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Some Convergence Theorems of Modified Proximal Point Algorithms for Nonexpansive Mappings in CAT(0) Spaces

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Abstract

In this paper, a new modified proximal point algorithm is proposed for finding a common element of the set of fixed points of a single-valued nonexpansive mapping, and the set of fixed points of a multivalued nonexpansive mapping, and the set of minimizers of convex and lower semicontinuous functions. We obtain convergence of the proposed algorithm to a common element of three sets in CAT(0) spaces.

Keywords: CAT(0) space, proximal point algorithm, fixed point, resolvent identity, functional analysis.
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1. Introduction

A metric space (X, d) is said to be a geodesic space, if it is connected geodesically. A geodesic path joining x to y in X is a mapping g from a closed interval $[0, l] \subseteq R$ to X such that $g(0) = x$, $g(l) = y$ and $d(g(s), g(t)) =\mid s - t \mid$ for all $s, t \in [0, l]$. In particular, the mapping g is an isometry and $d(x, y) = l$. The image of g is called as a geodesic segment joining x and y, which is uniquely denoted by $[x, y]$. We denote the unique point $z \in [x, y]$ such that

$$d(x, z) = kd(x, y) \text{ and } d(y, z) = (1 - k)d(x, y),$$

by $(1 - k)x \oplus ky$, where $0 \leq k \leq 1$.

A geodesic space is called as a $CAT(0)$ space, if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane R^2. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d)

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consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane R^2 such that

$$d_{R^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$$

for all $i, j \in \{1, 2, 3\}$. Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle in R^2. Then the triangle $\triangle$ is said to satisfy the $CAT(0)$ inequality if

$$d(x, y) \leq d_{R^2}(\bar{x}, \bar{y}),$$

for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$.

The useful inequality of a $CAT(0)$ space is the (CN) inequality[1], that is, if z, x, y are points in a $CAT(0)$ space and if $\frac{x \oplus y}{2}$ is the midpoint of a geodesic segment $[x, y]$, then the $CAT(0)$ inequality implies

$$d^2(z, \frac{x \oplus y}{2}) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y), \tag{CN}$$

which equals to the following inequality[2]

$$d^2(z, \lambda x \oplus (1-\lambda)y) \leq \lambda d^2(z, x) + (1-\lambda)d^2(z, y) - \lambda(1-\lambda)d^2(x, y), \tag{CN^*}$$

for any $\lambda \in [0, 1]$, where $\lambda x \oplus (1-\lambda)y$ denotes a unique point in $[x, y]$. Moreover, if X is a $CAT(0)$ space and $x, y \in X$, then for any $\lambda \in [0, 1]$, there exists a unique point $\lambda x \oplus (1-\lambda)y \in [x, y]$ such that

$$d(z, \lambda x \oplus (1-\lambda)y) \leq \lambda d(z, x) + (1-\lambda)d(z, y), \ for\ any\ z \in X. \tag{1.1}$$

In 2013, the proximal point algorithm was introduced by *Bačák* [3] into $CAT(0)$ spaces. For any x_1 in a $CAT(0)$ space X, a sequence $\{x_n\}$ generated by

$$x_{n+1} = argmin_{y \in X}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \tag{1.2}$$

where $\lambda_n > 0$ for all $n \in N$. If f has a minimizer, then the sequence $\{x_n\}$ $\Delta-$converges to its minimizer.

For all $\lambda > 0$, in a complete $CAT(0)$ space X, the $Moreau - Yosida$ resolvent of f [4] is defined as follows:

$$J_\lambda(x) = argmin_{y \in X}[f(y) + \frac{1}{2\lambda}d^2(y, x)],$$

where $f : X \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function.

The set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $argmin_{y \in X} f(y)$ of minimizers of f , which is found in reference [5]. For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive [6].

The following algorithm is proposed by *Suthep Suantai et.al*[7] in 2017 as follows:

$$\begin{cases} z_n = argmin_{y \in C}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ y_n = \beta_n z_n \oplus (1-\beta_n)w_n, \quad w_n \in Sz_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \quad \forall n \in N, \end{cases} \tag{1.3}$$

where T is a single-valued nonexpansive mapping, S is a multi-valued nonexpansive mapping, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Inspired by the above work, in this paper, we come up with a new modified algorithm, which improved and extended the results[7].

2. Preliminaries

We collect some definitions, lemmas, which will be used in next section.

Definition2.1[8] Let D be a nonempty closed subset of a $CAT(0)$ space X and let $CB(D)$, $CC(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets, closed convex subsets and compact convex subsets of D, respectively. The $Pompeiu - Hausdorff$ distance on $CB(D)$ is defined by

$$H(A,B) = max\{sup_{x\in A} dist(x,B), sup_{y\in B} dist(y,A)\}$$

for $A, B \in CB(D)$, where $dist(x,D) = inf\{d(x,y) : y \in D\}$ is the distance from a point x to a subset D.

Definition2.2[7] A single-valued mapping $T : D \to D$ is said to be *semicompact* if for any sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in D$. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in D : x = Tx\}$.

Definition2.3[7] A multi-valued mapping $S : D \to CB(D)$ is said to be

(1) *nonexpansive* if $H(Sx, Sy) \le d(x, y)$ for all $x, y \in D$;

(2) $hemi - compact$ if for any sequence $\{x_n\}$ in D such that

$$\lim_{n\to\infty} dist(x_n, Sx_n) = 0,$$

there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in D$.

An element $x \in D$ is called a fixed point of S if $x \in Sx$. The set of all fixed points of S is denoted by $F(S)$, that is, $F(S) = \{x \in D : x \in Sx\}$.

Definition2.4[7] Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by $r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n)$. The *asymptotic radius* of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the *asymptotic center* of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. In a complete $CAT(0)$ space, the asymptotic center $A(\{x_n\})$ consists of exactly one point[9].

Definition2.5[7] A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to $\Delta - converge$ to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta - lim_{n\to\infty} x_n = x$ and call x as $\Delta - limit$ of $\{x_n\}$.

It is easy to see that $CAT(0)$ spaces satisfy $Opial$ condition, which is known in Banach spaces theory as $Opial$ property, that is, given $\{x_n\} \subseteq X$ such that the sequence $\{x_n\}$ $\Delta-$converges to $x \in X$ and given $y \in X$ with $x \neq y$, then the following inequality holds

$$\lim_{n\to\infty} \inf d(x_n, x) < \lim_{n\to\infty} \inf d(x_n, y).$$

Lemma2.6[10] Every bounded sequence in a $CAT(0)$ space has a Δ-convergent subsequence.

Lemma2.7[11] Let D be a nonempty closed convex subset of a $CAT(0)$ space X. If $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.

Lemma2.8[2] If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

Lemma2.9[2] Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : D \to D$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in D such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x$, then $x = Tx$.

Lemma2.10[6] Let (X, d) be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then the following identity holds:

$$J_\lambda x = J_\mu\left(\frac{\lambda-\mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x\right), \forall x \in X, \lambda > \mu > 0,$$

where J_λ is the $Moreau - Yosida$ resolvent of f.

Lemma2.11[12] Let (X,d) be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in X$ and some $\lambda > 0$, then the following inequality holds:

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x,y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \le f(y),$$

where J_λ is the $Moreau-Yosida$ resolvent of f.

3. Main results

Next, we give the results of proposed algorithms in this section.

Theorem3.1 Suppose that the following conditions are satisfied:

(1) Let X be a complete $CAT(0)$ space and D be a nonempty closed convex subset of X;

(2) Let $T : D \to D$ be a single-valued nonexpansive mapping, $S : D \to CB(D)$ be a multi-valued nonexpansive mapping, and $f : D \to (-\infty, \infty]$ be a convex and lower semi-continuous proper function;

(3) $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ are sequences in $(0,1)$ with $0 < a \le \alpha_n, \beta_n, \gamma_n \le b < 1$ for all $n \in N$ and for some a, b are positive constants in $[0,1]$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in N$ and some λ;

(4) Suppose that $\Omega = F(T) \cap F(S) \cap argmin_{y\in D} f(y)$ is nonempty and $Sq = \{q\}$ for all $q \in \Omega$;

(5) Suppose that J_λ is semi-compact or T is semi-compact or S is hemi-compact.

For any $x_1 \in D$, the sequence $\{x_n\}$ generated in the following manner:

$$\begin{cases} z_n = argmin_{y\in D}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ t_n = \gamma_n z_n \oplus (1-\gamma_n)w_n, \ w_n \in Sz_n, \\ y_n = \beta_n z_n \oplus (1-\beta_n)Tt_n, \\ x_{n+1} = \alpha_n t_n \oplus (1-\alpha_n)y_n, \ \forall n \in N, \end{cases} \tag{3.1}$$

then the sequence $\{x_n\}$ converges strongly to a point in Ω.

Proof. This proof will be divided into a few steps as follows.

(i) Let $q \in \Omega$. Then we have $Tq = q \in Sq$ and $f(q) \le f(y)$, for all $y \in D$. It follows that

$$f(q) + \frac{1}{2\lambda_n}d^2(q,q) \le f(y) + \frac{1}{2\lambda_n}d^2(y,q), \quad \forall y \in D.$$

Hence, $q = J_{\lambda_n} q$ for all $n \in N$. Since $z_n = J_{\lambda_n} x_n$, it follows by the nonexpansiveness of J_{λ_n} that

$$d(z_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \le d(x_n, q). \tag{3.2}$$

For $q \in \Omega$, by virtue of $Sq = \{q\}$, by (1.1) and (3.1)-(3.2), it shows that

$$\begin{aligned} d(t_n, q) &= d(\gamma_n z_n \oplus (1-\gamma_n)w_n, q) \\ &\le \gamma_n d(z_n, q) + (1-\gamma_n)d(w_n, q) \\ &\le \gamma_n d(z_n, q) + (1-\gamma_n)dist(Sz_n, q) \\ &\le \gamma_n d(z_n, q) + (1-\gamma_n)H(Sz_n, Sq) \\ &\le \gamma_n d(z_n, q) + (1-\gamma_n)d(z_n, q) \\ &= d(z_n, q) \\ &\le d(x_n, q). \end{aligned} \tag{3.3}$$

By (3.3), we have

$$\begin{aligned} d(y_n, q) &= d(\beta_n z_n \oplus (1-\beta_n)Tt_n, q) \\ &\le \beta_n d(z_n, q) + (1-\beta_n)d(Tt_n, q) \\ &\le \beta_n d(z_n, q) + (1-\beta_n)d(t_n, q) \\ &\le \beta_n d(z_n, q) + (1-\beta_n)d(z_n, q) \\ &= d(z_n, q) \\ &\le d(x_n, q) \end{aligned} \tag{3.4}$$

and we get

$$\begin{aligned} d(x_{n+1}, q) &= d(\alpha_n t_n \oplus (1-\alpha_n) y_n, q) \\ &\leq \alpha_n d(t_n, q) + (1-\alpha_n) d(y_n, q) \\ &\leq \alpha_n d(z_n, q) + (1-\alpha_n) d(y_n, q) \\ &\leq d(z_n, q) \\ &\leq d(x_n, q). \end{aligned} \tag{3.5}$$

Therefore, by (3.5), we obtain that the sequence $\{d(x_n, q)\}$ is decreasing and bounded. So, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \Omega$.

(ii) Let

$$\lim_{n\to\infty} d(x_n, q) = l \geq 0. \tag{3.6}$$

By lemma 2.11, we have

$$\frac{1}{2\lambda_n} d^2(z_n, q) - \frac{1}{2\lambda_n} d^2(x_n, q) + \frac{1}{2\lambda_n} d^2(z_n, x_n) \leq f(q) - f(z_n).$$

Since $f(q) \leq f(z_n)$ for all $n \in N$, we get

$$d^2(z_n, x_n) \leq d^2(x_n, q) - d^2(z_n, q). \tag{3.7}$$

From (3.5), we get

$$d(x_{n+1}, q) \leq d(z_n, q) \leq d(x_n, q).$$

So, we have

$$\lim_{n\to\infty} d(x_{n+1}, q) \leq \lim_{n\to\infty} d(z_n, q) \leq \lim_{n\to\infty} d(x_n, q).$$

This implies that

$$\lim_{n\to\infty} d(z_n, q) = l. \tag{3.8}$$

By virtue of $(3.6) - (3.8)$, it shows that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.9}$$

Because of $0 < a \leq \alpha_n \leq b < 1$, also by (3.5) we get

$$d(x_{n+1}, q) \leq \alpha_n d(x_n, q) + (1-\alpha_n) d(y_n, q)$$

and change it as

$$\begin{aligned} d(y_n, q) &\geq \frac{1}{1-\alpha_n}[d(x_{n+1}, q) - \alpha_n d(x_n, q)] \\ &\geq \frac{1}{1-b}[d(x_{n+1}, q) - b d(x_n, q)], \end{aligned} \tag{3.10}$$

thus, we have

$$\lim_{n\to\infty} \inf d(y_n, q) \geq \lim_{n\to\infty} \inf\{\frac{1}{1-b}[d(x_{n+1}, q) - b d(x_n, q)]\} = l$$

and by (3.4), we obtain

$$\lim_{n\to\infty} \sup d(y_n, q) \leq \lim_{n\to\infty} \sup d(x_n, q) = l.$$

Then, we have

$$\lim_{n\to\infty} d(y_n, q) = l. \tag{A^*}$$

Similarity, by (3.5), we also get

$$d(x_{n+1}, q) \leq \alpha_n d(t_n, q) + (1-\alpha_n) d(y_n, q)$$

and also change it as

$$\begin{aligned} d(t_n, q) &\geq \frac{1}{\alpha_n}[d(x_{n+1}, q) - (1-\alpha_n)d(y_n, q)] \\ &\geq \frac{1}{a}[d(x_{n+1}, q) - (1-a)d(y_n, q)] \\ &\geq \frac{1}{a}[d(x_{n+1}, q) - (1-a)d(x_n, q)]. \end{aligned}$$

So, we have

$$\lim_{n\to\infty} \inf d(t_n, q) \geq \lim_{n\to\infty} \inf\{\frac{1}{a}[d(x_{n+1}, q) - (1-a)d(x_n, q)]\} = l.$$

and by (3.3), this show

$$\lim_{n\to\infty} \sup d(t_n, q) \leq \lim_{n\to\infty} \sup d(x_n, q) = l$$

Then, we obtain

$$\lim_{n\to\infty} d(t_n, q) = l. \tag{B^*}$$

Also from the inequality (CN^*), $Sq = \{q\}$ and (3.1) – (3.3), we have

$$\begin{aligned} d^2(t_n, q) &= d^2(\gamma_n z_n \oplus (1-\gamma_n)w_n, q) \\ &\leq \gamma_n d^2(z_n, q) + (1-\gamma_n)d^2(w_n, q) - \gamma_n(1-\gamma_n)d^2(z_n, w_n) \\ &\leq \gamma_n d^2(z_n, q) + (1-\gamma_n)dist^2(q, Sz_n) - \gamma_n(1-\gamma_n)d^2(z_n, w_n) \\ &\leq \gamma_n d^2(z_n, q) + (1-\gamma_n)H^2(Sq, Sz_n) - \gamma_n(1-\gamma_n)d^2(z_n, w_n) \\ &\leq \gamma_n d^2(z_n, q) + (1-\gamma_n)d^2(z_n, q) - \gamma_n(1-\gamma_n)d^2(z_n, w_n) \\ &\leq d^2(x_n, q) - \gamma_n(1-\gamma_n)d^2(z_n, w_n). \end{aligned} \tag{3.11}$$

By (3.1) – (3.4), we get

$$\begin{aligned} d^2(y_n, q) &= d^2(\beta_n z_n \oplus (1-\beta_n)Tt_n, q) \\ &\leq \beta_n d^2(z_n, q) + (1-\beta_n)d^2(Tt_n, q) - \beta_n(1-\beta_n)d^2(z_n, Tt_n) \\ &\leq \beta_n d^2(x_n, q) + (1-\beta_n)d^2(t_n, q) - \beta_n(1-\beta_n)d^2(z_n, Tt_n) \\ &\leq d^2(x_n, q) - \beta_n(1-\beta_n)d^2(z_n, Tt_n). \end{aligned} \tag{3.12}$$

Similarly, by (3.1) – (3.5), we have

$$\begin{aligned} d^2(x_{n+1}, q) &= d^2(\alpha_n t_n \oplus (1-\alpha_n)y_n, q) \\ &\leq \alpha_n d^2(t_n, q) + (1-\alpha_n)d^2(y_n, q) - \alpha_n(1-\alpha_n)d^2(t_n, y_n) \\ &\leq \alpha_n d^2(x_n, q) + (1-\alpha_n)d^2(y_n, q) - \alpha_n(1-\alpha_n)d^2(t_n, y_n) \\ &\leq d^2(x_n, q) - \alpha_n(1-\alpha_n)d^2(t_n, y_n). \end{aligned} \tag{3.13}$$

Because of $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, and from (3.6), and $(A^*), (B^*)$, this shows that

$$0 \leq \gamma_n(1-\gamma_n)d^2(z_n, w_n) \leq d^2(x_n, q) - d^2(t_n, q) \to 0(n\to\infty),$$

$$0 \leq \beta_n(1-\beta_n)d^2(z_n, Tt_n) \leq d^2(x_n, q) - d^2(y_n, q) \to 0(n\to\infty),$$

$$0 \leq \alpha_n(1-\alpha_n)d^2(t_n, y_n) \leq d^2(x_n, q) - d^2(x_{n+1}, q) \to 0(n\to\infty).$$

Thus, we obtain that

$$\lim_{n\to\infty} d(z_n, w_n) = \lim_{n\to\infty} d(z_n, Tt_n) = \lim_{n\to\infty} d(t_n, y_n) = 0. \tag{3.14}$$

In fact, because $t_n = \gamma_n z_n \oplus (1-\gamma_n)w_n$, we get

$$\begin{aligned} d(t_n, x_n) &= d(\gamma_n z_n \oplus (1-\gamma_n)w_n, x_n) \\ &\le \gamma_n d(z_n, x_n) + (1-\gamma_n)d(w_n, x_n) \\ &\le \gamma_n d(z_n, x_n) + (1-\gamma_n)\{d(w_n, z_n) + d(z_n, x_n)\} \\ &\to 0(n \to \infty). \end{aligned} \tag{3.15}$$

By the nonexpansiveness of T, and this together (3.14) with (3.15) shows that

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, z_n) + d(z_n, Tt_n) + d(Tt_n, Tx_n) \\ &\le d(x_n, z_n) + d(z_n, Tt_n) + d(t_n, x_n) \\ &\to 0(n \to \infty). \end{aligned} \tag{3.16}$$

Immediately, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

(iii) Because of nonexpansiveness of S, also from (3.10) and (3.14), we get

$$\begin{aligned} dist(x_n, Sx_n) &\le d(x_n, z_n) + dist(z_n, Sz_n) + H(Sz_n, Sx_n) \\ &\le d(x_n, z_n) + dist(z_n, Sz_n) + d(z_n, x_n) \\ &\le 2d(x_n, z_n) + d(z_n, w_n) \\ &\to 0(n \to \infty). \end{aligned}$$

It is easy to see that

$$\lim_{n\infty} dist(x_n, Sx_n) = 0.$$

(iv) By $\lambda_n > \lambda > 0$, lemma 2.10 and nonexpansiveness of J_λ, and $z_n = J_{\lambda_n} x_n$, we have

$$\begin{aligned} d(x_n, J_\lambda x_n) &\le d(x_n, z_n) + d(z_n, J_\lambda x_n) \\ &\le d(x_n, z_n) + d(J_{\lambda_n} x_n, J_\lambda x_n) \\ &= d(x_n, z_n) + d(J_\lambda(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n), J_\lambda x_n) \\ &\le d(x_n, z_n) + \frac{\lambda_n - \lambda}{\lambda_n} d(J_{\lambda_n} x_n, x_n) + \frac{\lambda}{\lambda_n} d(x_n, x_n) \\ &= (2 - \frac{\lambda}{\lambda_n}) d(x_n, z_n) \\ &\to 0(n \to \infty). \end{aligned}$$

This also shows that

$$\lim_{n\infty} d(x_n, J_\lambda x_n) = 0.$$

(v) Suppose that the mapping S is hemi-compact. By the step of (iii), we get $\lim_{n\to\infty} dist(x_n, Sx_n) = 0$. From the hemi-compactness of S and we have that there exists a subsequence $\{u_n\}$ of $\{x_n\}$, which strongly converges to an element q in D. Furthermore, by the above$(ii) - (iv)$, we have

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0, \lim_{n\to\infty} dist(u_n, Su_n) = 0 \ and \lim_{n\to\infty} d(u_n, J_\lambda u_n) = 0.$$

It follows by the nonexpansiveness of T and the nonexpansiveness of J_λ so that $q = Tq = J_\lambda q$, we get

$$q \in F(T) \cap F(J_\lambda) = F(T) \cap argmin_{y\in D} f(y).$$

By the nonexpansiveness of S, we have

$$\begin{aligned} dist(q, Sq) &\leq d(q, u_n) + dist(u_n, Su_n) + H(Su_n, Sq) \\ &\leq 2d(q, u_n) + dist(u_n, Su_n) \\ &\to 0 (n \to \infty). \end{aligned}$$

It shows that $dist(q, Sq) = 0$. This implies that $q \in Sq$. Therefore, we get $q \in F(S)$. By (3.16), we have

$$q \in F(T) \cap F(S) \cap argmin_{y \in D} f(y) = \Omega.$$

Through the double extract subsequence principle, it shows that the sequence $\{x_n\}$ strongly converges to a point q in Ω.

This completes the proof. □

Theorem3.2 Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $T : D \to D$ be a nonexpansive single-valued mapping, $S : D \to KC(D)$ be a nonexpansive multi-valued mapping, and $f : D \to (-\infty, \infty]$ be a convex and lower semi-continuous proper function. Suppose that $\Omega = F(T) \cap F(S) \cap argmin_{y \in D} f(y)$ is nonempty and $Sp = \{p\}$ for all $p \in \Omega$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for all $n \in N$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in N$ and some λ. Then the sequence $\{x_n\}$ $\Delta-$converges to a point in Ω.

Proof. Let $\omega_\Delta(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $p \in \omega_\Delta(x_n)$. So there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{p\}$. By Lemmas 2.6 and 2.7, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$\Delta - \lim_{n \to \infty} v_n = v \in D. \tag{3.17}$$

From Theorem 3.1$(ii), (iv)$, we have

$$\lim_{n \to \infty} d(v_n, Tv_n) = 0$$

and

$$\lim_{n \to \infty} d(v_n, J_\lambda v_n) = 0.$$

Then, by the nonexpansiveness of T and J_λ, it implies by Lemma 2.9 that $v = Tv = J_\lambda v$. So, we get

$$v \in F(T) \cap F(J_\lambda) = F(T) \cap argmin_{y \in D} f(y). \tag{3.18}$$

Since S is compact valued, for each $n \in N$, there exist $r_n \in Sv_n$ and $\delta_n \in Sv$ such that $d(v_n, r_n) = dist(v_n, Sv_n)$ and $d(r_n, \delta_n) = dist(r_n, Sv)$. By the third step of Theorem 3.1, it follows that

$$\lim_{n \to \infty} d(v_n, r_n) = 0.$$

By the compactness of Sv, so there exists a subsequence $\{\delta_{n_i}\}$ of $\{\delta_n\}$ such that $\lim_{i \to \infty} \delta_{n_i} = \delta \in Sv$. Then we have

$$\begin{aligned} \lim_{i \to \infty} \inf d(v_{n_i}, \delta) &\leq \lim_{i \to \infty} \inf (d(v_{n_i}, r_{n_i}) + d(r_{n_i}, \delta_{n_i}) + d(\delta_{n_i}, \delta)) \\ &\leq \lim_{i \to \infty} \inf (d(v_{n_i}, r_{n_i}) + dist(r_{n_i}, Sv) + d(\delta_{n_i}, \delta)) \\ &\leq \lim_{i \to \infty} \inf (d(v_{n_i}, r_{n_i}) + H(Sv_{n_i}, Sv) + d(\delta_{n_i}, \delta)) \\ &\leq \lim_{i \to \infty} \inf (d(v_{n_i}, r_{n_i}) + d(v_{n_i}, v) + d(\delta_{n_i}, \delta)) \\ &= \lim_{i \to \infty} \inf d(v_{n_i}, v). \end{aligned}$$

By (3.17) and the uniqueness of asymptotic centers, we have $v = \delta \in Sv$. Thus, by (3.18), we get

$$v \in F(T) \cap F(S) \cap argmin_{y \in D} f(y) = \Omega.$$

It follows by the first step of Theorem 3.1 and Lemma 2.8 so that $p = v$, and hence $\omega_\Delta(x_n) \subseteq \Omega$.

Suppose that $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u^*\}$ and $A(\{x_n\}) = \{x\}$. Since $u^* \in \omega_\Delta(x_n) \subseteq \Omega$ and $\{d(x_n, u^*)\}$ converges, it implies by Lemma 2.8 that $x = u^*$, which shows that $\omega_\Delta(x_n)$ consists of exactly one point. This implies that $\{x_n\}$ $\Delta-$converges to a point in Ω.

This completes the proof. $\square$

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