\mu>0$, we integrate this inequality on (ℓ,t) after the multiplying by $e^{\mu t}$ where $\ell\geq T_0+1$, we have

$$\int\limits_{\Omega_1}|u-\mathcal{M}|^{\rho+2}dx\leq\|u-\mathcal{M}\|^{\rho+2}_{L_{\rho+2}(\Omega_1)}(\ell)e^{-\mu(t-\ell)}+\frac{\rho+2}{2c}\int\limits_{\ell}^{t}(e^{-\mu(t-s)}\int\limits_{\Omega_1}h_0^2dx)ds+$$

$$\frac{1}{\mu}(1-e^{\mu(l-t)})(\rho+2)k_0\mathcal{M}\text{meas}(\partial\Omega_1)$$

for any $\varepsilon > 0$, we can take $\mathcal{M}$ large enough such that

$$\frac{\rho+2}{2c}\int_{\ell}^{t} e^{-\mu(t-s)}\int_{\Omega_1} h_0^2 dxds < \frac{\varepsilon}{3}, \tag{4.15}$$

$$\frac{1}{\mu}(1-e^{\mu(l-t)})(\rho+2)k_0\mathcal{M}\text{meas}(\partial\Omega_1) < \frac{\varepsilon}{3}. \tag{4.16}$$

If we choose T_2 as the following,

$$T_2 := \frac{1}{\mu}\ln(\frac{3M_3^{\rho+2}}{\varepsilon}) + \ell$$

where M_3 is in the proof of Lemma 3. Then $\forall t > T_2$, we have

$$\|u-\mathcal{M}\|^{\rho+2}_{L_{\rho+2}(\Omega_1)}(\ell)e^{-\mu(t-\ell)} < \frac{\varepsilon}{3}. \tag{4.17}$$

From (4.15), (4.16), (4.17) we have

$$\int_{\Omega_1}|u-\mathcal{M}|^{\rho+2}dx < \varepsilon, \quad \forall t > T_2. \tag{4.18}$$

Repeating the same step above, just taking $|(u+\mathcal{M})_-|^{\rho}(u+\mathcal{M})_-$, we have that there exists $\mathcal{M}_4$ and T_2' such that

$$\int_{\Omega(u\leq-\mathcal{M})}|(u+\mathcal{M})_-|^{\rho+2}dx < \varepsilon, \quad \text{for} \quad \text{any} \quad t \geq T_2', \quad \mathcal{M} \geq \mathcal{M}_4 \tag{4.19}$$

where $(u+\mathcal{M})_-$ denotes the negative part of $(u+\mathcal{M})$, that is

$$(u+\mathcal{M})_- := \begin{cases} u+\mathcal{M}, & \text{if } u \leq -\mathcal{M}, \\ 0, & \text{if } u \geq -\mathcal{M}. \end{cases}$$

We obtain by using (4.18) and (4.19),

$$\int_{\Omega(|u|\geq\mathcal{M}')}|(|u|-\mathcal{M}'^{\rho+2}dx = \int_{\Omega(u\geq\mathcal{M}')}|(u-\mathcal{M}'^{\rho+2}dx + \int_{\Omega(u<-\mathcal{M}')}|(u+\mathcal{M}'^{\rho+2}dx < 2\varepsilon,$$

where $t \geq \max\{T_2, T_2'\}$, $\mathcal{M}^{'} = \max\{\mathcal{M}, \mathcal{M}_4\}$ then

$$\int_{\Omega(|u|\geq2\mathcal{M}')}|u|^{\rho+2}dx = \int_{\Omega(|u|\geq2\mathcal{M}')}((|u|-\mathcal{M}')+\mathcal{M}'^{\rho+2}dx$$

$$\leq 2^{\rho+1}(\int_{\Omega(|u|\geq2\mathcal{M}')}(|u|-\mathcal{M}'^{\rho+2}dx) + \int_{\Omega(|u|\geq2\mathcal{M}')}\mathcal{M}'^{\rho+2}dx)$$

$$\leq 2^{\rho+1}(\int\limits_{\Omega(|u|\geq\mathcal{M}')}(|u|-\mathcal{M}'^{\rho+2}dx+\int\limits_{\Omega(|u|\geq\mathcal{M}')}(|u|-\mathcal{M}'^{\rho+2}dx).$$

Thus, we have

$$\int\limits_{\Omega(|u|\geq 2\mathcal{M}')}|u|^{\rho+2}dx<2^{\rho+3}\varepsilon.$$

Last inequality completes the proof of lemma. □

For the proof of uniformly asymptotic compactness in $W_2^1(\Omega)$, first we will give some a priori estimate on u_t in $L_2(\Omega)$-norm.

Lemma 5. *Assume that the conditions of Theorem 3 are satisfied. Then for any $\tau\in\mathbb{R}$ and any bounded subset $B\subset W_2^1(\Omega)\cap L_{\rho+2}(\Omega)$ there exists a positive constant $T_1=T_1(B,\tau)>0$ such that:*

$$\int\limits_{\Omega}u_t^2dx\leq R \quad for \quad all \quad t\geq T_1, \quad u_\tau\in B \tag{4.20}$$

for any translation bounded $h_0(t)$ and $h_0'(t)$ in $L_2^{loc}(\mathbb{R},L_2(\Omega))$, where $u_t(s)=\frac{d}{dt}(U_{\sigma(t,\tau)}u_\tau)\mid_{t=s}$, R depends on $r_1,a_1,\|k\|_{L_{n-1}(\partial\Omega)}$.

Proof. We denote by $u_t=v$ and by differentiating (1.1),(1.2) in time, we get

$$v_t-\Delta v+va(x)(\rho+1)\,|u|^\rho-vb(x)(\nu+1)\,|u|^\nu=h_0'(t),$$

$$\frac{\partial v}{\partial\eta}+k(x')v=0,$$

multiplying the first equality by v integrating over Ω and using the conditions of Theorem 1 we obtain that

$$\frac{d}{dt}\int\limits_{\Omega}v^2dx\leq 2(b_1(\rho+1)+k_0c_3^2+\frac{1}{2})\int\limits_{\Omega}v^2dx+\|h_0'(t)\|^2_{L_2(\Omega)}. \tag{4.21}$$

We will show that there exist $T_1>0$ and $M_2'>0$ such that $\forall t\geq T_1$,

$$\int\limits_t^{t+1}\int\limits_{\Omega}v^2dxds\leq M_2'. \tag{4.22}$$

Integrating inequality (4.7) from t to $t+1$, we have:

$$\int\limits_t^{t+1}\|u_t\|^2_{L_2(\Omega)}ds+(\frac{1}{2}\|\nabla u\|^2_{L_2(\Omega)}+\int\limits_{\Omega}(\frac{a(x)|u|^{\rho+2}}{\rho+2}-\frac{b(x)|u|^{\nu+2}}{\nu+2})dx+\frac{1}{2}\int\limits_{\partial\Omega}k(x'^2dx')(t+1)=$$

$$(\frac{1}{2}\|\nabla u\|^2_{L_2(\Omega)}+\int\limits_{\Omega}(\frac{a(x)|u|^{\rho+2}}{\rho+2}-\frac{b(x)|u|^{\nu+2}}{\nu+2})dx+\frac{1}{2}\int\limits_{\partial\Omega}k(x'^2dx')(t)$$

$$+\int_t^{t+1}\int_\Omega h_0(t)u_t dxds,$$

here we use inequality (4.3), then $\forall t \geq T_0+1$ (which is the proof of Lemma 3), we obtain that:

$$\frac{1}{2}\int_t^{t+1}\|u_t\|^2_{L_2(\Omega)}ds+(\frac{1}{2}\|\nabla u\|^2_{L_2(\Omega)}+\int_\Omega(\frac{a(x)|u|^{\rho+2}}{\rho+2}-\frac{b(x)|u|^{\nu+2}}{\nu+2})dx+\frac{1}{2}\int_{\partial\Omega}k(x'^2dx')(t+1)\leq$$

$$M_2+\frac{1}{2}\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))}$$

then

$$\frac{1}{2}\int_t^{t+1}\|u_t\|^2_{L_2(\Omega)}ds+(\int_\Omega(\frac{a(x)|u|^{\rho+2}}{\rho+2}-\frac{b(x)|u|^{\nu+2}}{\nu+2})dx)(t+1)\leq$$

$$\frac{k_0c_3^2}{2}\|u\|^2_{L_2(\Omega)}(t+1)+M_2+\frac{1}{2}\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))},$$

applying Hölder inequality and using Lemma 3, we have the following inequality $\forall t\geq T_0+1:$

$$\frac{1}{2}\int_t^{t+1}\|u_t\|^2_{L_2(\Omega)}ds\leq\frac{k_0c_3^2}{2}r_1+M_2+\frac{1}{2}\|h_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega))}+\frac{b_1\text{meas}(\Omega)}{\nu+2}+\frac{M_1^{\rho+2}b_1}{\nu+2}.$$

Thus we obtain (4.22). Consequently combining (4.21) with (4.22) and uniform Gronwall lemma we obtain (4.20).

□

Corollary 2. *We assume that the conditions of Theorem 3 are satisfied. Then the processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ possesses a uniform global attractor $\mathcal{A}$ in $L_2(\Omega)$ for any translation bounded $h\in L_2^{\text{loc}}(\mathbb{R};L_2(\Omega))$.*

Proof. If we consider Lemma 3 and embedding $W_2^1(\Omega)\cap L_{\rho+2}(\Omega)\subset L_2(\Omega)$, we have an absorbing set in $L_2(\Omega)$. Asymptotic compactness property of the processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ is clear because of compact embedding $W_2^1(\Omega)\cap L_{\rho+2}(\Omega)\hookrightarrow L_2(\Omega)$. Hence, we have the existence of the uniform attractor $\mathcal{A}$ in $L_2(\Omega)$ immediately. □

Lemma 6. *We assume the conditions of Theorem 3, then the family of process $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ is uniform asymptotically compact in $W_2^1(\Omega)$.*

Proof. We need to show that for any $\{u_{\tau_n}\}\subset B_1$, $\sigma_n\subset\Sigma$ and $t_n\to\infty$, $\{U_{\sigma_n}(t_n,\tau_n)\}_{n=1}^\infty$ is precompact in $W_2^1(\Omega)$. Denote by $u_n^{\sigma_n}(t_n):=\{U_{\sigma_n}(t_n,\tau_n)\}u_{\tau_n}$. We need to prove that $\{u_n^{\sigma_n}(t_n)\}$ is a Cauchy sequence in $W_2^1(\Omega)$:

$$\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|^2_{W_2^1(\Omega)}=\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|^2_{L_2(\Omega)}+\|\nabla u_n^{\sigma_n}(t_n)-\nabla u_m^{\sigma_m}(t_m)\|^2_{L_2(\Omega)},$$

here

$$\|\nabla u_n^{\sigma_n}(t_n)-\nabla u_m^{\sigma_m}(t_m)\|^2_{L_2(\Omega)}=\int\limits_\Omega(\nabla u_n^{\sigma_n}(t_n)-\nabla u_m^{\sigma_m}(t_m))(\nabla u_n^{\sigma_n}(t_n)-\nabla u_m^{\sigma_m}(t_m))dx,$$

$$=-\int\limits_\Omega(\Delta u_n^{\sigma_n}(t_n)-\Delta u_m^{\sigma_m}(t_m))(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))dx-\int\limits_{\partial\Omega}k(x')(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))^2dx',$$

$$=-\int\limits_\Omega(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))(\frac{d}{dt}u_n^{\sigma_n}(t_n)-\frac{d}{dt}u_m^{\sigma_m}(t_m))dx$$

$$-\int\limits_\Omega(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))(a(x)|u_n^{\sigma_n}|^\rho u_n^{\sigma_n}-b(x)|u_n^{\sigma_n}|^\nu u_n-a(x)|u_m^{\sigma_m}(t_m)|^\rho u_m^{\sigma_m}(t_m)$$

$$+b(x)|u_m^{\sigma_m}(t_m)|^\nu u_m^{\sigma_m}(t_m))dx-\int\limits_\Omega(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))(\sigma_m-\sigma_n)dx$$

$$-\int\limits_{\partial\Omega}k(x')(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))^2dx',$$

by using conditions we have:

$$(1-k_0c_3^2)\|\nabla u_n^{\sigma_n}(t_n)-\nabla u_m^{\sigma_m}(t_m)\|^2_{L_2(\Omega)}$$
$$\leq\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L_2(\Omega)}\|\|\frac{d}{dt}u_n^{\sigma_n}(t_n)-\frac{d}{dt}u_m^{\sigma_m}(t_m)\|_{L_2(\Omega)}+(b_1(\rho+1)$$
$$+k_0c_3^2)\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|^2_{L_2(\Omega)}+\|\sigma_n-\sigma_m\|_{L_2(\Omega)}\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L_2(\Omega)}.$$

Consequently

$$\begin{aligned}&\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|^2_{W_2^1(\Omega)}\\ \leq\ &\frac{1}{1-k_0c_3^2}(\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L_2(\Omega)}\|\frac{d}{dt}(u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m))\|_{L_2(\Omega)})\\ &+(\frac{b_1(\rho+1)+k_0c_3^2}{1-k_0c_3^2}+1)\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|^2_{L_2(\Omega)}\\ &+\frac{1}{1-k_0c_3^2}\|\sigma_n-\sigma_m\|_{L_2(\Omega)}\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L_2(\Omega)}.\end{aligned}$$

Here if we use asymptotic compactness in $L_2(\Omega)$ and inequality (4.20) then we obtain asymptotic compactness in $W_2^1(\Omega)$. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. From Corollary 2 and Lemma 5, it is easy to verify that $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ has uniformly asymptotic compactness in $L_{\rho+2}(\Omega)$. If we consider also Lemma 3 and Lemma 6, we can obtain the existence of uniform global attractor in $W_2^1(\Omega)\cap L_{\rho+2}(\Omega)$ immediately by using Theorem 3.9(which is in [5]). □

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