Available online: May 23, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 451–456 (2019)
DOI: 10.31801/cfsuasmas.425624
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

NULL CURVES OF CONSTANT BREADTH IN MINKOWSKI 4-SPACE

BÜLENT ALTUNKAYA AND FERDAĞ KAHRAMAN AKSOYAK

Abstract. In this paper, we define null curves of constant breadth in Minkowski 4-space and obtain a characterization of these curves. Also we give an example for such curves.

1. Introduction

The concept about curves of constant breadth were introduced by Euler in 1780 [2]. After then, this subject were studied by many geometers. Köse [7] investigated some properties of curves of constant width in plane. Also Köse [8] showed that when a space curve is given, another space curve can be obtained such that the tangents at corresponding points of the curves are parallel in the opposite directions and the distance between these points is always constant. In [9] and [1], the concepts about the curves of constant breadth were extended to spaces E^4 and E^n , respectively. However, many mathematicians have been interested in studying curves of constant breadth in semi-Euclidean space [4],[5],[6], [11], [13], [14]. In [12], they proved that there does not exist null curves of constant breadth in Minkowski 3- space.

In this paper, we define null curves of constant breadth in Minkowski 4 - space and give a characterization of these curves. Also we obtain a differential equation for null curves whose tangents at the corresponding points are parallel in opposite direction and give an example for such curves.

2. Preliminaries

Let $\mathbb{E}_1^4$ be the 4- dimensional Minkowski space-time. Then the metric tensor g in $\mathbb{E}_1^4$ is given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

Received by the editors: March 06, 2017; Accepted: January 22, 2018.

2010 *Mathematics Subject Classification.* 53C50, 53B30.

Key words and phrases. Null curves, curves of constant breadth, Minkowski space.

where (x_1, x_2, x_3, x_4) is a standard rectangular coordinate system of $\mathbb{E}_1^4$. There are three casual characters in Minkowski space. Let v be a vector in $\mathbb{E}_1^4$. If $g(v,v) > 0$ or $v = 0$, then it is called spacelike vector, if $g(v,v) < 0$, then it is called timelike vector and if $g(v,v) = 0$ and $v \neq 0$ then it is called null vector. The norm of a vector v is defined by $||v|| = \sqrt{|g(v,v)|}$ and if $||v|| = 1$ then v is said to be unit vector. A curve β in $\mathbb{E}_1^4$ is called spacelike, timelike or null, if all of its velocity vectors are spacelike, timelike or null, respectively. If a spacelike (or a timelike, resp.) curve β is given by arc-length parameter s, then its velocity vector $\beta'(s)$ satisfy the equality $g(\beta'(s), \beta'(s)) = 1$ (or $g(\beta'(s), \beta'(s)) = -1$, resp.) [10]. But that case is different for null curves. If a null curve β is parameterized by arclength function s, then $g(\beta''(s), \beta''(s)) = 1$.

Let the Frenet frame along a null curve β in $\mathbb{E}_1^4$ be denoted by $\{T, N_1, N_2, N_3\}$. Then the Frenet formulae of β is given as [15]:

$$\begin{aligned} T' &= k_1 N_1, \\ N_1' &= k_2 T - k_1 N_2, \\ N_2' &= -k_2 N_1 + k_3 N_3, \\ N_3' &= -k_3 T, \end{aligned} \tag{1}$$

where T and N_2 are null vectors that hold $g(T, N_2) = 1$ and N_1 and N_3 are spacelike vectors. Also $k_1(s)$, $k_2(s)$ and $k_3(s)$ are the curvature functions. If α is a straight line, then the first curvature $k_1(s)$ vanishes but it is equal to 1 in other cases. In this paper, we assume that the null curve β is not a straight line.

3. Null Curves of Constant Breadth in Minkowski 4-Space $\mathbb{E}_1^4$

In this section, we define null curves of constant breadth in Minkowski 4-space and have a characterization of these curves. Also we give an example for these curves at the last of the section.

Definition 1. *Let $\beta : J \subset \mathbb{R} \to \mathbb{E}_1^4$ and $\gamma : \overline{J} \subset \mathbb{R} \to \mathbb{E}_1^4$ be null curves. If the curves β and γ have parallel tangents in opposite directions at the corresponding points $\beta(s)$ and $\gamma(\overline{s})$ and the distance between these corresponding points is always constant, then the pair of null curves β and γ is called null curves of constant breadth.*

Let β and γ be a null pair of curves which have parallel tangents in opposite directions. Then the position vector of γ at the point $\gamma(\overline{s})$ can be written as:

$$\gamma(\overline{s}) = \beta(s) + a(s)T(s) + a_1(s)N_1(s) + a_2(s)N_2(s) + a_3(s)N_3(s), \tag{2}$$

where a and a_i $(i = 1, 2, 3)$ are $C^\infty-$functions on J.

If we take the derivative of (2) with respect to s and use (1) , then we obtain

$$\begin{aligned} \overline{T}(\overline{s})\frac{d\overline{s}}{ds} \;=\; & (1 + a' + a_1 k_2 - a_3 k_3) T(s) + (a_1' + a - a_2 k_2) N_1(s) \\ & + (a_2' - a_1) N_2(s) + (a_3' + a_2 k_3) N_3(s), \end{aligned} \tag{3}$$

where $\overline{T}$ is the tangent vector of γ. Since $\overline{T} = -T$, by using (3), we get

$$\begin{aligned} 1 + a' + a_1 k_2 - a_3 k_3 &= -\frac{d\bar{s}}{ds}, && (4) \\ a_1' + a - a_2 k_2 &= 0, && (5) \\ a_2' - a_1 &= 0, && (6) \\ a_3' + a_2 k_3 &= 0. && (7) \end{aligned}$$

Since the tangents at the corresponding points $\beta(s)$ and $\gamma(\bar{s})$ of the null curves β and γ are parallel in opposite directions we can write

$$\overline{T} = -T.$$

If we take the derivative of this equation with respect to s and use (1) again, we have

$$\bar{N}_1(\bar{s})\frac{d\bar{s}}{ds} = -N_1(s).$$

Since the vectors $\bar{N}_1$ and N_1 are spacelike vectors, we have

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1.$$

On the other hand, by using (5), (6) and (7), we can rewrite (3) as follows:

$$\frac{d\gamma}{ds} = (1 + a' + a_1 k_2 - a_3 k_3)\, T\,(s)\,. \tag{8}$$

Differentiating (8) with respect to s, we have

$$\frac{d^2\gamma}{ds^2} = (1 + a' + a_1 k_2 - a_3 k_3)'\, T\,(s) + (1 + a' + a_1 k_2 - a_3 k_3)\, N_1(s). \tag{9}$$

There exists a regular map defined by $\varphi : J \to \overline{J}$

$$\bar{s} = \varphi(s) = \int_0^s g(\gamma''(t), \gamma''(t))^{\frac{1}{4}}\, dt, \qquad \text{for all } s \in J \tag{10}$$

where $\bar{s}$ denotes the pseudo-arc length parameter of the curve γ. By using (9) and (10), we have

$$\frac{d\bar{s}}{ds} = \sqrt{|(1 + a' + a_1 k_2 - a_3 k_3)|}.$$

So we can say that the value of $\frac{d\bar{s}}{ds}$ is positive and $\frac{d\bar{s}}{ds} = 1$.

Theorem 1. *Let $\beta : J \subset \mathbb{R} \to \mathbb{E}_1^4$ and $\gamma : \overline{J} \subset \mathbb{R} \to \mathbb{E}_1^4$ be null curves. If the tangents at the corresponding points $\beta(s)$ and $\gamma(\bar{s})$ of the curves β and γ are parallel in opposite direction then the following differential equation is satisfied*

$$\left(\frac{(a_2 k_2 - a_2'')' + a_2' k_2 + 2}{k_3}\right)' + a_2 k_3 = 0. \tag{11}$$

Proof. We assume that the tangents at the corresponding points $\beta(s)$ and $\gamma(\overline{s})$ of the curves β and γ are parallel in opposite directions. By using the equations (4),(5),(6) and (7) we obtain the differential equation given by (11). □

Remark 1. *The differential equation (11) is a characterization for null curves which have parallel tangents in opposite directions. Via its solution, the position vector of γ can be determined. The general solution of this differential equation has not yet been found. But it can be considered for some special cases.*

Corollary 1. *Let $\beta : J \subset \mathbb{R} \to \mathbb{E}_1^4$ be a null curve whose the curvatures are constant and $\gamma : \overline{J} \subset \mathbb{R} \to \mathbb{E}_1^4$ be a null curve. If the tangents at the corresponding points of the curves β and γ are parallel in opposite direction then we obtain a_2 as follows:*

$$a_2 = d_1 e^{\sqrt{\lambda} s} + d_2 e^{-\sqrt{\lambda} s} + d_3 \cos \sqrt{-\mu} s + d_4 \sin \sqrt{-\mu} s, \tag{12}$$

where $\lambda = c_2 + \sqrt{c_2^2 + c_3^2}$ and $\mu = c_2 - \sqrt{c_2^2 + c_3^2}$.

Proof. Let us suppose that β is a null curve whose the curvatures are constant. Then we can consider them as $k_2 = c_2$ and $k_3 = c_3$, where c_2 and c_3 are non-zero real constants. In that case we induced differential equation (11) to following linear differential equation.

$$a_2^{(4)} - 2c_2 a_2'' - c_3^2 a_2 = 0. \tag{13}$$

If we solve the differential equation is given by (13), we have (12). It completes the proof. □

Theorem 2. *Let $\beta : J \subset \mathbb{R} \to \mathbb{E}_1^4$ and $\gamma : \overline{J} \subset \mathbb{R} \to \mathbb{E}_1^4$ be null curves. Let the tangents at the corresponding points $\beta(s)$ and $\gamma(\overline{s})$ of the curves β and γ be parallel in opposite directions. Then β and γ are null curves of constant breadth if and only if the curve $a_2 = 0$, where a_2 is the component of γ in the direction $N_2(s)$.*

Proof. We assume that the tangents at the corresponding points $\beta(s)$ and $\gamma(\overline{s})$ of the curves β and γ are parallel in opposite directions and the distance between these points is always constant. Then we obtain

$$\|\gamma(\overline{s}) - \beta(s)\|^2 = 2a(s)\, a_2(s) + a_1^2(s) + a_3^2(s) = \text{constant}. \tag{14}$$

By differentiating (14) with respect to s and by using (4),(5),(6) and (7), we have $a_2 = 0$.

Conversely we assume that the component of γ in the direction $N_2(s)$ is equal to zero. In that case, by using the differential equations given by (4-7), we get $a = a_1 = 0$ and a_3 =constant and $a_3 k_3 = 2$. Then we can write $\gamma(\overline{s}) = \beta(s) + a_3 N_3(s)$. So, the distance between corresponding points of the curves β and γ is constant. It completes the proof. □

Corollary 2. *If β and γ are null curves of constant breadth then the curvatures of β are $k_2 = k_2(s)$ and $k_3 = \frac{2}{a_3}$ =constant.*

Corollary 3. *If β and γ are null curves of constant breadth then the curve γ is determined as $\gamma(\overline{s}) = \beta(s) + a_3 N_3(s)$, where $a_3 = \frac{2}{k_3}$ =constant.*

Example 1. *(This null curve is given in [3]) Let β be a null curve in E_1^4 given by*

$$\beta(s) = \frac{1}{\sqrt{2}} \left(\sinh(s), \cosh(s), \sin(s), \cos(s)\right).$$

The Frenet Frame of β is given by

$$\begin{aligned}
T(s) &= \frac{1}{\sqrt{2}} \left(\cosh(s), \sinh(s), \cos(s), -\sin(s)\right), \\
N_1(s) &= \frac{1}{\sqrt{2}} \left(\sinh(s), \cosh(s), -\sin(s), -\cos(s)\right), \\
N_2(s) &= \frac{1}{\sqrt{2}} \left(-\cosh s, -\sinh s, \cos(s), -\sin(s)\right), \\
N_3(s) &= \frac{1}{\sqrt{2}} \left(\sinh(s), \cosh(s), \sin(s), \cos(s)\right).
\end{aligned}$$

Then, we get the curvatures of β as follows:

$$k_1(s) = 1, \quad k_2(s) = 0, \quad k_3(s) = -1.$$

From Theorem (2), we obtain null curve γ as follows:

$$\gamma(s) = \frac{1}{\sqrt{2}} \left(-\sinh(s), -\cosh(s), -\sin(s), -\cos(s)\right).$$

So, β and γ are null curves of constant breadth and $\|\gamma(s) - \beta(s)\| = 2$. Also, it can be easily seen that the tangent of γ $\bar{T} = -T$.

References

[1] Akdoğan Z., Mağden A., Some characterization of curves of constant breadth in E^n space, *Turkish J. Math.*, 25, (2001), 433-444.

[2] Euler L., De curvis triangularibus, *Acta Acad. Petropol*, 1778 (II), (1780) 3-30.

[3] Kahraman F, Gök I, Ilarslan K., Generalized null Bertrand curves in Minkowski space-time. *Scientic Annals of Al.I. Cuza University of Iasi Mathematica*, 60, (2014), 489-502.

[4] Kocayiğit H., Önder, M., Space curves of constant breadth in Minkowski 3-space, *Ann. Mat. Pura. Appl.* 192 (5), (2013), 805-814.

[5] Kocayiğit, H., Zennure, Ç., Some characterizations of constant breadth spacelike curves in Minkowski 4-space E_1^4, *New Trends Math. Sci.* 3 (2), (2015), 1-12.

[6] Kocayiğit, H., Çetin M., Evirgen M., Timelike curves of constant breadth according to Bishop frame in Minkowski 3-space, *Iran J.Sci. Technol. Trans. A Sci.* 41, (2017), 45-49.

[7] Köse Ö., Some properties of ovals and curves of constant width in a plane, *Doğa Sci. J. Serial B* 8, (1984), 119-126.

[8] Köse Ö., On space curves of constant breadth, *Doğa Mat.* 10, (1986), 11-14.

[9] Mağden A. Köse Ö., On the curves of constant breadth in E^4 space, *Turkish J. Math.*, 21, (1997), 277-284.

[10] O'Neill B., Semi-Riemannian geometry with applications to relativity, Academic Press, New York, 1983.

[11] Önder M., Kocayiğit H., Candan E. Differential equations characterizing timelike and spacelike curves of constant breadth in Minkowski 3-space E_1^3, *J. Korean Math. Soc.* 48(4), (2011), 849-866.
[12] Özyılmaz E., Yılmaz S. and Nizamoğlu Ş., Simple closed null curves in Minkowski 3-space, *Sci. Magna,* 5(3), (2009), 40-43.
[13] Turgut M., Smarandache breadth pseudo null curves in Minkowski space-time, *Int. J. Math. Comb.* 1, (2009), 46-49.
[14] Yılmaz S., Turgut M., Partially null curves of constant breadth in semi-Riemannian space, *Modern App. Sci.*, 3 (3), (2009), 60-63.
[15] Walrave J., Curves and surfaces in Minkowski space, Thesis (Ph.D.), Katholieke Universiteit Leuven (Belgium), 1995.

Current address: Bülent ALTUNKAYA: Ahi Evran University, Department of Mathematics Education Kırşehir, Turkey
E-mail address: bulent.altunkaya@ahievran.edu.tr
ORCID Address: http://orcid.org/0000-0003-4633-034X
Current address: Ferdağ KAHRAMAN AKSOYAK: Ahi Evran University, Department of Mathematics Education Kırşehir, Turkey
E-mail address: ferdag.aksoyak@ahievran.edu.tr
ORCID Address: http://orcid.org/0000-0002-3186-5643