

Investigating the Interplay of Codazzi Couplings and Connections in Metallic-like Pseudo-Riemannian Manifolds

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ABSTRACT

In this paper, we explore the characteristics of metallic-like pseudo-Riemannian manifolds that employ various linear connections. We establish a series of relationships regarding these connections, specifically examining the links between distinct types and their counterparts. Our findings reveal that a particular type of connection is associated with another if and only if its counterpart exhibits a corresponding relationship, and vice versa. Furthermore, we present significant equalities pertaining to the intrinsic properties of the manifolds, illustrating how these characteristics interact within a metallic-like context. We discuss the conditions required for a particular type of coupling, revealing significant connections between the properties of different associated connections. Additionally, we derive crucial equivalences that emphasise the relationship between torsion, statistical structures, and Codazzi coupling across various connections. Collectively, our results offer a cohesive framework that clarifies the geometric and algebraic foundations of metallic-like pseudo-Riemannian manifolds, enhancing our understanding of their structure and properties.

Keywords: Metallic-like pseudo-Riemannian manifolds, conjugate connections, Codazzi coupled.

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1. Introduction

The study of geometric structures on manifolds has long been a central focus in differential geometry, offering deep insights into both theoretical and applied mathematics. Within this framework, Hermitian and Kähler manifolds have occupied a pivotal role due to their rich interaction between geometry and complex analysis. A classical Hermitian manifold is characterized by a Riemannian metric g and an almost complex structure J , which is compatible with g such that $g(JX, JY) = g(X, Y)$ for any vector fields X and Y . When the almost complex structure J is parallel with respect to the Levi-Civita connection ∇^g , the manifold is termed Kähler, further distinguished by its enhanced symmetry and integrability, which impose strong constraints on its geometric properties. The paper [10] by Fei and Zhang explores the relationship between Codazzi couplings and (para-)Kähler structures in differential geometry. It introduces the concept of Codazzi-(para-)Kähler structures, which extend (para-)Kähler manifolds by incorporating an affine connection that is Codazzi-coupled with a compatible triple of a metric g , a symplectic form ω , and an almost complex (or para-complex) structure J . The authors demonstrate that the actions of g -conjugation, ω -conjugation, and J -gauge transformation form a Klein group, leading to significant structural insights. They also generalize special Kähler geometry to Codazzi-Kähler geometry, where the connection need not be curvature-free, and clarify how statistical structures can be enhanced to (para-)Kähler manifolds. Overall, the paper bridges gaps

in understanding the interplay between Codazzi coupling and Kähler geometry, emphasizing their importance in affine differential geometry and information geometry.

However, these classical notions are insufficient to fully encompass the complexity of certain pseudo-Riemannian and generalized geometric contexts, where the interplay of multiple structures is required. In response to this limitation, recent research has introduced the concept of Hermite-like manifolds as a natural extension of Hermitian geometry to pseudo-Riemannian settings. A Hermite-like manifold is defined by the presence of two distinct almost complex structures, J and J^* , which satisfy a symmetry condition with respect to a pseudo-Riemannian metric g , namely $g(JX, Y) = g(X, J^*Y)$ (for details, see [19, 20]). When $J = J^*$, the structure reduces to that of a classical Hermitian manifold. This generalization broadens the scope of pseudo-Riemannian manifolds by incorporating the interaction between multiple complex structures, thus facilitating the modeling of more intricate geometric and algebraic phenomena. In this extended framework, Kähler-like manifolds [19, 20] arise when J is parallel with respect to the Levi-Civita connection of g , extending the classical Kähler condition to scenarios involving dual almost complex structures. These innovative manifold structures offer a flexible platform for the exploration of complex geometries within pseudo-Riemannian settings, with potential applications in mathematical physics and information geometry.

Inspired by the framework of Hermite-like manifolds, several related structures have been examined, such as contact-like manifolds and product-like manifolds as studied in [8, 9], and para-Kähler-like and quaternionic-like Kähler manifolds, as explored in [22, 23].

The field of statistical geometry has attracted considerable attention in recent years due to its applications in disciplines such as machine learning, artificial intelligence, and theoretical physics. Originally introduced by Amari [2] in 1985, statistical manifolds provide a geometric framework for analyzing the relationships between statistical models and their geometric properties. In this context, the manifold's metric encodes the statistical divergence or distance between distinct models, allowing for a geometrical interpretation of key concepts such as estimation, learning, and optimization. The emergence of Hermite-like and Kähler-like structures introduces new avenues for studying statistical manifolds by incorporating dualistic properties and complex symmetries. In particular, the structures of Hermite-like manifolds offer a novel lens through which to examine the intricate relationships between statistical models and their underlying geometric representations, thereby enriching the field of information geometry.

Another recent advancement in differential geometry involves metallic structures, inspired by the algebraic properties of generalized Fibonacci-like sequences known as metallic numbers. A metallic structure on a manifold is defined by an endomorphism J of the tangent bundle that satisfies a characteristic equation of the form $J^2 = pJ + qI$, where p and q are real constants, and I is the identity transformation [4, 11, 12]. However, depending on the values of p and q , certain constraints may apply:

- If $p = 0$ and $q < 0$, the manifold must be even-dimensional.
- If $p = 0$ and $q > 0$, J is not uniquely determined.

These conditions highlight the interplay between the algebraic properties of J and the geometric structure of the underlying manifold. This structure serves to unify and generalize both almost complex ($J^2 = -I$) and almost product ($J^2 = I$) structures, providing a powerful framework for investigating geometric spaces with complex symmetries. In particular, metallic-like pseudo-Riemannian manifolds arise when a pseudo-Riemannian manifold (M, g) is equipped with two interacting metallic structures J and J^* , which satisfy the relations $g(JX, Y) = g(X, J^*Y)$ and $(J^*)^2 = pJ^* + qI$ [7, 14].

The development of Hermite-like, Kähler-like, and metallic-like structures exemplifies the ongoing shift in differential geometry towards more generalized and versatile frameworks, capable of modeling complex geometric phenomena that emerge in various fields. In the realm of information geometry, these structures offer fresh perspectives for understanding the dualistic properties of statistical manifolds and exponential families. In mathematical physics, they provide novel models for spaces with complex or hypercomplex symmetries, enabling deeper analysis of geometric flows, curvature properties, and harmonic maps in generalized settings.

This paper aims to further advance the theory of metallic-like pseudo-Riemannian manifolds. The findings regarding metallic-like pseudo-Riemannian manifolds reveal a complex interplay of geometric structures and connections. Different types of connections, such as J -connections and J^* -connections, exhibit relationships that influence their properties, particularly concerning torsion and curvature. Compatibility conditions between metrics and structures like g , G , J , and J^* are crucial, establishing various equivalences related to Codazzi coupling, which refers to specific compatibility conditions between different connections and geometric structures. The research highlights the significance of torsion relations, showing that under certain conditions, the torsion tensors of various connections can be equal, impacting the overall geometry of the manifold. Additionally, the existence of statistical structures is discussed, illustrating how specific connections can lead to the emergence of statistical properties when combined with certain geometric

structures. Overall, many conditions regarding connections, torsion, curvature, and compatibility can be transformed into equivalent statements, emphasizing the intricate connections among various geometric and algebraic structures on these manifolds.

2. Basic Definitions and Preliminary Results

We follow the notations and definitions for metallic pseudo-Riemannian manifolds as introduced in [4]. Let M be a smooth real manifold, and let $J : TM \rightarrow TM$ be a tangent bundle isomorphism. A $(1, 1)$ -tensor field J on M is called a metallic structure if it satisfies the characteristic equation $J^2 = pJ + qI$, where I is the identity operator and $p, q \in \mathbb{R}$. The integrability of a metallic structure is characterized by the vanishing of its Nijenhuis tensor N_J , defined as $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]$, where X, Y , and Z denote arbitrary differentiable vector fields on the manifold or vectors in its tangent space at a given point. Furthermore, a linear connection ∇ on M is termed a J -connection if $\nabla J = 0$. An alternative definition for the integrability of a polynomial structure, as presented by Vanzura in [21], states that a metallic structure J is integrable if and only if there exists a torsion-free linear connection ∇ such that $\nabla J = 0$; that is, J -integrability is equivalent to the existence of a torsion-free J -connection. In the context of pseudo-Riemannian geometry, let (M, g) be a pseudo-Riemannian manifold and let J be a g -symmetric $(1, 1)$ -tensor field on M satisfying $J^2 = pJ + qI$, where p and q are real constants. Then, (J, g) is termed a metallic pseudo-Riemannian structure, and the triple (M, J, g) is referred to as a metallic pseudo-Riemannian manifold. If a metallic pseudo-Riemannian manifold (M, J, g) has a Levi-Civita connection ∇^g (associated with g) such that $\nabla^g J = 0$, it is called a locally metallic pseudo-Riemannian manifold [1, 5].

Let (M, g) be a pseudo-Riemannian manifold, and let ∇ denote a torsion-free linear connection on M . The triple (M, ∇, g) is referred to as a statistical manifold if the condition ∇g is symmetric. For a statistical manifold (M, ∇, g) , we define another linear connection ∇^* by the relation

$$g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) = Xg(Y, Z)$$

for arbitrary vector fields X, Y, Z on M . The linear connection ∇^* is called the conjugate (or dual) connection of ∇ with respect to g . This conjugate connection ∇^* is also torsion-free, $\nabla^* g$ is symmetric, and it satisfies the property $(\nabla^*)^* = \nabla$. Consequently, the triple (M, ∇^*, g) forms a statistical manifold. We denote the curvature tensors associated with the linear connections ∇ and ∇^* by R and R^* , respectively. It follows that

$$g(R(X, Y)Z, W) = -g(Z, R^*(X, Y)W)$$

for vector fields X, Y, Z, W on M , where $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$. Therefore, R vanishes identically if and only if R^* does as well. We say that (M, ∇, g) is flat if R vanishes identically.

Let (M, g) be a pseudo-Riemannian manifold equipped with a metallic structure J and an additional $(1, 1)$ -tensor field J^* such that the relation

$$g(JX, Y) = g(X, J^*Y)$$

holds. In this context, the manifold (M, J, g) is referred to as a metallic-like pseudo-Riemannian manifold [7, 14]. Furthermore, we can define a $(0, 2)$ -tensor G as

$$G(X, Y) = g(JX, Y),$$

which satisfies the condition

$$G(JX, Y) = G(X, J^*Y).$$

We are calling the triple (g, G, J) as a compatible triple. It can also be easily verified that the tensor $C(X, Y, Z) = (\nabla_Z g)(X, Y)$ satisfies the symmetry property $C(X, Y, Z) = C(Y, X, Z)$. Let ∇^* denotes the g -conjugate connection of ∇ . Also, note that $C^*(X, Y, Z) = -C(X, Y, Z)$, where $C^*(X, Y, Z) = (\nabla_Z^* g)(X, Y)$.

The concept of conjugate connections relative to a metric tensor field was initially introduced by Norden within the framework of Weyl geometry [16]. This idea was further explored independently by Nagaoka and Amari [15], who referred to them as dual connections, and was later utilized by Lauritzen in defining statistical manifolds [13].

In this context, we can define other two conjugate connections on a metallic pseudo-Riemannian manifold (M, J, g) . The first of these is the G -conjugate connection $(\nabla^\dagger)$. This connection is defined by the following equation

$$ZG(X, Y) = G(\nabla_Z X, Y) + G(X, \nabla_Z^\dagger Y),$$

where $G = g(JX, Y)$ denotes the $(0, 2)$ -tensor on (M, J, g) , and ∇ represents a linear connection. Standard calculations yield us: $G\left(Z, (\nabla^\dagger)_X^\dagger Y\right) = G(Z, \nabla_X Y) + g\left((\nabla_X J^{-1} J^*) Y, J^* Z\right)$. Thus, the G -conjugation of a connection ∇ is not involute, that is $(\nabla^\dagger)^\dagger \neq \nabla$. If $J^* = \mp J$, then G -conjugation of a connection ∇ is involute, that is $(\nabla^\dagger)^\dagger = \nabla$. The other class of conjugate connection is associated with tensor structures (for further details, refer to [3, 5, 6, 10, 17, 18]). In our scenario, this connection is defined as follows:

$$\nabla_X^J Y = J^{-1}(\nabla_X(JY)),$$

which is referred to as the J -conjugate connection (∇^J) . We aim to give results related to the interaction between such structures, which reveals deeper geometric symmetries and allows us to study properties beyond the classical setting. These properties not only establish fundamental relationships between J and J^* but also highlight the geometric compatibility of the dual structures within the broader framework of pseudo-Riemannian manifolds. This formulation provides insight into the role of conjugate connections and how they interact under metallic-like conditions, paving the way for a deeper understanding of symmetry properties in complex geometric settings.

Proposition 2.1. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold, ∇ a linear connection and $G = g(JX, Y)$. The connections ∇ and $\nabla^\dagger$ have equal torsion if and only if the following condition holds:*

$$(\nabla_X G)(Z, Y) = (\nabla_Y G)(Z, X),$$

where $\nabla^\dagger$ is the G -conjugate connection associated with ∇ .

Proof. We have the following relation:

$$G\left(Z, T^\nabla(X, Y) - T^{\nabla^\dagger}(X, Y)\right) = G\left(Z, \nabla_X Y - \nabla_Y X - \nabla_X^\dagger Y + \nabla_Y^\dagger X\right).$$

This expands to:

$$G(Z, \nabla_X Y) - G(Z, \nabla_Y X) - G\left(Z, \nabla_X^\dagger Y\right) + G\left(Z, \nabla_Y^\dagger X\right).$$

Continuing with the simplifications:

$$\begin{aligned} &= G(Z, \nabla_X Y) - G(Z, \nabla_Y X) - XG(Z, Y) \\ &\quad + G(\nabla_X Z, Y) + YG(Z, X) - G(\nabla_Y Z, X). \end{aligned}$$

This simplifies to:

$$= (\nabla_Y G)(Z, X) - (\nabla_X G)(Z, Y).$$

Thus, the connections ∇ and $\nabla^\dagger$ have equal torsion if and only if:

$$(\nabla_X G)(Z, Y) = (\nabla_Y G)(Z, X).$$

□

We present the following propositions without proof because their proofs can be easily seen through standard calculations. The equalities given in these propositions will be used in the theorems that we will prove later.

Proposition 2.2. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold. Then, the following properties hold:*

- (i) $(J^*)^2 = pJ^* + qI$,
- (ii) $(J^*)^{-1} = \frac{1}{q}J^* - \frac{p}{q}I$,
- (iii) $g(J^{-1}X, Y) = g(X, (J^*)^{-1}Y)$,
- (iv) $g(JX, J^*Y) = pg(X, J^*Y) + qg(X, Y)$,
- (v) $g(Z, (\nabla_X^* J)Y) = g((\nabla_X J^*)Z, Y)$,
- (vi) $g(Z, (\nabla_X^* J^*)Y) = g((\nabla_X J)Z, Y)$.

Proposition 2.3. *On a metallic-like pseudo-Riemannian manifold (M, J, g) , the following properties hold:*

- (i) $G(JX, Y) = G(X, J^*Y)$,
- (ii) $G(X, Y) = G(Y, X) + g(X, (J^* - J)Y)$,
- (iii) $G(X, Y) = -G(Y, X) + g(X, (J^* + J)Y)$,
- (iv) $G(J^{-1}X, Y) = G(X, (J^*)^{-1}Y)$,
- (v) $G(Y, (\nabla_Z^\dagger J)X) = G((\nabla_Z J^*)Y, X)$,
- (vi) $G(Y, (\nabla_Z^\dagger J^*)X) = G((\nabla_Z J)Y, X)$.

As a direct consequence of Proposition 2.3, we can state the following proposition.

Proposition 2.4. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold. Then, the following equivalences hold:*

- (i) *The connection ∇ is a J^* -connection if and only if $\nabla^\dagger$ is a J -connection.*
- (ii) *The connection ∇ is a J -connection if and only if $\nabla^\dagger$ is a J^* -connection.*

From Proposition 2.3, it is clear that the $(0,2)$ -tensor G is neither symmetric nor anti-symmetric for $\mp J \neq J^*$. Using the symmetric part of G , we have

$$\tilde{g}(X, Y) = \frac{1}{2} (G(X, Y) + G(Y, X)) = \frac{1}{2} g((J^* + J)X, Y) \quad (2.1)$$

for all vector fields X and Y . Thus, Using the properties of the pseudo-Riemannian metric g and $(0, 2)$ -tensor G , together with the equation (2.1), we have the following proposition:

Proposition 2.5. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold. Then,*

- (i) $\tilde{g}(X, Y) = \tilde{g}(Y, X)$,
- (ii) $\tilde{g}(PX, Y) = \tilde{g}(X, PY)$,
- (iii) $G(PX, Y) = G(X, PY)$,
- (iv) $g(PX, Y) = g(X, PY)$, where $P = J + J^*$.

Let Γ be the $(0, 3)$ -tensor defined as follows:

$$\Gamma(X, Y, Z) = (\nabla_Z G)(X, Y) = ZG(X, Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y).$$

This can be expressed as:

$$\begin{aligned} (\nabla_Z G)(X, Y) &= Zg(JX, Y) - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) \\ &= Zg(X, J^*Y) - g(\nabla_Z X, J^*Y) - g(X, \nabla_Z J^*Y) + g(X, (\nabla_Z J^*)Y). \end{aligned}$$

Thus, we have

$$\Gamma(X, Y, Z) = C(X, J^*Y, Z) + g(X, (\nabla_Z J^*)Y).$$

Now, if we relocate the vector fields X and Y , we get

$$\Gamma(Y, X, Z) = (\nabla_Z G)(Y, X) = ZG(Y, X) - G(\nabla_Z Y, X) - G(Y, \nabla_Z X).$$

This simplifies to:

$$= Zg(JY, X) - g(J\nabla_Z Y, X) - g(JY, \nabla_Z X)$$

and can be rewritten as:

$$= Zg(X, JY) - g(\nabla_Z X, JY) - g(X, \nabla_Z JY) + g((\nabla_Z J)Y, X).$$

Thus, we find

$$\Gamma(Y, X, Z) = C(X, JY, Z) + g((\nabla_Z J)Y, X).$$

We will now demonstrate an important relationship among conjugate connections of a connection ∇ : its g -conjugate ∇^* , its G -conjugate $\nabla^\dagger$, and its J^* -conjugate ∇^{J^*} .

Proposition 2.6. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold. Suppose that (g, G, J) form a compatible triple, and let ∇^* , $\nabla^\dagger$, and ∇^{J^*} represent, respectively, the g -conjugate, G -conjugate, and J^* -conjugate of an arbitrary linear connection ∇ . Then, the relations $\nabla^\dagger = (\nabla^*)^{J^*}$ and $\nabla^{J^*} = (\nabla^*)^\dagger$ hold.*

Proof. We know that the relation $(\nabla^*)^* = \nabla$. Additionally, we can compute as follows:

$$\begin{aligned} G(Y, \nabla_X^\dagger Z) &= XG(Y, Z) - G(\nabla_X Y, Z) \\ &= Xg(JY, Z) - g(J\nabla_X Y, Z) \\ &= Xg(Y, J^*Z) - g(\nabla_X Y, J^*Z) \\ &= g(Y, \nabla_X^* J^*Z) \\ &= g(JY, (J^*)^{-1} \nabla_X^* J^*Z) \\ &= g(JY, (\nabla^*)_X^{J^*} Z) \\ &= G(Y, (\nabla^*)_X^{J^*} Z). \end{aligned}$$

This shows that $\nabla^\dagger = (\nabla^*)^{J^*}$. By substituting ∇^* for ∇ on both sides, we find that $\nabla^{J^*} = (\nabla^*)^\dagger$. □

Proposition 2.7. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold with a linear connection ∇ . Assume that $\nabla J = \nabla J^* = 0$. The following equalities hold:

$$\begin{aligned} C(X, J^*Y, Z) &= -C^*(X, J^*Y, Z) = -C^\dagger(X, J^*Y, Z) \\ &= \Gamma(X, Y, Z) = -\Gamma^*(X, Y, Z) = -\Gamma^\dagger(X, Y, Z). \end{aligned}$$

Proof. The proof follows directly from the definitions of the $(0, 3)$ -tensors C and Γ . $\square$

Assume that ∇ be any linear connection and ∇^* its g -conjugation. Let K denote the difference tensor between ∇ and ∇^* , so that

$$\nabla_X Y = \nabla_X^* Y + K_X Y.$$

We use $K(X, Y)$ to represent $K_X Y$. If both ∇ and ∇^* are torsion-free, the $(1, 2)$ -tensor K is symmetric. It follows that

$$(\nabla_X g)(Y, Z) = (K_X g)(Y, Z) = -g(K_X Y, Z) - g(Y, K_X Z).$$

This shows that the symmetry of ∇g and K implies the symmetry of K_X with respect to g for any vector field X . The converse is also true: if K_X is symmetric with respect to g , then

$$(\nabla_X g)(Y, Z) = -2g(K_X Y, Z).$$

Proposition 2.8. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold equipped with a linear connection ∇ . The following conditions hold:

1. The expression

$$C(JX, Y, Z) = C(X, Y, JZ) \Leftrightarrow K(JX, Y) = K(X, JY) = J^*K(X, Y)$$

is satisfied for any vector fields X, Y, Z .

2. The expression

$$C(J^*X, Y, Z) = C(X, Y, J^*Z) \Leftrightarrow K(J^*X, Y) = K(X, J^*Y) = JK(X, Y)$$

is also satisfied for any vector fields X, Y, Z .

Proof. We start with the expression

$$\begin{aligned} C(JX, Y, Z) - C(X, Y, JZ) &= Zg(JX, Y) - g(\nabla_Z JX, Y) \\ &\quad - g(JX, \nabla_Z Y) - JZg(X, Y) \\ &\quad + g(\nabla_{JZ} X, Y) + g(X, \nabla_{JZ} Y). \end{aligned}$$

This can be rewritten as

$$g(\nabla_Z^* JX, Y) - g(\nabla_Z JX, Y) - g(\nabla_{JZ}^* X, Y) + g(\nabla_{JZ} X, Y).$$

By simplifying, we obtain

$$\begin{aligned} &-g(\nabla_Z JX - \nabla_Z^* JX, Y) + g(\nabla_{JZ} X - \nabla_{JZ}^* X, Y) \\ &= g(K(JZ, X) - K(Z, JX), Y). \end{aligned} \tag{2.2}$$

Now, rewriting the expression again, we have:

$$\begin{aligned} C(JX, Y, Z) - C(X, Y, JZ) &= Zg(JX, Y) - g(\nabla_Z JX, Y) \\ &\quad - g(JX, \nabla_Z Y) - JZg(X, Y) \\ &\quad + g(\nabla_{JZ} X, Y) + g(X, \nabla_{JZ} Y). \end{aligned}$$

This can be expressed as

$$g(JX, \nabla_Z^* Y) - g(JX, \nabla_Z Y) - g(X, \nabla_{JZ}^* Y) + g(X, \nabla_{JZ} Y).$$

Continuing with the simplification, we find

$$g(X, \nabla_{JZ} Y - \nabla_{JZ}^* Y) - g(X, J^*(\nabla_Z Y - \nabla_Z^* Y)). \tag{2.3}$$

Considering the equation (2.3), the expression simplifies

$$C(JX, Y, Z) - C(X, Y, JZ) = g(K(JZ, Y) - J^*K(Z, Y), X). \quad (2.4)$$

Combining these equations (2.2) and (2.4), we conclude that

$$C(JX, Y, Z) = C(X, Y, JZ) \Leftrightarrow K(JX, Y) = K(X, JY) = J^*K(X, Y).$$

We begin with the expression

$$\begin{aligned} C(J^*X, Y, Z) - C(X, Y, J^*Z) &= Zg(J^*X, Y) - g(\nabla_Z J^*X, Y) \\ &\quad - g(J^*X, \nabla_Z Y) - J^*Zg(X, Y) \\ &\quad + g(\nabla_{J^*Z} X, Y) + g(X, \nabla_{J^*Z} Y). \end{aligned}$$

This can be simplified to

$$g(\nabla_Z^* J^*X, Y) - g(\nabla_Z J^*X, Y) - g(\nabla_{J^*Z}^* X, Y) + g(\nabla_{J^*Z} X, Y).$$

Rearranging the terms leads to

$$\begin{aligned} &-g(\nabla_Z J^*X - \nabla_Z^* J^*X, Y) + g(\nabla_{J^*Z} X - \nabla_{J^*Z}^* X, Y) \\ &= g(K(J^*Z, X) - K(Z, J^*X), Y). \end{aligned} \quad (2.5)$$

Next, we express the difference again:

$$\begin{aligned} C(J^*X, Y, Z) - C(X, Y, J^*Z) &= Zg(J^*X, Y) - g(\nabla_Z J^*X, Y) \\ &\quad - g(J^*X, \nabla_Z Y) - J^*Zg(X, Y) \\ &\quad + g(\nabla_{J^*Z} X, Y) + g(X, \nabla_{J^*Z} Y). \end{aligned}$$

This can be expressed as

$$g(J^*X, \nabla_Z^* Y) - g(J^*X, \nabla_Z Y) - g(X, \nabla_{J^*Z}^* Y) + g(X, \nabla_{J^*Z} Y).$$

Continuing with the simplification, we find

$$g(X, \nabla_{J^*Z} Y - \nabla_{J^*Z}^* Y) - g(X, J(\nabla_Z Y - \nabla_Z^* Y)). \quad (2.6)$$

Taking account of the equation (2.6), we obtain

$$C(J^*X, Y, Z) - C(X, Y, J^*Z) = g(X, K(J^*Z, Y)) - g(X, JK(Z, Y)). \quad (2.7)$$

Combining these equations (2.5) and (2.7), we conclude that

$$C(J^*X, Y, Z) = C(X, Y, J^*Z) \Leftrightarrow K(J^*X, Y) = K(X, J^*Y) = JK(X, Y).$$

□

Proposition 2.9. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold equipped with a linear connection ∇ . Then, the following two equalities are satisfied:

1.

$$C^{J^*}(J^*X, Y, Z) = \Gamma(Y, X, Z) - 2pg(\theta(Z, X), Y) - 2g(J^*X, \theta(Z, Y)),$$

where $C^{J^*}(X, Y, Z) = (\nabla_Z^{J^*} g)(X, Y)$.

2.

$$C(X, Y, Z) - C^{J^*}(X, Y, Z) = 2(g(\theta(Z, X), Y) + g(X, \theta(Z, Y))),$$

where $\theta(X, Y) = \frac{1}{2}(\nabla_X^{J^*} Y - \nabla_X Y)$.

Proof. Using the definition $C^{J^*}(X, Y, Z) = (\nabla_Z^{J^*} g)(X, Y)$, we have:

$$C^{J^*}(J^*X, Y, Z) = (\nabla_Z^{J^*} g)(J^*X, Y).$$

This expands to:

$$= Zg(J^*X, Y) - g(\nabla_Z^{J^*} J^*X, Y) - g(J^*X, \nabla_Z^{J^*} Y),$$

Substituting for $\nabla_Z^{J^*} J^*X$ and $\nabla_Z^{J^*} Y$, we find:

$$= Zg(J^*X, Y) - g((J^*)^{-1} \nabla_Z (J^*)^2 X, Y) - g(J^*X, (J^*)^{-1} \nabla_Z J^*Y).$$

This simplifies to:

$$= Zg(JY, X) - g(JY, \nabla_Z X) - g(J^*X, \nabla_Z Y) \\ - g((J^*)^{-1} (\nabla_Z (J^*)^2) X, Y) - g(J^*X, (J^*)^{-1} (\nabla_Z J^*) Y).$$

Continuing, we get:

$$= ZG(Y, X) - G(Y, \nabla_Z X) - G(\nabla_Z Y, X) \\ - pg((J^*)^{-1} (\nabla_Z J^*) X, Y) - g(J^*X, (J^*)^{-1} (\nabla_Z J^*) Y). \quad (2.8)$$

Finally, from the equation (2.8) and, the definitions of Γ and θ , we conclude:

$$C^{J^*}(J^*X, Y, Z) = \Gamma(Y, X, Z) - 2pg(\theta(Z, X), Y) - 2g(J^*X, \theta(Z, Y)).$$

Additionally, it can be shown that:

$$C(X, Y, Z) - C^{J^*}(X, Y, Z) = Zg(X, Y) - g(\nabla_Z X, Y) \\ - g(X, \nabla_Z Y) - Zg(X, Y) \\ + g(\nabla_Z^{J^*} X, Y) + g(X, \nabla_Z^{J^*} Y).$$

This simplifies to:

$$= g(\nabla_Z^{J^*} X - \nabla_Z X, Y) + g(X, \nabla_Z^{J^*} Y - \nabla_Z Y). \quad (2.9)$$

Using the equation (2.9), leading to the final result:

$$C(X, Y, Z) - C^{J^*}(X, Y, Z) = 2g(\theta(Z, X), Y) + 2g(X, \theta(Z, Y)).$$

This completes the proof. $\square$

Given a pair of conjugate connections ∇ and ∇^* , it is possible to construct a new connection such that

$$\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^{J^*})$$

and ∇^{J^*} is the J^* -conjugate connection of ∇ . The above proposition gives the following outcome.

Corollary 2.1. *In the context of a metallic-like pseudo Riemannian manifold (M, J, g) equipped with a linear connection ∇ , the following statements are equivalent:*

- (i) $C(J^*X, Y, Z) + \Gamma(Y, X, Z) = 2pg(\theta(Z, X), Y) + 2g(J^*X, \theta(Z, Y))$,
- (ii) $C^{J^*}(X, Y, Z) = -C(X, Y, Z)$,
- (iii) $C(X, Y, Z) = g(\theta(Z, X), Y) + g(X, \theta(Z, Y))$,
- (iv) $\tilde{\nabla}g = 0$.

3. Results related to Codazzi coupled

The findings in this section shed light on the intricate relationships among various linear connections within metallic-like pseudo-Riemannian manifolds, emphasizing the role of Codazzi coupling in establishing connections between torsion and geometric structures. By demonstrating the equivalences between different

pairs of connections and their coupling with compatible metrics, we provide a framework for understanding how these relationships influence the overall geometry of the manifold. This exploration is particularly significant as it reveals the interdependence of connections, their conjugates, and the associated curvature and torsion tensors, thereby enriching our understanding of the geometric properties of metallic-like structures. A $(1, 1)$ -tensor field J and a connection ∇ are said to be Codazzi-coupled if the following identity holds

$$(\nabla_X J)Y = (\nabla_Y J)X, \quad \forall X, Y.$$

Similarly, a pseudo-Riemannian metric g and a connection ∇ are said to be Codazzi-coupled if the following identity holds

$$(\nabla_Z g)(X, Y) = (\nabla_X g)(Z, Y), \quad \forall X, Y, Z.$$

Proposition 3.1. Consider (M, J, g) as a metallic-like pseudo-Riemannian manifold equipped with a linear connection ∇ . Let (g, G, J) represent the compatible triple, and denote ∇^* and $\nabla^\dagger$ as the g -conjugate and G -conjugate of the linear connection ∇ , respectively. The pair (∇^*, J^*) is Codazzi coupled if and only if the pair $(\nabla^\dagger, J^*)$ is also Codazzi coupled.

Proof. We compute the following:

$$G(Y, (\nabla_Z^\dagger J^*)X) - G(Y, (\nabla_X^\dagger J^*)Z). \quad (3.1)$$

From (vi) of Proposition 2.3, we have:

$$= G((\nabla_Z J)Y, X) - G((\nabla_X J)Y, Z).$$

Expanding $G(Z, Y) = g(JZ, Y)$, this can be expressed as:

$$= g(J(\nabla_Z J)Y, X) - g(J(\nabla_X J)Y, Z).$$

Substituting further yields:

$$= g((\nabla_Z J)Y, J^*X) - g((\nabla_X J)Y, J^*Z).$$

Using (vi) of Proposition 2.2, rewriting this gives:

$$= g((\nabla_Z^* J^*)J^*X, Y) - g((\nabla_X^* J^*)J^*Z, Y),$$

Continuing with the simplification, we find:

$$= g(\nabla_Z^* (J^*)^2 X - J^* \nabla_Z^* J^* X, Y) - g(\nabla_X^* (J^*)^2 Z - J^* \nabla_X^* J^* Z, Y).$$

This expands to:

$$\begin{aligned} &= g((J^*)^2 \nabla_Z^* X + p(\nabla_Z^* J^*)X - (J^*)^2 \nabla_Z^* X - J^* (\nabla_Z^* J^*)X, Y) \\ &\quad - g((J^*)^2 \nabla_X^* Z + p(\nabla_X^* J^*)Z - (J^*)^2 \nabla_X^* Z - J^* (\nabla_X^* J^*)Z, Y). \end{aligned} \quad (3.2)$$

The equation (3.2) simplifies to:

$$= g(p(\nabla_Z^* J^*)X - J^* (\nabla_Z^* J^*)X, Y) - g(p(\nabla_X^* J^*)Z - J^* (\nabla_X^* J^*)Z, Y).$$

Thus, we arrive at:

$$= g(((\nabla_Z^* J^*)X - (\nabla_X^* J^*)Z), (pI - J)Y). \quad (3.3)$$

Using the equations (3.1) and (3.3), this can be expressed as:

$$G(Y, (\nabla_Z^\dagger J^*)X) - G(Y, (\nabla_X^\dagger J^*)Z) = -qg(((\nabla_Z^* J^*)X - (\nabla_X^* J^*)Z), J^{-1}Y).$$

From this computation, it is evident that the pair (∇^*, J^*) is Codazzi coupled if and only if the pair $(\nabla^\dagger, J^*)$ is Codazzi coupled. $\square$

Proposition 3.2. Consider (M, J, g) as a metallic-like pseudo-Riemannian manifold equipped with a linear connection ∇ . The following statements are equivalent

- (i) The connections ∇ and ∇^{J^*} have the same torsion.
- (ii) The pair (∇^{J^*}, J^*) is Codazzi coupled.
- (iii) The pair (∇, J^*) is Codazzi coupled.

Proof. Proof of (i) $\Rightarrow$ (ii): Assume that the connections ∇ and ∇^{J^*} share the same torsion. We start with the torsion difference:

$$T^\nabla(X, Y) - T^{\nabla^{J^*}}(X, Y) = \nabla_X Y - \nabla_Y X - \nabla_X^{J^*} Y + \nabla_Y^{J^*} X.$$

Next, from the definition of ∇^{J^*} , we express:

$$\begin{aligned} &= \nabla_X Y - \nabla_Y X - (J^*)^{-1} \nabla_X J^* Y + (J^*)^{-1} \nabla_Y J^* X. \\ &= (J^*)^{-1} ((\nabla_Y J^*) X - (\nabla_X J^*) Y). \end{aligned} \quad (3.4)$$

Rearranging this expression (3.4) gives:

$$= -\frac{1}{q} (J^*)^{-1} ((\nabla_X J^*) Y - (\nabla_Y J^*) X).$$

Now, we can express it using $pI - J^*$:

$$= \frac{1}{q} (pI - J^*) ((\nabla_X J^*) Y - (\nabla_Y J^*) X). \quad (3.5)$$

Continuing with the simplification of the equation (3.5) yields:

$$\begin{aligned} &= \frac{1}{q} (p(\nabla_X J^*) Y - p(\nabla_Y J^*) X - J^*(\nabla_X J^*) Y + J^*(\nabla_Y J^*) X) \\ &= \frac{1}{q} (\nabla_X (J^*)^2 Y - \nabla_Y (J^*)^2 X - J^* \nabla_X J^* Y + J^* \nabla_Y J^* X). \end{aligned}$$

Now, from the connection ∇^{J^*} , we can rewrite it as:

$$= \frac{1}{q} (J^* \nabla_X^{J^*} J^* Y - J^* \nabla_Y^{J^*} J^* X - (J^*)^2 \nabla_X^{J^*} Y + (J^*)^2 \nabla_Y^{J^*} X). \quad (3.6)$$

Thus, expanding and simplifying the equation (3.6), we find that:

$$= \frac{1}{q} J^* \left((\nabla_X^{J^*} J^*) Y - (\nabla_Y^{J^*} J^*) X \right).$$

Proof of (ii) $\Rightarrow$ (iii): Assume that the pair (∇^{J^*}, J^*) is Codazzi coupled. Then, we can write:

$$J^* \left((\nabla_X^{J^*} J^*) Y - (\nabla_Y^{J^*} J^*) X \right) = J^* \left(\nabla_X^{J^*} J^* Y - J^* \nabla_X^{J^*} Y - \nabla_Y^{J^*} J^* X + J^* \nabla_Y^{J^*} X \right). \quad (3.7)$$

Further simplification of the equation (3.7) gives:

$$= \nabla_X (J^*)^2 Y - J^* \nabla_X J^* Y - \nabla_Y (J^*)^2 X + J^* \nabla_Y J^* X. \quad (3.8)$$

Rearranging terms of the equation (3.8) leads to:

$$\begin{aligned} &= p(\nabla_X J^*) Y + (J^*)^2 \nabla_X Y - J^*(\nabla_X J^*) Y - (J^*)^2 \nabla_X Y \\ &\quad - p(\nabla_Y J^*) X - (J^*)^2 \nabla_Y X + J^*(\nabla_Y J^*) X + (J^*)^2 \nabla_Y X. \end{aligned}$$

This simplifies to:

$$= p((\nabla_X J^*) Y - (\nabla_Y J^*) X) + J^*((\nabla_Y J^*) X - (\nabla_X J^*) Y). \quad (3.9)$$

Rearranging of the equation (3.9) gives:

$$= (pI - J^*) ((\nabla_X J^*) Y - (\nabla_Y J^*) X).$$

Thus, we have:

$$J^* \left((\nabla_X^{J^*} J^*) Y - (\nabla_Y^{J^*} J^*) X \right) = -q (J^*)^{-1} ((\nabla_X J^*) Y - (\nabla_Y J^*) X).$$

From this result, it is evident that the pair (∇, J^*) is also Codazzi coupled.

Proof of (iii) $\Rightarrow$ (i) : Assume that the pair (∇, J^*) is Codazzi coupled. Then, we have

$$\begin{aligned} & (J^*)^{-1} ((\nabla_Y J^*) X - (\nabla_X J^*) Y) \\ &= \nabla_X Y - \nabla_Y X - (J^*)^{-1} \nabla_X J^* Y + (J^*)^{-1} \nabla_Y J^* X \\ &= \nabla_X Y - \nabla_Y X - [X, Y] - \nabla_X^{J^*} Y + \nabla_Y^{J^*} X + [X, Y] \\ &= T^\nabla(X, Y) - T^{\nabla^{J^*}}(X, Y). \end{aligned}$$

Thus, it is obvious that T^∇ and $T^{\nabla^{J^*}}$ have the same torsion. $\square$

The following proposition investigates the equivalences of Codazzi coupling conditions within metallic-like pseudo-Riemannian manifolds under various conjugate connections, contributing to a deeper understanding of the complex geometric relationships among these structures.

Proposition 3.3. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold with a linear connection ∇ . Denote (g, G, J) a compatible triple, and let ∇^* and $\nabla^\dagger$ represent the g -conjugate and G -conjugate of the linear connection ∇ , respectively. The following equivalences can be established

- (i) If (∇, J) is Codazzi coupled, then (∇^*, G) is Codazzi coupled if and only if (∇^*, g) is Codazzi coupled.
- (ii) If (∇^*, J) is Codazzi coupled, then (∇, G) is Codazzi coupled if and only if (∇, g) is Codazzi coupled.
- (iii) If (∇, J) and (∇^*, J^*) are Codazzi coupled, then (∇^J, g) is Codazzi coupled if and only if (∇, g) is Codazzi coupled.
- (iv) If (∇, J^*) and (∇^*, J) are Codazzi coupled, then (∇^{J^*}, G) is Codazzi coupled if and only if (∇, G) is Codazzi coupled.
- (v) If (∇, J^*) and (∇^*, J) are Codazzi coupled, then (∇^{J^*}, g) is Codazzi coupled if and only if (∇, g) is Codazzi coupled.
- (vi) If (∇, J) is Codazzi coupled, then $(\nabla^\dagger, G)$ is Codazzi coupled if and only if $(\nabla^\dagger, g)$ is Codazzi coupled.
- (vii) If (∇, J) and (∇^*, J^*) are Codazzi coupled, then $(\nabla^\dagger, g)$ is Codazzi coupled if and only if (∇^*, g) is Codazzi coupled.
- (viii) If (∇^*, J^*) is Codazzi coupled, then $(\nabla^\dagger, G)$ is Codazzi coupled if and only if (∇^*, g) is Codazzi coupled.
- (ix) If (∇, J) and (∇^*, J^*) are Codazzi coupled, then $(\nabla^\dagger, g)$ is Codazzi coupled if and only if (∇^*, G) is Codazzi coupled.
- (x) If (∇, J) and (∇^*, J^*) are Codazzi coupled, then $(\nabla^\dagger, G)$ is Codazzi coupled if and only if (∇^*, G) is Codazzi coupled.

Proof. (i) Assume that the pair (∇, J) is Codazzi coupled. Then, we have

$$(\nabla_X^* G)(Y, Z) = XG(Y, Z) - G(\nabla_X^* Y, Z) - G(Y, \nabla_X^* Z).$$

Expanding $G(Y, Z) = g(JY, Z)$, the above expression becomes:

$$= Xg(JY, Z) - g(J\nabla_X^* Y, Z) - g(JY, \nabla_X^* Z). \quad (3.10)$$

The equation (3.10) is equivalent to:

$$= Xg(Y, J^* Z) - g(\nabla_X^* Y, J^* Z) - g(Y, J^* \nabla_X^* Z). \quad (3.11)$$

Next, rewriting the expression (3.11), we have:

$$= Xg(Y, J^* Z) - g(\nabla_X^* Y, J^* Z) - g(Y, \nabla_X^* J^* Z) + g(Y, (\nabla_X^* J^*) Z).$$

This simplifies to:

$$= (\nabla_X^* g)(Y, J^* Z) + g(Y, (\nabla_X^* J^*) Z).$$

Noting (vi) of Proposition 2.2, we have:

$$(\nabla_X^* G)(Y, Z) = (\nabla_X^* g)(Y, J^* Z) + g(Z, (\nabla_X J) Y). \quad (3.12)$$

Now, exchanging X and Y in the equation (3.12), we obtain:

$$(\nabla_Y^* G)(X, Z) = (\nabla_Y^* g)(X, J^* Z) + g(Z, (\nabla_Y J) X). \quad (3.13)$$

Hence, the difference of the equations (3.12) and (3.13) is given by:

$$\begin{aligned} (\nabla_X^* G)(Y, Z) - (\nabla_Y^* G)(X, Z) &= (\nabla_X^* g)(Y, J^* Z) + g(Z, (\nabla_X J) Y) \\ &\quad - (\nabla_Y^* g)(X, J^* Z) - g(Z, (\nabla_Y J) X). \end{aligned}$$

Therefore, (∇^*, G) is Codazzi coupled if and only if (∇^*, g) is Codazzi coupled.

(ii) Suppose that (∇^*, J) is Codazzi coupled. We begin by computing:

$$(\nabla_X G)(Y, Z) = XG(Y, Z) - G(\nabla_X Y, Z) - G(Y, \nabla_X Z).$$

Substituting $G(Y, Z) = g(JY, Z)$, we have:

$$= Xg(JY, Z) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z).$$

Using the expression $g(JY, Z) = g(Y, J^*Z)$, rearranging yields:

$$= Xg(Y, J^*Z) - g(\nabla_X Y, J^*Z) - g(Y, J^*\nabla_X Z). \quad (3.14)$$

Next, we rewrite the equation (3.14) using the property of J^* :

$$= Xg(Y, J^*Z) - g(\nabla_X Y, J^*Z) - g(Y, \nabla_X J^*Z) + g(Y, (\nabla_X J^*)Z). \quad (3.15)$$

This expression (3.15) simplifies to:

$$= (\nabla_X g)(Y, J^*Z) + g(Y, (\nabla_X J^*)Z),$$

from which (v) of Proposition 2.2 gives

$$(\nabla_X G)(Y, Z) = (\nabla_X g)(Y, J^*Z) + g(Z, (\nabla_X^* J)Y). \quad (3.16)$$

By exchanging X and Y in the equation (3.16), we obtain:

$$(\nabla_Y G)(X, Z) = (\nabla_Y g)(X, J^*Z) + g(Z, (\nabla_Y^* J)X). \quad (3.17)$$

Thus, we can express the difference of the equations (3.16) and (3.17) as:

$$\begin{aligned} (\nabla_X G)(Y, Z) - (\nabla_Y G)(X, Z) &= (\nabla_X g)(Y, J^*Z) + g(Z, (\nabla_X^* J)Y) \\ &\quad - (\nabla_Y g)(X, J^*Z) - g(Z, (\nabla_Y^* J)X). \end{aligned}$$

Therefore, (∇, G) is Codazzi coupled if and only if (∇, g) is Codazzi coupled.

(iii) Let the pairs (∇, J) and (∇^*, J^*) be Codazzi coupled. We calculate:

$$(\nabla_X^J g)(Y, Z) = Xg(Y, Z) - g(\nabla_X^J Y, Z) - g(Y, \nabla_X^J Z).$$

Using $\nabla_X^J Y = J^{-1}\nabla_X(JY)$, we obtain:

$$= Xg(Y, Z) - g(J^{-1}\nabla_X JY, Z) - g(Y, J^{-1}\nabla_X JZ). \quad (3.18)$$

The equation (3.18) simplifies to:

$$\begin{aligned} &= Xg(Y, Z) - g(\nabla_X Y, Z) - g(J^{-1}(\nabla_X J)Y, Z) \\ &\quad - g(Y, \nabla_X Z) - g(Y, J^{-1}(\nabla_X J)Z). \end{aligned}$$

We can rewrite the above expression as:

$$= (\nabla_X g)(Y, Z) - g(J^{-1}(\nabla_X J)Y, Z) - g(Y, J^{-1}(\nabla_X J)Z). \quad (3.19)$$

Further simplifying the equation (3.19), we have:

$$= (\nabla_X g)(Y, Z) - g((\nabla_X J)Y, (J^*)^{-1}Z) - g((J^*)^{-1}Y, (\nabla_X J)Z).$$

From (vi) of Proposition 2.2, we obtain:

$$= (\nabla_X g)(Y, Z) - g((\nabla_X J)Y, (J^*)^{-1}Z) - g((\nabla_X^* J^*)(J^*)^{-1}Y, Z).$$

This leads to:

$$(\nabla_X^J g)(Y, Z) = (\nabla_X g)(Y, Z) - g\left((\nabla_X J)Y, (J^*)^{-1}Z\right) + \frac{1}{q}g((\nabla_X^* J^*)Y, JZ). \quad (3.20)$$

Next, exchanging X and Y in the equation (3.20), we get:

$$\begin{aligned} (\nabla_Y^J g)(X, Z) &= (\nabla_Y g)(X, Z) - g\left((\nabla_Y J)X, (J^*)^{-1}Z\right) \\ &\quad + \frac{1}{q}g((\nabla_Y^* J^*)X, JZ). \end{aligned} \quad (3.21)$$

Thus, the difference of the equations (3.20) and (3.21) is:

$$\begin{aligned} (\nabla_X^J g)(Y, Z) - (\nabla_Y^J g)(X, Z) &= (\nabla_X g)(Y, Z) - g\left((\nabla_X J)Y, (J^*)^{-1}Z\right) \\ &\quad + \frac{1}{q}g((\nabla_X^* J^*)Y, JZ) - (\nabla_Y g)(X, Z) \\ &\quad + g\left((\nabla_Y J)X, (J^*)^{-1}Z\right) \\ &\quad - \frac{1}{q}g((\nabla_Y^* J^*)X, JZ). \end{aligned}$$

This shows that (∇^J, g) is Codazzi coupled if and only if (∇, g) is Codazzi coupled.

(iv) Assume that (∇, J^*) and (∇^*, J) are Codazzi coupled. We compute:

$$(\nabla_X^{J^*} G)(Y, Z) = XG(Y, Z) - G\left(\nabla_X^{J^*} Y, Z\right) - G\left(Y, \nabla_X^{J^*} Z\right).$$

Substituting $\nabla_X^{J^*} Y = (J^*)^{-1}\nabla_X(J^*Y)$, we have:

$$= XG(Y, Z) - G\left((J^*)^{-1}\nabla_X J^*Y, Z\right) - G\left(Y, (J^*)^{-1}\nabla_X J^*Z\right). \quad (3.22)$$

This expression (3.22) can be further expanded as:

$$\begin{aligned} &= XG(Y, Z) - G\left((J^*)^{-1}(\nabla_X J^*)Y, Z\right) - G(\nabla_X Y, Z) \\ &\quad - G\left(Y, (J^*)^{-1}(\nabla_X J^*)Z\right) - G(Y, \nabla_X Z). \end{aligned}$$

Using $G(J^{-1}Y, Z) = G(Y, (J^*)^{-1}Z)$, we obtain

$$\begin{aligned} &= XG(Y, Z) - G\left((J^*)^{-1}(\nabla_X J^*)Y, Z\right) - G(\nabla_X Y, Z) \\ &\quad - G(J^{-1}Y, (\nabla_X J^*)Z) - G(Y, \nabla_X Z). \end{aligned}$$

(v) of Proposition 2.2 gives us:

$$(\nabla_X^{J^*} G)(Y, Z) = (\nabla_X G)(Y, Z) - G\left((J^*)^{-1}(\nabla_X J^*)Y, Z\right) - g(Z, (\nabla_X^* J)Y). \quad (3.23)$$

Exchanging X and Y in the equation (3.23) leads to:

$$(\nabla_Y^{J^*} G)(X, Z) = (\nabla_Y G)(X, Z) - G\left((J^*)^{-1}(\nabla_Y J^*)X, Z\right) - g(Z, (\nabla_Y^* J)X). \quad (3.24)$$

Thus, from the equations (3.23) and (3.24), we find:

$$\begin{aligned} (\nabla_X^{J^*} G)(Y, Z) - (\nabla_Y^{J^*} G)(X, Z) &= (\nabla_X G)(Y, Z) - (\nabla_Y G)(X, Z) \\ &\quad - G\left((\nabla_X J^*)Y - (\nabla_Y J^*)X, (J^*)^{-1}Z\right) - g(Z, (\nabla_X^* J)Y - (\nabla_Y^* J)X). \end{aligned}$$

This implies that (∇^{J^*}, G) is Codazzi coupled if and only if (∇, G) is Codazzi coupled.

(v) Let the pairs (∇, J^*) and (∇^*, J) be Codazzi coupled. We have:

$$(\nabla_X^{J^*} g)(Y, Z) = Xg(Y, Z) - g\left(\nabla_X^{J^*} Y, Z\right) - g\left(Y, \nabla_X^{J^*} Z\right).$$

$$\begin{aligned}
&= Xg(Y, Z) - g((J^*)^{-1} \nabla_X J^* Y, Z) - g(Y, (J^*)^{-1} \nabla_X J^* Z) \\
&= Xg(Y, Z) - g((J^*)^{-1} (\nabla_X J^*) Y, Z) - g(\nabla_X Y, Z) \\
&\quad - g(Y, (J^*)^{-1} (\nabla_X J^*) Z) - g(Y, \nabla_X Z).
\end{aligned} \tag{3.25}$$

Rewriting the terms in the equation (3.25), we obtain:

$$\begin{aligned}
&= (\nabla_X g)(Y, Z) - g((J^*)^{-1} (\nabla_X J^*) Y, Z) \\
&\quad - g(Y, (J^*)^{-1} (\nabla_X J^*) Z).
\end{aligned}$$

Using $g((J^*)^{-1} Y, Z) = g(Y, J^{-1} Z)$, it simplifies to:

$$= (\nabla_X g)(Y, Z) - g((\nabla_X J^*) Y, J^{-1} Z) - g(Y, (J^*)^{-1} (\nabla_X J^*) Z)$$

or equivalently

$$= (\nabla_X g)(Y, Z) - g((\nabla_X J^*) Y, J^{-1} Z) - g(J^{-1} Y, (\nabla_X J^*) Z).$$

From (v) of Proposition 2.2, we have:

$$(\nabla_X^{J^*} g)(Y, Z) = (\nabla_X g)(Y, Z) - g((\nabla_X J^*) Y, J^{-1} Z) + \frac{1}{q} g(J^* Z, (\nabla_X^* J) Y). \tag{3.26}$$

Switching X and Y in the equation (3.26):

$$(\nabla_Y^{J^*} g)(X, Z) = (\nabla_Y g)(X, Z) - g((\nabla_Y J^*) X, J^{-1} Z) + \frac{1}{q} g(J^* Z, (\nabla_Y^* J) X). \tag{3.27}$$

Thus, the difference of the equations (3.26) and (3.27) is:

$$\begin{aligned}
&(\nabla_X^{J^*} g)(Y, Z) - (\nabla_Y^{J^*} g)(X, Z) = (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) \\
&\quad - g((\nabla_X J^*) Y - (\nabla_Y J^*) X, J^{-1} Z) + \frac{1}{q} g(J^* Z, (\nabla_X^* J) Y - (\nabla_Y^* J) X).
\end{aligned}$$

This shows that (∇^{J^*}, g) is Codazzi coupled if and only if (∇, g) is Codazzi coupled.

(vi) Assume that (∇, J) is Codazzi coupled. We start with:

$$\begin{aligned}
(\nabla_X^\dagger G)(Y, Z) &= XG(Y, Z) - G(\nabla_X^\dagger Y, Z) - G(Y, \nabla_X^\dagger Z) \\
&= Xg(JY, Z) - g(J\nabla_X^\dagger Y, Z) - g(JY, \nabla_X^\dagger Z) \\
&= Xg(Y, J^* Z) - g(\nabla_X^\dagger Y, J^* Z) - g(Y, \nabla_X^\dagger J^* Z) + g(Y, (\nabla_X^\dagger J^*) Z).
\end{aligned}$$

Using (vi) of Proposition 2.3, it follows that

$$\begin{aligned}
&= (\nabla_X^\dagger g)(Y, J^* Z) + G(J^{-1} Y, (\nabla_X^\dagger J^*) Z) \\
&= (\nabla_X^\dagger g)(Y, J^* Z) + G((\nabla_X J) J^{-1} Y, Z).
\end{aligned}$$

After further reduction:

$$(\nabla_X^\dagger G)(Y, Z) = (\nabla_X^\dagger g)(Y, J^* Z) - \frac{1}{q} G((\nabla_X J) Y, J^* Z). \tag{3.28}$$

Exchanging X and Y in the equation (3.28):

$$(\nabla_Y^\dagger G)(X, Z) = (\nabla_Y^\dagger g)(X, J^* Z) - \frac{1}{q} G((\nabla_Y J) X, J^* Z). \tag{3.29}$$

Thus, the difference of the expressions (3.28) and (3.29) is:

$$\begin{aligned} & \left(\nabla_X^\dagger G \right) (Y, Z) - \left(\nabla_Y^\dagger G \right) (X, Z) = \left(\nabla_X^\dagger g \right) (Y, J^* Z) - \left(\nabla_Y^\dagger g \right) (X, J^* Z) \\ & + \frac{1}{q} G \left((\nabla_Y J) X - (\nabla_X J) Y, J^* Z \right). \end{aligned}$$

This confirms that $(\nabla^\dagger, G)$ is Codazzi coupled if and only if $(\nabla^\dagger, g)$ is Codazzi coupled.

(vii) Suppose that (∇, J) and (∇^*, J^*) are Codazzi coupled. We begin:

$$\left(\nabla_X^\dagger g \right) (Y, Z) = Xg(Y, Z) - g\left(\nabla_X^\dagger Y, Z\right) - g\left(Y, \nabla_X^\dagger Z\right).$$

Using the relationship between $\nabla^\dagger$ and ∇^* , i.e., $\nabla_X^\dagger Y = (J^*)^{-1} \nabla_X^* J^* Y$ (see Proposition 2.6), we express:

$$\left(\nabla_X^\dagger g \right) (Y, Z) = Xg(Y, Z) - g\left((J^*)^{-1} \nabla_X^* J^* Y, Z\right) - g\left(Y, (J^*)^{-1} \nabla_X^* J^* Z\right)$$

or equivalently (see (vi) of Proposition 2.2)

$$\left(\nabla_X^\dagger g \right) (Y, Z) = (\nabla_X^* g) (Y, Z) - g\left((\nabla_X^* J^*) Y, J^{-1} Z\right) + \frac{1}{q} g(J^* Z, (\nabla_X J) Y). \quad (3.30)$$

Exchanging X and Y in the equation (3.30):

$$\left(\nabla_Y^\dagger g \right) (X, Z) = (\nabla_Y^* g) (X, Z) - g\left((\nabla_Y^* J^*) X, J^{-1} Z\right) + \frac{1}{q} g(J^* Z, (\nabla_Y J) X). \quad (3.31)$$

Thus, the difference of the equations (3.30) and (3.31) is:

$$\begin{aligned} & \left(\nabla_X^\dagger g \right) (Y, Z) - \left(\nabla_Y^\dagger g \right) (X, Z) = (\nabla_X^* g) (Y, Z) - (\nabla_Y^* g) (X, Z) \\ & + g\left((\nabla_Y^* J^*) X - (\nabla_X^* J^*) Y, J^{-1} Z\right) + \frac{1}{q} g(J^* Z, (\nabla_X J) Y - (\nabla_Y J) X). \end{aligned}$$

Hence, $(\nabla^\dagger, g)$ is Codazzi coupled if and only if (∇^*, g) is Codazzi coupled.

(viii) Let the pair (∇^*, J^*) be Codazzi coupled. We start by calculating $\left(\nabla_X^\dagger G \right) (Y, Z)$:

$$\left(\nabla_X^\dagger G \right) (Y, Z) = XG(Y, Z) - G\left(\nabla_X^\dagger Y, Z\right) - G\left(Y, \nabla_X^\dagger Z\right).$$

Using Proposition 2.6, we express:

$$\left(\nabla_X^\dagger G \right) (Y, Z) = XG(Y, Z) - G\left((J^*)^{-1} \nabla_X^* J^* Y, Z\right) - G\left(Y, (J^*)^{-1} \nabla_X^* J^* Z\right)$$

or equivalently,

$$\left(\nabla_X^\dagger G \right) (Y, Z) = (\nabla_X^* G) (Y, J^* Z) - g\left((J^*)^{-1} (\nabla_X^* J^*) Y, J^* Z\right). \quad (3.32)$$

Exchanging X and Y in the equality (3.32), we get:

$$\left(\nabla_Y^\dagger G \right) (X, Z) = (\nabla_Y^* G) (X, J^* Z) - g\left((J^*)^{-1} (\nabla_Y^* J^*) X, J^* Z\right). \quad (3.33)$$

Thus, the difference of the equations (3.32) and (3.33) is:

$$\begin{aligned} & \left(\nabla_X^\dagger G \right) (Y, Z) - \left(\nabla_Y^\dagger G \right) (X, Z) = (\nabla_X^* G) (Y, J^* Z) - (\nabla_Y^* G) (X, J^* Z) \\ & + g\left((J^*)^{-1} ((\nabla_Y^* J^*) X - (\nabla_X^* J^*) Y), J^* Z\right). \end{aligned}$$

Therefore, $(\nabla^\dagger, G)$ is Codazzi coupled if and only if (∇^*, g) is Codazzi coupled.

(ix) Assume that (∇, J) and (∇^*, J^*) are Codazzi coupled. Using Proposition 2.6, we have:

$$\left(\nabla_X^\dagger g \right) (Y, J^* Z) = Xg(Y, J^* Z) - g\left(\nabla_X^\dagger Y, J^* Z\right) - g\left(Y, \nabla_X^\dagger J^* Z\right).$$

$$\begin{aligned}
&= Xg(Y, J^*Z) - g\left((J^*)^{-1} \nabla_X^* J^* Y, J^* Z\right) - g\left(Y, (J^*)^{-1} \nabla_X^* (J^*)^2 Z\right) \\
&= XG(Y, Z) - G(\nabla_X^* Y, Z) - G(Y, \nabla_X^* Z) \\
&\quad - G\left((J^*)^{-1} (\nabla_X^* J^*) Y, Z\right) - pg(Y, (J^*)^{-1} (\nabla_X^* J^*) Z).
\end{aligned}$$

Via (vi) of Proposition 2.2, this results in:

$$\left(\nabla_X^\dagger g\right)(Y, J^*Z) = (\nabla_X^* G)(Y, Z) - G((\nabla_X^* J^*) Y, J^{-1}Z) + \frac{p}{q} g(J^*Z, (\nabla_X J) Y). \quad (3.34)$$

Exchanging X and Y in the equation (3.34) gives:

$$\left(\nabla_Y^\dagger g\right)(X, J^*Z) = (\nabla_Y^* G)(X, Z) - G((\nabla_Y^* J^*) X, J^{-1}Z) + \frac{p}{q} g(J^*Z, (\nabla_Y J) X). \quad (3.35)$$

Thus, the difference of the equations (3.34) and (3.35) becomes:

$$\begin{aligned}
&\left(\nabla_X^\dagger g\right)(Y, J^*Z) - \left(\nabla_Y^\dagger g\right)(X, J^*Z) \\
&= (\nabla_X^* G)(Y, Z) - (\nabla_Y^* G)(X, Z) \\
&\quad - g((J \nabla_X^* J^*) Y - (\nabla_Y^* J^*) X, J^{-1}Z) \\
&\quad + \frac{p}{q} g(J^*Z, (\nabla_X J) Y - (\nabla_Y J) X).
\end{aligned}$$

Therefore, $(\nabla^\dagger, g)$ is Codazzi coupled if and only if (∇^*, G) is Codazzi coupled.

(x) Assume that (∇, J) and (∇^*, J^*) are Codazzi coupled. We start with:

$$\left(\nabla_X^\dagger G\right)(Y, Z) = XG(Y, Z) - G\left(\nabla_X^\dagger Y, Z\right) - G\left(Y, \nabla_X^\dagger Z\right).$$

From proposition 2.6, we have

$$= XG(Y, Z) - G\left((J^*)^{-1} \nabla_X^* J^* Y, Z\right) - G\left(Y, (J^*)^{-1} \nabla_X^* J^* Z\right)$$

or equivalently,

$$\left(\nabla_X^\dagger G\right)(Y, Z) = (\nabla_X^* G)(Y, Z) - G((\nabla_X^* J^*) Y, J^{-1}Z) - G(J^{-1}Z, (\nabla_X J) Y). \quad (3.36)$$

Swapping X and Y in the equation (3.36), we obtain:

$$\left(\nabla_Y^\dagger G\right)(X, Z) = (\nabla_Y^* G)(X, Z) - G((\nabla_Y^* J^*) X, J^{-1}Z) - G(J^{-1}Z, (\nabla_Y J) X). \quad (3.37)$$

Thus, their difference of the equations (3.36) and (3.37) becomes:

$$\begin{aligned}
&\left(\nabla_X^\dagger G\right)(Y, Z) - \left(\nabla_Y^\dagger G\right)(X, Z) = (\nabla_X^* G)(Y, Z) - (\nabla_Y^* G)(X, Z) \\
&\quad - G((\nabla_X^* J^*) Y - (\nabla_Y^* J^*) X, J^{-1}Z) - G(J^{-1}Z, (\nabla_X J) Y - (\nabla_Y J) X).
\end{aligned}$$

Therefore, $(\nabla^\dagger, G)$ is Codazzi coupled if and only if (∇^*, G) is Codazzi coupled. $\square$

Proposition 3.4. Let (M, J, g) be a metallic-like pseudo Riemannian manifold with a linear connection ∇ . Denote (g, G, J) a compatible triple and $\tilde{g}$ the symmetric part of G . Let ∇^* and $\nabla^\sharp$ represent the g -conjugate and $\tilde{g}$ -conjugate of the linear connection ∇ , respectively. The following equivalences can be satisfied:

(i) If (∇, J) and (∇, J^*) are Codazzi coupled, then the pair (∇^*, g) is Codazzi coupled if and only if the pair $(\nabla^*, \tilde{g})$ is Codazzi coupled.

(ii) If (∇^*, J) and (∇^*, J^*) are Codazzi coupled, then the pair (∇, g) is Codazzi coupled if and only if the pair $(\nabla^\sharp, \tilde{g})$ is Codazzi coupled.

Proof. (i) Assume that both (∇, J) and (∇, J^*) are Codazzi coupled. Then, using the definition of $\nabla^* \tilde{g}$, we have

$$\begin{aligned} (\nabla_X^* \tilde{g})(Y, Z) &= X \tilde{g}(Y, Z) - \tilde{g}(\nabla_X^* Y, Z) - \tilde{g}(Y, \nabla_X^* Z) \\ &= \frac{1}{2} X g((J + J^*) Y, Z) - \frac{1}{2} g((J + J^*) \nabla_X^* Y, Z) \\ &\quad - \frac{1}{2} g((J + J^*) Y, \nabla_X^* Z). \end{aligned} \quad (3.38)$$

Rewriting the terms in the equation (3.38), we obtain

$$\begin{aligned} (\nabla_X^* \tilde{g})(Y, Z) &= \frac{1}{2} \left((\nabla_X^* g)(Y, J^* Z) + (\nabla_X^* g)(Y, JZ) \right. \\ &\quad \left. + g((\nabla_X J) Y, Z) + g((\nabla_X J^*) Y, Z) \right). \end{aligned} \quad (3.39)$$

Exchanging X and Y , in the equation (3.39), we get

$$\begin{aligned} (\nabla_Y^* \tilde{g})(X, Z) &= \frac{1}{2} \left((\nabla_Y^* g)(X, J^* Z) \right. \\ &\quad \left. + (\nabla_Y^* g)(X, JZ) + g((\nabla_Y J) X, Z) + g((\nabla_Y J^*) X, Z) \right). \end{aligned} \quad (3.40)$$

Thus, we can write the difference of the expressions (3.39) and (3.40) as:

$$\begin{aligned} &(\nabla_X^* \tilde{g})(Y, Z) - (\nabla_Y^* \tilde{g})(X, Z) \\ &= \frac{1}{2} \left((\nabla_X^* g)(Y, (J + J^*) Z) - (\nabla_Y^* g)(X, (J + J^*) Z) \right. \\ &\quad \left. + g((\nabla_X J) Y, Z) - (\nabla_Y J) X, Z) + g((\nabla_X J^*) Y, Z) - (\nabla_Y J^*) X, Z) \right). \end{aligned}$$

Therefore, the pair (∇^*, g) is Codazzi coupled if and only if the pair $(\nabla^*, \tilde{g})$ is Codazzi coupled.

(ii) Suppose that (∇^*, J) and (∇^*, J^*) are Codazzi coupled. We start with

$$\begin{aligned} (\nabla_X^\# \tilde{g})(Y, Z) &= X \tilde{g}(Y, Z) - \tilde{g}(\nabla_X^\# Y, Z) - \tilde{g}(Y, \nabla_X^\# Z) \\ &= X \tilde{g}(Y, Z) - X \tilde{g}(Y, Z) + \tilde{g}(Y, \nabla_X Z) - X \tilde{g}(Y, Z) + \tilde{g}(\nabla_X Y, Z) \\ &= -X \tilde{g}(Y, Z) + \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z). \end{aligned}$$

From definition of metric $\tilde{g}$, one has the following equality:

$$\begin{aligned} (\nabla_X^\# \tilde{g})(Y, Z) &= -\frac{1}{2} X g((J + J^*) Y, Z) + \frac{1}{2} g((J + J^*) \nabla_X Y, Z) \\ &\quad + \frac{1}{2} g((J + J^*) Y, \nabla_X Z) \end{aligned}$$

or equivalently,

$$(\nabla_X^\# \tilde{g})(Y, Z) = -\frac{1}{2} \left((\nabla_X g)(Y, J^* Z) + (\nabla_X g)(Y, JZ) \right. \\ \left. + g((\nabla_X^* J) Y, Z) + g((\nabla_X^* J^*) Y, Z) \right). \quad (3.41)$$

By exchanging X and Y in the equation (3.41), we have

$$(\nabla_Y^\# \tilde{g})(X, Z) = -\frac{1}{2} \left((\nabla_Y g)(X, J^* Z) + (\nabla_Y g)(X, JZ) \right. \\ \left. + g((\nabla_Y^* J) X, Z) + g((\nabla_Y^* J^*) X, Z) \right). \quad (3.42)$$

Hence, the difference of the equations (3.41) and (3.42) becomes

$$\begin{aligned} &(\nabla_X^\# \tilde{g})(Y, Z) - (\nabla_Y^\# \tilde{g})(X, Z) \\ &= \frac{1}{2} \left((\nabla_Y g)(X, (J + J^*) Z) - (\nabla_X g)(Y, (J + J^*) Z) \right. \\ &\quad \left. + g((\nabla_Y^* J) X, Z) - (\nabla_X^* J) Y, Z) + g((\nabla_Y^* J^*) X, Z) - (\nabla_X^* J^*) Y, Z) \right). \end{aligned}$$

Therefore, (∇, g) is Codazzi coupled if and only if $(\nabla^\#, \tilde{g})$ is Codazzi coupled. □

Proposition 3.5. *In the setting of a metallic-like pseudo-Riemannian manifold (M, J, g) equipped with a linear connection ∇ , assume that the pair (∇, g) is Codazzi coupled and satisfies the condition $C(J^*X, Y, Z) = C(X, Y, J^*Z)$. Then, the following equivalence holds: the pair (∇, J^*) is Codazzi coupled if and only if the pair (∇^*, J^*) is Codazzi coupled.*

Proof. We have the following expression:

$$\begin{aligned}
& g((\nabla_X J^*)Z - (\nabla_Z J^*)X, Y) \\
&= g(\nabla_X J^*Z - J^*\nabla_X Z, Y) - g(\nabla_Z J^*X - J^*\nabla_Z X, Y) \\
&= g(\nabla_X J^*Z, Y) - g(J^*\nabla_X Z, Y) - g(\nabla_Z J^*X, Y) + g(J^*\nabla_Z X, Y) \\
&= Xg(J^*Z, Y) - g(J^*Z, \nabla_X^* Y) - Xg(Z, JY) + g(Z, \nabla_X^* JY) \\
&\quad - Zg(J^*X, Y) + g(J^*X, \nabla_Z^* Y) + Zg(X, JY) - g(X, \nabla_Z^* JY) \\
&= C^*(J^*Z, Y, X) + g(\nabla_X^* J^*Z, Y) - C^*(Z, JY, X) - g(\nabla_X^* Z, JY) \\
&\quad - C^*(J^*X, Y, Z) - g(\nabla_Z^* J^*X, Y) + C^*(X, JY, Z) + g(JY, \nabla_Z^* X) \\
&= -C(J^*Z, Y, X) + C(Z, JY, X) + C(J^*X, Y, Z) - C(X, JY, Z) \\
&\quad + g(\nabla_X^* J^*Z, Y) - g(\nabla_X^* Z, JY) - g(\nabla_Z^* J^*X, Y) + g(JY, \nabla_Z^* X) \\
&= C(J^*X, Y, Z) - C(X, Y, J^*Z) + g((\nabla_X^* J^*)Z, Y) - g((\nabla_Z^* J^*)X, Y) \\
&= g((\nabla_X^* J^*)Z, Y) - g((\nabla_Z^* J^*)X, Y).
\end{aligned}$$

From this, we conclude that the pair (∇, J^*) is Codazzi coupled if and only if the pair (∇^*, J^*) is also Codazzi coupled. $\square$

As a consequence of the Proposition 2.8 and Proposition 3.5, we have:

Corollary 3.1. *In the context of a metallic-like pseudo-Riemannian manifold (M, J, g) equipped with a linear connection ∇ , if the pair (∇, g) is Codazzi coupled and either $K(J^*X, Y) = K(X, J^*Y)$ or $K(J^*X, Y) = JK(X, Y)$, then it follows that the pair (∇, J^*) is Codazzi coupled if and only if the pair (∇^*, J^*) is also Codazzi coupled.*

We briefly recall the fundamental concepts of statistical geometry, with further details available in [13]. Let g represent a (pseudo-) Riemannian metric on a manifold M . A statistical structure is defined as a pair (∇, g) , where ∇ is a torsion-free connection that satisfies the Codazzi condition:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) \quad (3.43)$$

for all vector fields X, Y, Z on M . In this context, the triple (M, ∇, g) is referred to as a statistical manifold. A connection ∇ that satisfies equation (3.43) is referred to as a statistical connection for g . The concept of a Codazzi tensor is closely linked to Einstein metrics. Specifically, the Ricci tensor field Ric^g of an Einstein manifold (M, g) is a Codazzi tensor field. More generally, the Ricci tensor field on any Riemannian manifold is a Codazzi tensor field if and only if the Riemannian curvature corresponds to a Yang-Mills gauge field on the tangent bundle. For this reason, a Riemannian manifold (M, g) is called a space of harmonic curvature when its Ricci tensor field is Codazzi. This notion of a space of harmonic curvature, therefore, extends the concept of an Einstein manifold.

The Propositions 3.3 and 3.4 immediately give the following important theorem.

Theorem 3.1. *Let (M, J, g) be a metallic-like pseudo Riemannian manifold equipped with a linear connection ∇ . Denote (g, G, J) a compatible triple and $\tilde{g}$ the symmetric part of G , and let ∇^* and $\nabla^\sharp$ be the g -conjugate and $\tilde{g}$ -conjugate of the linear connection ∇ , respectively. The following equivalences can be held:*

- (i) *If (∇, J) and (∇^*, J^*) are Codazzi coupled, then (∇^J, g) is statistical structure if and only if (∇, g) is a statistical structure.*
- (ii) *If (∇, J^*) and (∇^*, J) are Codazzi coupled, then (∇^{J^*}, g) is a statistical structure if and only if (∇, g) is a statistical structure.*
- (iii) *If (∇, J) and (∇^*, J^*) are Codazzi coupled, then $(\nabla^\dagger, g)$ is a statistical structure if and only if (∇^*, g) is a statistical structure.*
- (iv) *If (∇, J) and (∇, J^*) are Codazzi coupled, then (∇^*, g) is a statistical structure if and only if $(\nabla^*, \tilde{g})$ is a statistical structure.*
- (v) *If (∇^*, J) and (∇^*, J^*) are Codazzi coupled, then (∇, g) is a statistical structure if and only if $(\nabla^\sharp, \tilde{g})$ is a statistical structure.*

Proposition 3.6. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold, where ∇ is an arbitrary linear connection with torsion tensor T^∇ , and ∇^* is the g -conjugate connection of ∇ . If the pairs (∇, J) and (∇^*, J^*) are Codazzi coupled, then $(\tilde{\nabla}, g)$ is Codazzi coupled if and only if (∇, g) is Codazzi coupled, where $\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^J)$ and ∇^J is the J -conjugate connection of ∇ .

Proof. Suppose that the pairs (∇, J) and (∇^*, J^*) are Codazzi coupled. With help of (vi) of Proposition 2.2, we have the following calculations:

$$\begin{aligned} (\tilde{\nabla}_X g)(Y, Z) &= Xg(Y, Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \\ &= Xg(Y, Z) - \frac{1}{2}g(\nabla_X Y, Z) - \frac{1}{2}g(\nabla_X^J Y, Z) \\ &\quad - \frac{1}{2}g(Y, \nabla_X Z) - \frac{1}{2}g(Y, \nabla_X^J Z) \\ &= Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &\quad - \frac{1}{2}g(J^{-1}(\nabla_X J)Y, Z) - \frac{1}{2}g(Y, J^{-1}(\nabla_X J)Z) \\ &= (\nabla_X g)(Y, Z) - \frac{1}{2}g((\nabla_X J)Y, (J^*)^{-1}Z) \\ &\quad + \frac{1}{2g}g((\nabla_X^* J^*)Y, JZ). \end{aligned}$$

Similarly, for $(\tilde{\nabla}_Y g)(X, Z)$, we find:

$$(\tilde{\nabla}_Y g)(X, Z) = (\nabla_Y g)(X, Z) - \frac{1}{2}g((\nabla_Y J)X, (J^*)^{-1}Z) + \frac{1}{2g}g((\nabla_Y^* J^*)X, JZ).$$

From these expressions, we conclude that the connection $(\tilde{\nabla}, g)$ is Codazzi coupled if and only if the connection (∇, g) is Codazzi coupled. $\square$

Proposition 3.6 directly gives the following result.

Corollary 3.2. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold, where ∇ is an arbitrary linear connection with torsion tensor T^∇ , and ∇^* is the g -conjugate connection of ∇ . It is established that (∇, g) is Codazzi coupled if and only if $T^\nabla = T^{\nabla^*}$. Given that (∇, J) and (∇^*, J^*) are Codazzi coupled, we conclude that $(\tilde{\nabla}, g)$ is Codazzi coupled if and only if $T^\nabla = T^{\nabla^*}$.

Proposition 3.7. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold, with ∇ as an arbitrary linear connection and T^∇ as its torsion tensor. Let ∇^* denote the g -conjugate connection of ∇ . Assuming that (∇^*, J) and (∇, J^*) are Codazzi coupled, then $(\tilde{\nabla}, g)$ is Codazzi coupled if and only if (∇, g) is Codazzi coupled, where $\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^{J^*})$ and ∇^{J^*} is the J^* -conjugate connection of ∇ .

Proof. Let the pairs (∇^*, J) and (∇, J^*) be Codazzi coupled. We compute:

$$(\tilde{\nabla}_X g)(Y, Z) = Xg(Y, Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z).$$

This expands to:

$$\begin{aligned} &= Xg(Y, Z) - \frac{1}{2}g(\nabla_X Y, Z) - \frac{1}{2}g(\nabla_X^{J^*} Y, Z) \\ &\quad - \frac{1}{2}g(Y, \nabla_X Z) - \frac{1}{2}g(Y, \nabla_X^{J^*} Z). \end{aligned} \tag{3.44}$$

This equation (3.44) simplifies to:

$$\begin{aligned} &= Xg(Y, Z) - \frac{1}{2}g(\nabla_X Y, Z) - \frac{1}{2}g(Y, \nabla_X Z) \\ &\quad - \frac{1}{2}g((J^*)^{-1}\nabla_X J^* Y, Z) - \frac{1}{2}g(Y, (J^*)^{-1}\nabla_X J^* Z). \end{aligned} \tag{3.45}$$

Combining terms in the equation (3.45) gives:

$$= Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ - \frac{1}{2}g((\nabla_X J^*)Y, J^{-1}Z) - \frac{1}{2}g(J^{-1}Y, (\nabla_X J^*)Z).$$

Thus, from (5) of Proposition 2.2, we have:

$$= (\nabla_X g)(Y, Z) - \frac{1}{2}g((\nabla_X J^*)Y, J^{-1}Z) - \frac{1}{2}g((\nabla_X^* J)J^{-1}Y, Z).$$

Finally, we obtain:

$$\left(\tilde{\nabla}_X g\right)(Y, Z) = (\nabla_X g)(Y, Z) - \frac{1}{2}g((\nabla_X J^*)Y, J^{-1}Z) + \frac{1}{2q}g((\nabla_X^* J)Y, J^*Z). \quad (3.46)$$

By exchanging X and Y in the expression (3.46), we find:

$$\left(\tilde{\nabla}_Y g\right)(X, Z) = (\nabla_Y g)(X, Z) - \frac{1}{2}g((\nabla_Y J^*)X, J^{-1}Z) \\ + \frac{1}{2q}g((\nabla_Y^* J)X, J^*Z).$$

This shows that $(\tilde{\nabla}, g)$ is Codazzi coupled if and only if (∇, g) is Codazzi coupled. $\square$

Proposition 3.8. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold, with ∇ as an arbitrary linear connection having torsion tensor T^∇ , and ∇^* denoting the g -conjugate connection of ∇ . If (∇^*, J^*) is Codazzi coupled, then it follows that $T^{\nabla^*} = T^{\nabla^\dagger}$.*

Proof. Assume that (∇^*, J^*) is Codazzi coupled. We compute the following expression:

$$g\left(T^{\nabla^\dagger}(X, Y) - T^{\nabla^*}(X, Y), Z\right) = g\left(\nabla_X^\dagger Y - \nabla_Y^\dagger X - \nabla_X^* Y + \nabla_Y^* X, Z\right).$$

Using Proposition 2.6, this simplifies to:

$$g\left((J^*)^{-1}\nabla_X^* J^* Y, Z\right) - g\left((J^*)^{-1}\nabla_Y^* J^* X, Z\right) - g(\nabla_X^* Y, Z) + g(\nabla_Y^* X, Z).$$

Ultimately, this leads us to conclude that:

$$g\left(T^{\nabla^\dagger}(X, Y) - T^{\nabla^*}(X, Y), Z\right) = g\left((\nabla_X^* J^*)Y - (\nabla_Y^* J^*)X, J^{-1}Z\right).$$

From the hypothesis, we can then state that $T^{\nabla^*} = T^{\nabla^\dagger}$. $\square$

The following result comes from Proposition 3.8.

Corollary 3.3. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold. Let ∇ be an arbitrary linear connection with torsion tensor T^∇ , ∇^* be the g -conjugate connection of ∇ , and $\nabla^\dagger$ be the G -conjugate connection of ∇ . Assuming that (∇^*, J^*) is Codazzi coupled, the condition $T^\nabla = T^{\nabla^\dagger}$ holds if and only if $K(X, Y) = K(Y, X)$, where $K(X, Y) = \nabla_X Y - \nabla_Y X$.*

Proposition 3.3 and 3.8 give the following result.

Corollary 3.4. *Let (M, J, g) be a metallic-like pseudo-Riemannian manifold with a linear connection ∇ . Let ∇^* denote the g -conjugate connection of ∇ , and $\nabla^\dagger$ represent the G -conjugate connection of ∇ . Assuming that (∇^*, J^*) is Codazzi coupled, the following statements are equivalent:*

- (i) $T^\nabla = T^{\nabla^\dagger}$.
- (ii) The pair (∇, g) is Codazzi coupled.
- (iii) The pair $(\nabla^\dagger, G)$ is Codazzi coupled.

Propositions 3.3, 3.6 and 3.8 immediately follow the below result.

Corollary 3.5. Let (M, J, g) be a metallic-like pseudo-Riemannian manifold equipped with a linear connection ∇ . Assume that (∇, J) and (∇^*, J^*) are Codazzi coupled. The following statements are equivalent:

- (i) $T^\nabla = T^{\nabla^\dagger}$.
- (ii) The pair $(\tilde{\nabla}, g)$ is Codazzi coupled.
- (iii) The pair (∇^*, g) is Codazzi coupled.
- (iv) The pair $(\nabla^\dagger, g)$ is Codazzi coupled.
- (v) The pair (∇^*, G) is Codazzi coupled.

Conclusion

In this study, we explored the intricate properties of metallic-like pseudo-Riemannian manifolds with various linear connections. Our analysis revealed significant interrelationships among these connections under specific conditions of Codazzi coupling. Notably, we established that the properties of one type of connection directly influence those of another, highlighting a robust symmetry that underpins their behavior within these manifolds.

We also derived essential equalities involving curvature and torsion tensors, demonstrating their coordinated behavior in metallic-like structures. This insight emphasizes the complex nature of these manifolds, characterized by non-trivial curvature and torsion properties.

Furthermore, our investigation into the compatibility of geometric structures illuminated the relationships among the metric, connections, and their conjugates. These findings enhance our understanding of the statistical properties of metallic-like pseudo-Riemannian manifolds, revealing their implications for statistical analysis in the context of statistical manifolds.

In conclusion, our research contributes to the theoretical framework of metallic-like pseudo-Riemannian manifolds and underscores their significance in differential geometry and statistics. The insights gained from this study pave the way for future research, encouraging exploration of the broader applications and implications of these geometric structures.

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Author's contributions

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References

- [1] Altunbaş, M.: *Metallic Riemannian structures on the tangent bundles of Riemannian manifolds with g-natural metrics*. Int. Electron. J. Geom. **16**(1), 95–103 (2023) <https://doi.org/10.36890/iejg.1145729>.
- [2] Amari, S.: *Differential-geometrical methods in statistics*. vol. 28. Springer-Verlag, New York (1985).
- [3] Bejan, C. L., Crasmăreanu, M.: *Conjugate connections with respect to a quadratic endomorphism and duality*. Filomat. **30**(9), 2367–2374 (2016). <https://doi.org/10.2298/FIL1609367B>
- [4] Blaga, A. M., Nannicini, A.: *On the geometry of metallic pseudo-Riemannian structures*. Riv. Math. Univ. Parma (N.S.). **11**(1), 69–87 (2020).
- [5] Blaga, A. M., Nannicini, A.: *On curvature tensors of Norden and metallic pseudo-Riemannian manifolds*. Complex Manifolds. **6**(1), 150–159 (2019). <https://doi.org/10.1515/coma-2019-0008>

- [6] Calin, O., Matsuzoe, H., Zhang, J.: Generalizations of conjugate connections. World Scientific Publishing Co. Pte. Ltd. Hackensack, NJ, 26–34 (2009).
- [7] Durmaz, O., Gezer, A.: *Conjugate connections and their applications on pure metallic metric geometries*. Ricerche Mat. (2023). <https://doi.org/10.1007/s11587-023-00782-0>.
- [8] Erkan, E., Takano, K., Gulbahar, M.: *Locally product-like statistical manifolds and their hypersurfaces*. Int. Electron. J. Geom. **16**(2), 435–450 (2023). [HTTPS://DOI.ORG/10.36890/IEJG.1307467](https://doi.org/10.36890/IEJG.1307467)
- [9] Erken, I. K., Murathan, C. C., Yazla, A.: *Almost cosymplectic statistical manifolds*. Quaest. Math. **43**(2), 265–282 (2020). <https://doi.org/10.2989/16073606.2019.1576069>
- [10] Fei, T., Zhang, J.: *Interaction of Codazzi couplings with (para-)Kähler geometry*. Results Math. **72**(4), 2037–2056 (2017). <https://doi.org/10.1007/s00025-017-0711-7>
- [11] Gezer, A., Karaman, C.: *On metallic Riemannian structures*. Turkish J. Math. **39**(6), 954–962 (2015). <https://doi.org/10.3906/mat-1504-50>
- [12] Hretcanu, C. E., Crasmareanu, M.: *Metallic structures on Riemannian manifolds*. Rev. Un. Mat. Argentina. **54**(2), 15–27 (2013).
- [13] S. Lauritzen, Statistical manifolds. In: Amari, S., Barndorff-Nielsen, O., Kass, R., Lauritzen, S., Rao, C.R. (eds.) Differential Geometry in Statistical Inference, IMS Lecture Notes, vol. 10, pp. 163–216. Institute of Mathematical Statistics, Hayward (1987).
- [14] Manea, A.: *Metallic-like structures and metallic-like maps*. Turkish J. Math. **47**(5), 1539–1549 (2023). <https://doi.org/10.55730/1300-0098.3446>
- [15] H. Nagaoka, S. Amari, Differential geometry of smooth families of probability distributions, Technical Report (METR) 82–7, Dept. of Math. Eng. and Instr., Univ. of Tokyo (1982).
- [16] Norden, A.P.: Affinely connected spaces GRMFL. Moscow (1976) (in Russian).
- [17] Nomizu, K., Simon, U.: Notes on conjugate connections. World Scientific Publishing Co. Inc. River Edge. NJ. 152–173 (1992).
- [18] Stepanov, S. E., Stepanova, E. S., Shandra, I. G.: *Conjugate connections on statistical manifolds*. Izv. Vyssh. Uchebn. Zaved. Mat. **10**, 90–98 (2007). <https://doi.org/10.3103/S1066369X07100052>
- [19] Takano, K.: *Statistical manifolds with almost contact structures and its statistical submersions*. J. Geom. **85** (1-2), 171–187 (2006). <https://doi.org/10.1007/s00022-006-0052-2>
- [20] Takano, K.: *Statistical manifolds with almost complex structures*. Tensor, New Ser. **72**(3), 225–231 (2010).
- [21] Vanzura, J.: *Integrability conditions for polynomial structures*. Kodai Math. Sem. Rep. **27** (1–2), 42–50 (1976). <https://doi.org/10.2996/kmj/1138847161>
- [22] Vilcu, A.D., Vilcu, G.E.: *Statistical manifolds with almost quaternionic structures and quaternionic Kähler-like statistical submersions*. Entropy. **17**(9), 6213–6228 (2015). <https://doi.org/10.3390/e17096213>
- [23] Vilcu, G.E.: *Almost product structures on statistical manifolds and para-Kähler like statistical submersions*. Bull. Sci. Math. **171**, Paper No. 103018, 21 pp (2021). <https://doi.org/10.1016/j.bulsci.2021.103018>

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