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Neighbor Rupture Degree of Some Middle Graphs

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Abstract: Networks have an important place in our daily lives. Internet networks, electricity networks, water networks, transportation networks, social networks and biological networks are some of the networks we run into every aspects of our lives. A network consists of centers connected by links. A network is represented when centers and connections modelled by vertices and edges, respectively. In consequence of the failure of some centers or connection lines, measurement of the resistance of the network until the communication interrupted is called vulnerability of the network. In this study, neighbor rupture degree which is a parameter that explores the vulnerability values of the resulting graphs due to the failure of some centers of a communication network and its neighboring centers becoming nonfunctional were applied to some middle graphs and neighbor rupture degree of the $M(C_n)$, $M(P_n)$, $M(K_{1,n})$, $M(W_n)$, $M(P_n \times K_2)$ and $M(C_n \times K_2)$ have been found.

Bazı Middle Grafların Komşu Rupture Derecesi

Anahtar Kelimeler
Graf teorisi,
Zedelenebilirlik,
Komşu rupture derecesi,
Middle graflar

Özet: Ağların günlük hayatımızda önemli bir yeri vardır. İnternet ağları, elektrik ağları, su şebekeleri, ulaşım ağları, sosyal ağlar, biyolojik ağlar gibi hayatımızın her alanında karşımıza çıkmaktadırlar. Bir ağ, bağlantılar ile birbirine bağlı olan merkezlerden oluşur. Merkezler tepeler ve bağlantılar da ayrıtlar ile modellendiğinde bir graf bir ağı temsil etmektedir. Bir ağın bazı merkezlerinin veya bağlantı hatlarının bozulması sonucunda, ağdaki iletişim kesilene kadar geçen süredeki ağın dayanma gücünün ölçümüne o ağın zedelenebilirlik değeri denir. Bir ağın zedelenebilirliğinin belirlenmesinde, tanımlanmış çeşitli zedelenebilirlik parametreleri kullanılmaktadır. Bu parametrelerden bazıları; bağlantılılık sayısı (connectivity), dayanıklılık sayısı (toughness), bütünlük değeri (integrity), kararlılık değeri (tenacity), saçılma sayısı (scattering number) ve rupture derecesidir.
Bu çalışmada bir iletişim ağının bazı merkezlerinin bozulmasıyla kendisine komşu merkezlerin de işlevsiz hale gelmesi sonucu oluşan grafların zedelenebilirlik değerlerini inceleyen bir parametre olan komşu rupture derecesi bazı middle graflara uygulanmış ve elde edilen $M(C_n)$, $M(P_n)$, $M(K_{1,n})$, $M(W_n)$, $M(P_n \times K_2)$ ve $M(C_n \times K_2)$ graflarının komşu rupture dereceleri elde edilmiştir.

1. Introduction

Today, with the development of technology, expectations about speed and reliability in transportation and communication networks have increased even more. Communication networks consists of connection lines connecting centers. These networks can be modeled with the help of graphs. In consequence of the failure of some centers of connection lines, measurement of the resistance of the network until the communication interrupted is called vulnerability of a network. Several parameters have been introduced in the calculation of the vulnerability in graphs. In connectivity [1], toughness [2], integrity [3], tenacity [4], scattering number [5] and rupture degree [6]. But they have just taken into account the vertices dismissed from the graph. When the centers of a network are disrupted, the network in which the neighbors of these centers affected is called a spy network. In a spy network, the neighbors of the disrupted vertices are unreliable. For this reason, the neighborhoods should be taken into consideration in spy networks [7]. Neighbor connectivity [8], neighbor integrity [9, 10], neighbor scattering number [11], neighbor toghness [12], neighbor isolated tenacity [13], and neighbor rupture degree [7] are some of the vulnerability parameters related to the spy networks.
In this paper, neighbor rupture degree which is a parameter that explores the vulnerability values of the resulting graphs due to the failure of some centers of a communication network and its neighboring centers becoming nonfunctional were studied. The neighbor rupture degree

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of a noncomplete connected graph G is defined to be

$$Nr(G) = \max_{S \subset V(G)} \{\omega(G/S) - |S| - c(G/S); \omega(G/S) \geq 1\}$$

where S is any vertex subversion strategy of a graph G, that is the elements of S and all its neighbor have been deleted from G, $\omega(G/S)$ is the number of connected components in G/S and $c(G/S)$ is the maximum order of the components of G/S [7].

2. Neighbor Rupture Degree of Middle Graphs

The middle graph $M(G)$ of a graph obtained from G by inserting a new vertex into every edge of G and joining these new vertices by edges which lie on adjacent edges of G [14]. There are many studies about the vulnerability of middle graphs such as integrity [15], rupture degree [16], bondage number [17] and edge tenacity [18].
In this study, the neighbor rupture degree and the vulnerability of middle graphs are considered. The neighbor rupture degree of middle graphs of cycle, path, star and wheel graphs are studied and general results are derived. Also cartesian product of the complete graph K_2 by path graph P_n and cycle graph C_n are obtained.

Theorem 2.1. *Let C_n be a cycle graph of order n and let $M(C_n)$ be the middle graph of C_n. Then the neighbor rupture degree of $M(C_n)$ is*

$$Nr(M(C_n)) = -1.$$

Proof. Let S be a subversion strategy of $M(C_n)$ and $|S| = r$ be the number of removing vertices of $M(C_n)$.
Case 1: Let $n \not\equiv 0 \ (mod 6)$
Subcase 1: If $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$, then we have $\omega(M(C_n)/S) \leq r$ and $c(M(C_n)/S) \geq \lfloor \frac{2n-3r}{r} \rfloor$.
Hence,

$$\omega(M(C_n)/S) - |S| - c(M(C_n)/S) \leq r - r - \frac{2n-3r}{r}$$

and we get

$$Nr(M(C_n)) \leq \max_r \{3 - \frac{2n}{r}\}.$$

The function $f(r) = 3 - \frac{2n}{r}$ is an increasing function and takes its maximum value at $\lfloor \frac{n}{2} \rfloor$.
Thus, we obtain

$$Nr(M(C_n)) \leq f(\lfloor \frac{n}{2} \rfloor) \leq -1. \tag{1}$$

Subcase 2: If $r > \lfloor \frac{n}{2} \rfloor$, then we have $\omega(M(C_n)/S) \leq r - 1$, $c(M(C_n)/S) \geq 1$. Hence,

$$\omega(M(C_n)/S) - |S| - c(M(C_n)/S) \leq r - 1 - r - 1 \leq -2 \tag{2}$$

According to the definition of neighbor rupture degree we take maximum of (1) and (2). Thus we get,

$$Nr(M(C_n)) \leq -1 \tag{3}$$

On the other hand, there exist S^* such that $|S^*| = \lfloor \frac{n}{2} \rfloor$, $\omega(M(C_n)/S) = \lfloor \frac{n}{2} \rfloor$ and $c(M(C_n)/S) = 1$, then

$$\omega(M(C_n)/S) - |S| - c(M(C_n)/S) \geq -1$$

and we get

$$Nr(M(C_n)) \geq -1 \tag{4}$$

By (3) and (4) we obtain the result

$$Nr(M(C_n)) = -1. \tag{5}$$

Case 2: Let $n \equiv 0 \ (mod 6)$
Subcase 1: If $0 \leq r \leq \frac{n}{3}$, then we get $\omega(M(C_n)/S) \leq r$ and $c(M(C_n)/S) \geq \frac{2n-5r}{r}$. Thus,

$$\omega(M(C_n)/S) - |S| - c(M(C_n)/S) \leq r - r - \frac{2n-5r}{r}$$

and

$$Nr(M(C_n)) \leq \max_r \{5 - \frac{2n}{r}\}.$$

The function $f(r) = 5 - \frac{2n}{r}$ is an increasing function and takes its maximum value ar $r = \frac{n}{3}$. Hence, we have

$$f(\frac{n}{3}) = 5 - \frac{2n}{\frac{n}{3}} = -1.$$

As a result we get,

$$Nr(C_n) \leq -1. \tag{6}$$

Subcase 2: If $r > \frac{n}{3}$ then, we obtain $\omega(M(C_n)/S) \leq r - 1$ and $c(M(C_n)/S) \geq 1$. Hence,

$$\omega(M(C_n)/S) - |S| - c(M(C_n)/S) \leq r - 1 - r - 1 = -2. \tag{7}$$

According to the definition neighbor rupture degree we take maximum of (6) and (7). Thus we get

$$Nr(M(C_n)) \leq -1. \tag{8}$$

There exist S^* such that $|S^*| = \frac{n}{3}$, $\omega(M(C_n)/S^*) = \frac{n}{3}$ and $c(M(C_n)/S^*) = 1$ then,

$$Nr(M(C_n)) \geq \frac{n}{3} - \frac{n}{3} - 1 = -1. \tag{9}$$

By (8) and (9) we get the result,

$$Nr(M(C_n)) = -1 \tag{10}$$

From case (1) and case (2) the proof is completed. □

Theorem 2.2. *Let P_n be a path of order n and let $M(P_n)$ be the middle graph of P_n. Then the neighbor rupture degree of $M(P_n)$ is*

$$Nr(M(P_n)) = 0.$$

Proof. Let S be a subversion strategy of $M(P_n)$ and let the vertices of $M(P_n)$ be labeled as $\{1, 2, 3, ..., 2n-1\}$. If $|S| = r$ vertices are subverted from $M(P_n)$, then the survival subgraph $M(P_n/S)$ contains at most $r+1$ components having at least one vertex. Since $\omega(M(P_n)/S) \leq r+1$ and $c(M(P_n)/S) \geq 1$, we get $\omega(M(P_n)/S) - |S| - c(M(P_n)/S) \leq r + 1 - r - 1 = 0$. Thus,

$$Nr(M(P_n)) \leq 0. \tag{11}$$

On the other hand, there exists a subversion strategy S^* such that $|S^*| = \lfloor \frac{n-1}{2} \rfloor$ and by the definition we have

$Nr(M(P_n)) \geq \omega(M(P_n)/S^*) - |S^*| - c(M(P_n)/S^*)$.

If n is odd, then let $S^* = \{3,7,11,...,2n-3\}$ and if n is even let $S^* = \{3,7,11,...,2n-9,2n-4\}$. After the subversion of S^*, the survival subgraph $M(P_n)/S^*$ contains only isolated vertices as components, i.e. $M(P_n)/S^* = \{1,5,9,...,2n-1\}$ where n is odd and $M(P_n)/S^* = \{1,5,9,...,2n-7,2n-1\}$ where n is even.

Therefore $|S^*| = \lfloor\frac{n-1}{2}\rfloor$, $\omega(M(P_n)/S^*) = \lfloor\frac{n-1}{2}\rfloor + 1$ and $c(M(P_n)/S^*) = 1$. Thus we get, $\omega(M(P_n)/S^*) - |S^*| - c(M(P_n)/S^*)$

$$= \lfloor\frac{n-1}{2}\rfloor + 1 - \lfloor\frac{n-1}{2}\rfloor - 1 = 0$$

and

$$Nr(M(P_n)) \geq 0. \tag{12}$$

By (11) and (12), we get the result. □

Theorem 2.3. *Let $K_{1,n}$ be a star graph of order $n+1$ and let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then the neighbor rupture degree of $M(K_{1,n})$ is*

$$Nr(M(K_{1,n})) = n-2.$$

Proof. Let S be a subversion strategy of $M(K_{1,n})$ and let $|S| = r$.

Case 1: If $r = 1$, then the number of components after the subversion of any vertex is at most n having at least one vertex. Since $\omega(M(K_{1,n})/S) \leq n$ and $c(M(K_{1,n})/S) \geq 1$ we get,

$$\omega(M(K_{1,n})/S) - |S| - c(M(K_{1,n})/S) \leq n-1-1 = n-2.$$

Thus,

$$Nr(M(K_{1,n})) \leq n-2. \tag{13}$$

Case 2: If $r \geq 2$, then the number of components that the survival subgraph $M(K_{1,n})/S)$ can have is at most $n-1$ having at least one vertex.
Hence $\omega(M(K_{1,n})/S)) \leq n-1$, $c(M(K_{1,n})/S)) \geq 1$ and we have,

$$\begin{aligned} \omega(M(K_{1,n})/S)) - |S| - c(M(K_{1,n})/S)) &\leq n-1-2-1 \\ &= n-4. \end{aligned}$$

Therefore,

$$Nr(M(K_{1,n})) \leq n-4. \tag{14}$$

By taking the maximum of (13) and (14) we get,

$$Nr(M(K_{1,n})) \leq n-2. \tag{15}$$

On the other hand, there exist a subversion strategy S^* such that $S^* = \{v|deg(v) = n \; and \; v \in V(M(K_{1,n}))\}$. Thus the survival subgraph is $M(K_{1,n})/S^* = \{u_1,u_2,...,u_n\}$, the only vertices of $K_{1,n}$ having degree 1.
Since these vertices are independent in $M(K_{1,n})$, we have $\omega(M(K_{1,n})/S^*) = n$ and $c(M(K_{1,n})/S^*) = 1$. By the definition of neighbor rupture degree

$$w(M(K_{1,n})/S^*) - |S^*| - c(M(K_{1,n})/S^*) \geq n-1-1 = n-2.$$

Thus,

$$Nr(M(K_{1,n})) \geq n-2. \tag{16}$$

By (15) and (16)we get the result. □

Theorem 2.4. *Let W_n be a wheel graph of order n and let $M(W_n)$ be the middle graph of W_n. Then the neighbor rupture degree of $M(W_n)$ with $n > 6$ is*

$$Nr(M(W_n)) = -1.$$

Proof. Let S be a subversion strategy of $M(W_n)$ and $|S| = r$ be the number of removing vertices of $M(W_n)$.

Case 1: If $0 \leq r \leq \lfloor\frac{n-1}{2}\rfloor$, then we get $\omega(M(W_n)/S) \leq r$ and $c(M(W_n)/S) \geq \lfloor\frac{2n-2-3r}{r}\rfloor$.
Hence,

$$\omega(M(W_n)/S) - |S| - c(M(W_n)/S) \leq r - r - \lfloor\frac{2n-2-3r}{r}\rfloor$$

and

$$Nr(M(W_n)) \leq \max_r\{3 - \frac{2n-2}{r}\}$$

The function $f(r) = 3 - \frac{2n-2}{r}$ is an increasing function and takes its maximum value at $\lfloor\frac{n-1}{2}\rfloor$. Thus, we get

$$f(\lfloor\frac{n-1}{2}\rfloor) \leq -1$$

and

$$Nr(M(W_n))) \leq -1. \tag{17}$$

Case 2: If $r > \lfloor\frac{n-1}{2}\rfloor$, then we obtain $\omega(M(W_n)/S) \leq r-1$ and $c(M(W_n)/S) \geq 1$. Hence,

$$\omega(M(W_n)/S) - |S| - c(M(W_n)/S) \leq r-1-r-1 \leq -2. \tag{18}$$

According to the definition of neighbor rupture degree we take the maximum of (17) and (18). Thus we get,

$$Nr(M(W_n)) \leq -1. \tag{19}$$

Moreover, it can be easily seen that there is a subversion strategy S^* of $M(W_n)$ such that $|S^*| = \lfloor\frac{n-1}{2}\rfloor$, $\omega(M(W_n)/S^*) = \lfloor\frac{n-1}{2}\rfloor$ and $c(M(W_n)/S^*) = 1$. Thus,

$$\begin{aligned} \omega(M(W_n)/S^*) - |S^*| - c(M(W_n)/S^*) &\geq \lfloor\frac{n-1}{2}\rfloor - \lfloor\frac{n-1}{2}\rfloor - 1 \\ &= -1. \end{aligned} \tag{20}$$

By (19) and (20) we get the result

$$Nr(M(W_n) = -1.$$

From case 1 and case 2, the proof is completed. □

Theorem 2.5. *Let $P_n \times K_2$ be the cartesian product of a path graph P_n and K_2. $M(P_n \times K_2)$ be the middle graph of this graph with $n > 2$. Then,*

$$Nr(M(P_n \times K_2)) = \begin{cases} -2, & n \equiv 2 \; (mod3); \\ -1, & otherwise. \end{cases}$$

Proof. Let S be a subversion strategy of $M(P_n \times K_2)$ and $|S| = r$ be the number of removing vertices of $M(P_n \times K_2)$.

Case 1: Let $n \equiv 2 \; (mod3)$.

Subcase 1: If $0 \leq r \leq \lfloor \frac{2n}{3} \rfloor + 1$, then we get $\omega(M(P_n \times K_2)/S) \leq r-1$ and $c(M(P_n \times K_2)/S) \geq \lfloor \frac{5n-6r-2}{r-1} \rfloor$. Thus,

$$\omega(M(P_n \times K_2)/S) - |S| - c(M(P_n \times K_2)/S)$$
$$= r-1-r-\frac{5n-6r-2}{r-1} \leq 5-\frac{5n-8}{r-1}$$
$$Nr(M(P_n \times K_2)) \leq \max_r\{5-\frac{5n-8}{r-1}\}.$$

The function $f(r) = 5 - \frac{5n-8}{r-1}$ is an increasing function and takes its maximum value at $\lfloor \frac{2n}{3} \rfloor + 1$. Hence, we have,

$$f(\lfloor \frac{2n}{3} \rfloor + 1) \leq -2.$$
$$Nr(M(P_n \times K_2)) \leq -2. \quad (21)$$

Subcase 2: If $r > \lfloor \frac{2n}{3} \rfloor + 1$, then we have $\omega(M(P_n \times K_2)/S) \leq r-2$ and $c(M(P_n \times K_2)/S) \geq 1$. Hence,

$$\omega(M(P_n \times K_2)/S) - |S| - c(M(P_n \times K_2)/S) \leq r-2-r-1 \leq -3 \quad (22)$$

According to the definition of neighbor rupture degree we take the maximum of (21) and (22). Thus we get

$$Nr(M(P_n \times K_2)) \leq -2 \quad (23)$$

In addition to this, there exist S^* such that $|S^*| = \lfloor \frac{2n}{3} \rfloor + 1$, $\omega(M(P_n \times K_2)/S^*) = \lfloor \frac{2n}{3} \rfloor$ and $c(M(P_n \times K_2)/S^*) = 1$ then

$$\omega(M(P_n \times K_2)/S^*) - |S^*| - c(M(P_n \times K_2)/S^*) \geq -2$$

and
$$Nr(M(P_n \times K_2)) \geq -2. \quad (24)$$

From (23) and (24) we obtain the result

$$Nr(M(P_n \times K_2)) = -2. \quad (25)$$

Case 2: Let $n \not\equiv 2\ (mod 3)$.

Subcase 1: If $0 \leq r \leq \lceil \frac{2n}{3} \rceil$, then we get $\omega(M(P_n \times K_2)) \leq r$ and $c(M(P_n \times K_2)/S) \geq \lfloor \frac{5n-2-6r}{r} \rfloor$. Hence,

$$\omega(M(P_n \times K_2)/S) - |S| - c(M(P_n \times K_2)/S)$$
$$\leq r-r-\frac{5n-6r-2}{r} = 6-\frac{5n-2}{r}$$

and
$$Nr(M(P_n \times K_2)) \leq \max_r\{6-\frac{5n-2}{r}\}.$$

The function $f(r) = 6 - \frac{5n-2}{r}$ is an increasing function and takes maximum value at $r = \lceil \frac{2n}{3} \rceil$. Thus, we get

$$f(\lceil \frac{2n}{3} \rceil) = 6 - \lfloor \frac{5n-2}{\frac{2n}{3}} \rfloor \leq -1.$$

and
$$Nr(M(P_n \times K_2)) \leq -1. \quad (26)$$

Subcase 2: If $r > \lceil \frac{2n}{3} \rceil$ then, we obtain $\omega(M(P_n \times K_2)/S) \leq r-1$ and $c(M(P_n \times K_2)/S) \geq 1$. Hence,

$$\omega(M(P_n \times K_2)/S) - |S| - c(M(P_n \times K_2)/S) \leq r-1-r-1 \leq -2. \quad (27)$$

According to the definition neighbor rupture degree we take maximum of (26) and (27). Thus we get,

$$Nr(M(P_n \times K_2)) \leq -1. \quad (28)$$

It is obvious that there is a subversion strategy S^* of $M(P_n \times K_2)$ such that $|S^*| = \lceil \frac{2n}{3} \rceil$. Then $\omega(M(P_n \times K_2)/S^*) = \lceil \frac{2n}{3} \rceil$ and $c(M(P_n \times K_2)/S^*) = 1$. Thus,

$$\omega(M(P_n \times K_2)/S^*) - |S^*| - c(M(P_n \times K_2)/S^*) \geq \lceil \frac{2n}{3} \rceil - \lceil \frac{2n}{3} \rceil - 1 \geq -1 \quad (29)$$

According to (28) and (29), we get the result

$$Nr(M(P_n \times K_2) = -1.$$

The proof is completed. □

Theorem 2.6. *Let $C_n \times K_2$ be the cartesian product of a cycle C_n and K_2. $M(C_n \times K_2)$ be the middle of this graph with $n > 3$. Then,*

$$Mr(M(C_n \times K_2)) = \begin{cases} -1, & \text{if } n \text{ is even;} \\ -2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let S be a subversion strategy of $M(C_n \times K_2)$ and $|S| = r$ be the number of removing vertices of $M(C_n \times K_2)$.

Case 1: If n is even:

Subcase 1: If $0 \leq r \leq n$, then we have $\omega(M(C_n \times K_2)/S) \leq r$ and $c(M(C_n \times K_2)/S) \geq \frac{5n-4r}{r}$. Therefore,

$$\omega(M(C_n \times K_2)/S) - |S| - c(M(C_n \times K_2)/S) \leq r-r-\frac{5n-4r}{r}$$
$$= 4-\frac{5n}{r}.$$
$$Nr(M(C_n \times K_2)) \leq \max_r\{4-\frac{5n}{r}\}.$$

The function $f(r) = 4 - \frac{5n}{r}$ is an increasing function and takes its maximum value at $r = n$. Thus, we have

$$f(n) = 4 - \frac{5n}{n} = -1.$$

and
$$Nr(M(C_n \times K_2)) \leq -1. \quad (30)$$

Subcase 2: If $r > n$, then we obtain $\omega(M(C_n \times K_2)/S) \leq r-1$, $c(M(C_n \times K_2)/S) \geq 1$. Hence,

$$\omega(M(C_n \times K_2)/S) - |S| - c(M(C_n \times K_2)/S) \leq r-1-r-1 = -2$$

and

$$Nr(M(C_n \times K_2)) \leq -2. \quad (31)$$

In compliance with the definition of neighbor rupture degree, we get maximum of (30) and (31). Thus, we obtain,

$$Nr(M(C_n \times K_2)) \leq -1. \quad (32)$$

In other respects, there exist S^* such that $|S^*| = n$, $\omega(M(C_n \times K_2)/S^*) = n$ and $c(M(C_n \times K_2)/S^*) = 1$ then

$$\omega(M(C_n \times K_2)/S^*) - |S^*| - c(M(C_n \times K_2)/S^*) \geq n-n-1 = -1. \quad (33)$$

According to (32) and (33) we obtain the result.

$$Nr(M(C_n \times K_2)) = -1.$$

Case 2: If n is odd:

Subcase 1: If $0 \leq r \leq n$, then we get $\omega(M(C_n \times K_2)/S) \leq r-1$ and $c(M(C_n \times K_2)/S) \geq \frac{5n-4r-1}{r-1}$. Thus,

$$\omega(M(C_n \times K_2)/S) - |S| - c(M(C_n \times K_2)/S)$$
$$\leq r-1-r-\frac{5n-4r-1}{r-1} = 3-\frac{5(n-1)}{r-1}.$$
$$Nr(M(C_n \times K_2)) \leq \max_r\{3-\frac{5(n-1)}{r-1}\}.$$

The function $f(r) = 3-\frac{5(n-1)}{r-1}$ is an increasing function and gets its maximum value at $r = n$. Therefore, we have

$$f(n) = 3-\frac{5(n-1)}{n-1} = -2$$

and

$$Nr(M(C_n \times K_2)) \leq -2 \quad (34)$$

Subcase 2: If $r > n$, then we get $\omega(M(C_n \times K_2)/S) \leq r-2$ and $c(M(C_n \times K_2)/S) \geq 1$. So,

$$\omega(M(C_n \times K_2)/S) - |S| - c(M(C_n \times K_2)/S) \leq r-2-r-1 = -3.$$

and

$$Nr(M(C_n \times K_2)) \leq -3. \quad (35)$$

We take maximum of (34) and (35) from the definition of neighbor rupture degree and we get

$$Nr(M(C_n \times K_2)) \leq -2. \quad (36)$$

Besides, it can be easily seen that there is a subversion strategy S^* of $M(C_n \times K_2)$ such that $|S^*| = n$, $\omega(M(C_n \times K_2)/S^*) = n-1$ and $c(M(C_n \times K_2)/S^*) = 1$ then,

$$\omega(M(C_n \times K_2)/S^*) - |S^*| - c(M(C_n \times K_2)/S^*) \geq n-1-n-1 = -2$$

and

$$Nr(M(C_n \times K_2)) \geq -2. \quad (37)$$

According to (36) and (37) we obtain the result.

$$Nr(M(C_n \times K_2)) = -2.$$

□

3. Conclusion

In this paper, a vulnerability parameter for spy networks, neighbor rupture degree is studied. This parameter is applied to the middle graphs of cycle, path, star and wheel graphs and some general results are obtained. Also the neighbor rupture degree of the cartesian product graphs $P_n \times K_2$ and $C_n \times K_2$ are derived.

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