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Rings in Which Every Quasi-nilpotent Element is Nilpotent

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Abstract. A ring R is called a QN-ring if R satisfies the equation $Q(R) = N(R)$. In this paper, we present some fundamental properties of the class of QN-rings. It is shown that, for R being a 2-primal (nil-semicommutative) ring, R is a QN-ring if and only if $Q(R)$ is a nil ideal; if R is a QN-ring, then $R/J(R)$ is a semiprime ring; if R is a QN-ring and $R/J(R)$ is nil-semicommutative, then R is a feckly reduced ring. We also show that if $T_n(R, \alpha)$ is a QN-ring, then R is a QN-ring.

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1. Introduction and Basic Concepts

In this paper, a given ring R is assumed to be associative with a unity element. We use the notation $U(R)$ to represent the group of units of R, $J(R)$ for the Jacobson radical of R, $N(R)$ for the set of nilpotent elements, and $N_*(R)$ and $N^*(R)$ to denote the lower and upper radicals of R, respectively. The basic properties of these sets can be found in [9].

According to [6], an element $a \in R$ is called *quasi-nilpotent* if $1 - ax$ is invertible for every element $x \in R$ that commutes with a. The set of all quasi-nilpotent elements in R is denoted as $Q(R)$. It is clear that nilpotent elements and elements of the Jacobson radical are quasi-nilpotent, though the converse is not generally true.

In recent years, many authors have studied classes of rings related to the set of nilpotent elements $N(R)$. It is known that for any ring R, the following inclusions always hold:

$$N_*(R) \subseteq N^*(R) \subseteq J(R), \quad N(R) \subseteq Q(R).$$

When R is a left Artinian ring, $N_*(R) = N^*(R)$, and when R is commutative, $N_*(R) = N^*(R) = N(R)$. A ring satisfying $N^*(R) = N(R)$ is called a *2-primal* ring; satisfying $N^*(R) = N(R)$ is called an *NI-ring*; satisfying $N(R) = J(R)$ is called an *NJ-ring*. Additionally, other classes of rings have been constructed similarly [2,5,6,9]. Moreover, the generalization of the set Q(R) has been studied and developed by some authors. They have proposed a ring structure for the class of rings through this generalized set [7, 10].

The study of these classes of rings sheds light on many aspects related to classical ring classes in particular and ring structure theory in general. Based on the above, we define the class of QN-rings as the class of rings satisfying $Q(R) = N(R)$. In this paper, we provide some examples and characterizations of QN-rings as a foundation for further research. We show that if R is a 2-primal ring, then R is a QN-ring if and only if $Q(R)$ is a nil ideal of R (Theorem

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2.12). We also prove that if R is a QN-ring, then $R/J(R)$ is a semiprime ring (Theorem 2.15). Every strongly π-regular ring is a QN-ring (Proposition 2.28).

Let α be an endomorphism of the ring R and n be a positive integer. The ring $T_n(R, \alpha)$ is defined as follows:

$$\left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \in T_n(R) : a_i \in R \right\}.$$

We show that if $T_n(R, \alpha)$ is a QN-ring, then R is a QN-ring (Theorem 3.2). Moreover, it is shown that if R is a ring and M is a bimodule over R such that $rm = mr$ for all $m \in M$ and $r \in R$, then the trivial extension $T(R, M)$ is a QN-ring if and only if R is a QN-ring (Theorem 3.4).

2. Rings in Which $N(R)$ Contain All Quasi-nilpotents

First, we introduce the concept of the QN-ring class and provide some examples of that ring class.

Definition 2.1. A ring is called *QN* if the set of nilpotent elements and the set of quasi-nilpotent elements are equal, i.e., $Q(R) = N(R)$.

Here are some specific examples of QN-rings.

Example 2.2. (1) The ring $\mathbb{Z}$ is the QN-ring.

(2) Every division ring is a QN-ring. In fact, $N(R) = Q(R) = 0$.

(3) Every Boolean ring is a QN-ring. Note that the $Q(R)$ contains no idempotent elements except for 0. Moreover, for every $x \in N(R)$, there exists a positive integer m such that $x^m = 0$. Since R is a Boolean ring, we have $x^2 = x$ and so $N(R) = 0$.

(4) Let $S = M_2(\mathbb{F}_2)$ and $K = \mathbb{F}_2[[x]]$. Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in S$, then $A \notin J(S)$ but $A \in Q(S)$. On the other hand, $x \in Q(K)$ but $x \notin N(K)$. Thus, K is not QN.

Lemma 2.3. *If R is a semilocal QN-ring, then R is a two-sided perfect ring.*

Proof. Assume that R is a semilocal QN-ring. Then, we have $J(R) \subseteq Q(R) = N(R)$. It means that $J(R)$ is the nil ideal. We deduce that R is left and right perfect. □

The following properties demonstrate the closure of the QN-ring class under direct product and isomorphism.

Proposition 2.4. *The class of QN-rings is closed under ring isomorphisms and direct products.*

Proof. Let $f : R \to S$ be a ring isomorphism, and R be a QN-ring. We will prove that S is also a QN-ring. Indeed, it is easy to see that an element a is nilpotent in R if and only if $f(a)$ is nilpotent in S. Indeed, for an element $a \in Q(R)$, we have, for every $f(x) \in S$ satisfying $f(a)f(x) = f(x)f(a)$, that $f(ax) = f(xa)$, and hence $ax = xa$. We have that $a \in Q(R)$ and obtain that $1 - ax$ is invertible in R, so $1 - f(a)f(x) = f(1 - ax)$ is also invertible in S, implying $f(a) \in Q(S)$. Thus, the class of QN-rings is closed under ring isomorphisms. The final statement of the proposition is derived from the properties $Q\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} Q(R_i)$ and $N\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} N(R_i)$. □

For a subring S of a ring R, the set

$$\mathcal{R}[R, S] := \{(r_1, \ldots, r_n, s, s, \ldots) : r_i \in R, s \in S, n \geq 1\},$$

with addition and multiplication defined componentwise, is a ring.

As an immediate consequence, we yield:

Corollary 2.5. *The ring $\mathcal{R}[R, S]$ is a QN ring if and only if R and S are QN-rings.*

Remark 2.6. Let R be a ring and S be a subring of R. We see that every nilpotent element of S in R is also nilpotent in R, i.e., $N(R) \cap S = N(S)$. For quasi-nilpotent elements, there exist quasi-nilpotent elements in R that are not quasi-nilpotent in S, and vice versa. Therefore, in general, QN-rings are not closed under subrings. Next, we establish a condition for subrings of QN-rings to also be QN-rings.

Recall that a ring homomorphism $f : S \to R$ is called *local* if for every x that is not invertible in S, $f(x)$ is also not invertible in R. A subring S of a ring R is called a *rationally closed subring* if $U(S) = U(R) \cap S$. This is equivalent to the inclusion monomorphism $\iota : S \to R$ being a local ring homomorphism.

The following result preserves the QN-ring class under rationally closed subrings. Note that if S is a rationally closed subring of R, then $Q(S) \subseteq Q(R) \cap S$.

Theorem 2.7. *Let R be a QN-ring and S be a rationally closed subring of R. Then, S is a QN-ring.*

Proof. Consider $a \in Q(R) \cap S$, i.e. $a \in S$ and $1 - ax \in U(R)$ for all $x \in R$ with $ax = xa$. Note that for every $x \in S$ with $ax = xa$, we have $1 - ax \in S$ because S is a subring. Therefore, $1 - ax \in U(R) \cap S = U(S)$ under the assumption that S is a rationally closed subring of R. Thus, $a \in Q(S)$. Along with the condition $Q(S) \subseteq Q(R) \cap S$, we obtain $Q(S) = Q(R) \cap S$. Combined with Remark 2.6, we conclude that S is a QN-ring. □

Remark 2.8. The condition $Q(S) \subseteq Q(R) \cap S$ in Theorem 2.7 cannot be omitted. Consider the following example: Let $\mathbb{Z}_{(2)}$ be the set of rational numbers such that the denominator is an odd integer. Define $R = M_2(\mathbb{Q})$ and $S = M_2(\mathbb{Z}_{(2)})$. Then, R is a QN-ring and S is a subring of R. The element $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is quasi-nilpotent in S, but A is not quasi-nilpotent in R. Indeed, consider the matrix $B = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$. One can check that $AB = BA$ and $1 - AB = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is non-invertible. Thus, A is not quasi-nilpotent in R. Therefore, the condition $Q(S) \subseteq Q(R) \cap S$ does not hold, and S cannot be a QN-ring because A is quasi-nilpotent without being nilpotent in R.

Corollary 2.9. *Let R be a QN-ring and e be an idempotent element in R. Then the subring eRe is also a QN-ring.*

Proof. The result follows from the property $Q(eRe) = eRe \cap Q(R)$. □

Let S and T be any rings, ${}_S M_T$ a bimodule, and the formal triangular matrix R defined as:

$$R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}.$$

Corollary 2.10. *If $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ is a QN-ring, then S and T are QN-rings.*

Proof. Take $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $S \cong eRe$, $T \cong fRf$. It follows from Corollary 2.9 that S and T are QN. □

Corollary 2.11. *If there exists a ring R such that $M_n(R)$ is a QN-ring, then R is QN.*

Recall that a ring R is termed 2-*primal*, provided that $N(R) = N_*(R)$. A ring R is called *nil-semicommutative* if for all $a, b \in N(R)$, $ab = 0$ implies $aRb = 0$. From the definition of nil-semicommutative rings, it implies that every semicommutative ring is nil-semicommutative.

The following result presents the properties of QN-rings within the class of 2-primal (nil-semicommutative) rings. Moreover, the set $Q(R)$ is a nil ideal of the ring. In this case, $Q(R)$ coincides with the Jacobson radical of the ring.

Theorem 2.12. *Let R be a 2-primal ring. Then, the following statements are equivalent:*

(1) *R is a QN-ring.*
(2) *$Q(R)$ is a nil ideal of R.*

Proof. (1) ⇒ (2) Assume that R is a 2-primal ring. Then, $N(R) = N_*(R)$ is the ideal of R and so $N(R) \subseteq J(R)$. On the other hand, since R is a QN-ring, we have

$$J(R) \subseteq Q(R) = N(R) \subseteq J(R)$$

Thus, $Q(R) = J(R) = N(R)$ is a nil ideal of R.

(2) ⇒ (1) We have that $Q(R)$ is a nil ideal and obtain that $Q(R) \subseteq N(R)$. We deduce that $Q(R) = N(R)$. □

Since every nil-semicommutative ring is 2-primal, we have the following result.

Corollary 2.13. *Let R be a nil-semicommutative ring. Then the following statements are equivalent:*

(1) *R is a QN-ring.*
(2) *$Q(R)$ is a nil ideal of R.*

Corollary 2.14. *If R is a nil-semicommutative QN-ring, then $Q(R) = J(R)$ is a nil ideal of R.*

The following result addresses the semiprime property of the quotient ring of a QN-ring.

Theorem 2.15. *If R is a QN-ring, then $R/J(R)$ is a semiprime ring.*

Proof. Call $\bar{R} = R/J(R)$. Let $\bar{a} = a + J(R)$ be an element of $\bar{R}$ such that $\bar{a}\bar{R}\bar{a} = 0$ and so $a^2 \in J(R)$. It follows that $aRa \subseteq J(R)$. We show that $a \in J(R)$ Indeed, from $aRa \subseteq J(R)$, it immediately infers that $(aR)^2 \leq J(R)$. We have that R is a QN-ring and obtain that

$$(aR)^2 \subseteq Q(R) = N(R).$$

This means aR is a nil right ideal of R, and hence $aR \leq J(R)$. It follows that $a \in J(R)$. We deduce that $R/J(R)$ is a semiprime ring. □

Corollary 2.16. *If R is a semiprimitive QN-ring, then R is a semiprime ring.*

A ring R is called *reduced* if R has no nonzero nilpotent elements. Recall that a ring R is *feckly reduced* if $R/J(R)$ is reduced.

The following results highlight that the nilpotent elements of QN-rings in certain special classes of rings are trivial.

Theorem 2.17. *If R is a QN-ring and $R/J(R)$ is a nil-semicommutative ring, then R is a feckly reduced ring.*

Proof. Let $\overline{R} = R/J(R)$. Assume that $\bar{x}^2 = 0$. Since $R/J(R)$ is nil-semicommutative, we have $\bar{x}\overline{R}\bar{x}\overline{R} = 0$. This implies $xRxR \subseteq J(R) \subseteq Q(R)$. By assumption, R is a QN-ring, so xR is a nil right ideal of R. Thus, $xR \leq J(R)$, and consequently, $x \in J(R)$. □

Recall that, a ring R is said to be a *Dedekind-finite* if $ab = 1$ implies $ba = 1$ for any $a, b \in R$.

Corollary 2.18. *If R is a QN-ring and $R/J(R)$ is a nil-semicommutative ring, then R is Directly finite.*

The following results identify classes of rings that are QN-rings. These classes of rings have been extensively studied in recent years.

Proposition 2.19. *Let R be a ring. If every non-invertible element of R is nilpotent, then R is a QN-ring.*

Proof. From the assumption, we have $R \setminus U(R) = N(R)$. For every $x \in R$ such that x is not nilpotent, it follows that $x \in U(R)$. Then there exists $r \in R$ such that $xr = rx = 1$. This implies $x \notin Q(R)$. Hence, we obtain $Q(R) \subseteq N(R)$ or equivalently $N(R) = Q(R)$. □

Recall that a ring R is called a *UU-ring* if R satisfies the equality $U(R) = 1 + N(R)$ (see [3]). A ring R is called a *weak UU-ring* if it satisfies the equality $U(R) = N(R) \pm 1$ (see [4]).

Proposition 2.20. *Every weak UU-ring is a QN-ring.*

Proof. Consider $x \in Q(R)$ and $x \notin N(R)$. Since $1 + x \in U(R) = N(R) \pm 1$, $1 + x = q - 1$ for some $q \in N(R)$ and so $2 + x \in N(R)$. Similarly, since $1 + x^2 \in U(R)$, it follows that $2 + x^2 \in N(R)$. Hence, we have $2 + x^2 - (2 + x) = x(x - 1) \in N(R)$. Since $x - 1 \in U(R)$, we have $x \in N(R)$, which is a contradiction. Thus, $Q(R) \subseteq N(R)$. □

Recall a ring R a *UQ-ring* if $U(R) = 1 + Q(R)$ (see [6]).

Next, we give a necessary and sufficient condition to a QN-ring to be a UU-ring.

Proposition 2.21. *The following statements are equivalent for a ring R.*

(1) *R is a UU-ring.*
(2) *R is a QN-ring and UQ-ring.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a UU-ring. From Proposition 2.28, it imfers that R is a QN-ring. It follows that $U(R) = 1 + N(R) = 1 + Q(N)$. We deduce that R is a UQ-ring.

(2) $\Rightarrow$ (1) Assume that R is a QN-ring and UQ-ring. Then, we have

$$U(R) = 1 + Q(N) = 1 + N(R).$$

We deduce that R is a UU-ring. □

Example 2.22. Let $K = \mathbb{F}_2[[x]]$. Then, K is a UQ ring but not a UU ring, since $1 + x \in U(K)$ but $x \notin N(K)$. Thus, K is not a QN-ring.

A ring R is called a *strongly nil-good* ring if every element $x \in R$ can be expressed as $x = n + u$, where $n \in N(R)$ and $u \in U(R)$ and $nu = un$.

Proposition 2.23. *Every strongly nil-good ring is a QN-ring*

Proof. We show that $Q(R) = 0$. Indeed, let $x \in Q(R)$ and $x \neq 0$. Then, $x = a + b$, where $a \in N(R)$ and $b \in U(R)$ and $ab = ba$. This implies $a = x - b$. Notice that $ab = ba$, and hence $xb = bx$. It follows that $ab^{-1} = xb^{-1} - 1$. We have that $xb^{-1} = b^{-1}x$ and $x \in Q(R)$ and obtain that $ab^{-1} \in U(R)$. From this, it follows that $a \in U(R)$, thus $a = 0$. Therefore, we have $x = b \in U(R)$, which is a contradiction. Hence $Q(R) = 0$ or R is a QN-ring. □

Recall that a ring R is called *strongly weakly nil-clean* if every element of R is the sum or difference of a nilpotent element and an idempotent element that commute with each other [1, 8].

Proposition 2.24. *Every strongly weakly nil-clean ring is QN.*

Proof. For each $x \in Q(R)$, we have $x = q + e$ or $x = q - e$, where $q \in N(R)$ and $e^2 = e \in R$, and $qe = eq$. Suppose $q^n = 0$, then $(x + e)^n = 0$, or $(e - x)^n = 0$. Note that $xe = ex$, which implies $e = e^n \in Q(R)$. Hence, we obtain $e = 0$, and thus $x = q$ is nilpotent. Therefore, R is a QN-ring. □

Lemma 2.25 ([5, Lemma 4.1]). *Let R be a ring, I an ideal of R such that $I \leq J(R)$. If for every $\bar{q} \in Q(R/I)$, then $q \in Q(R)$.*

Proposition 2.26. *Let I be an ideal of R such that $I \leq J(R)$. If R is a QN-ring, then R/I is also a QN-ring.*

Proof. Let $\bar{q} \in Q(R/I)$. Then we have $q \in Q(R)$ by Lemma 2.25. Since R is a QN-ring, it follows that $Q(R) = N(R)$, and therefore q is nilpotent. □

Recall that an element x in a ring is called *left π-regular* if the chain $Rx \supseteq Rx^2 \supseteq Rx^3 \supseteq \cdots$ terminates, and x is called *right π-regular* if the chain $xR \supseteq x^2R \supseteq x^3R \supseteq \cdots$ terminates. An element x is called *strongly π-regular* if it is both left and right π-regular. A ring R is called strongly π-regular if all elements of R are strongly π-regular.

Lemma 2.27. *Let R be a ring and $a \in R$. The following statements are equivalent:*

(1) *a is strongly π-regular;*
(2) *There exist $b \in R$ and an integer $n \geq 1$ such that $a^n = a^{n+1}b$ and $ab = ba$;*
(3) *There exists an integer $n \geq 1$ such that $a^n = eu = ue$, where $e^2 = e$ and $u \in U(R)$.*

Proposition 2.28. *Every strongly π-regular ring is a QN-ring.*

Proof. Let R be a strongly π-regular ring. We will show that $Q(R) \subseteq N(R)$. Assume $a \in Q(R)$. By Lemma 2.27, there exist an integer $n \geq 1$ and $b \in R$ such that $a^n = a^{n+1}b$ and $ab = ba$. Since $a \in Q(R)$, it follows that $1 - ab \in U(R)$, the group of invertible elements in R. Furthermore, we have $a^n(1 - ab) = a^n - a^{n+1}b = 0$. As $(1 - ab)$ is invertible, this implies that $a^n = 0$. Hence, $a \in N(R)$, or equivalently $Q(R) \subseteq N(R)$. □

Corollary 2.29. *Every reduced regular ring is a QN-ring.*

Example 2.30. The ring R of square matrices of degree n over a division ring is a QN-ring. Indeed, R is a simple Artinian ring, hence the descending chain condition $AR \geq A^2R \geq \cdots$ is satisfied. According to Lemma 2.27 and Proposition 2.28, R is a QN-ring.

3. Extensions of QN-rings

Proposition 3.1. *Let R be a ring and let G be a group. If RG is a QN-ring, then R is a QN-ring.*

Proof. Since R is a rationally closed subring of RG, it follows directly from Theorem 2.7 that R is a QN-ring. □

Let α be an endomorphism of the ring R and n be a positive integer. The ring $T_n(R,\alpha)$ is defined as follows:

$$\left\{\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \in T_n(R) : a_i \in R\right\}.$$

With addition defined component-wise and multiplication defined by:

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix}\begin{pmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ 0 & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ 0 & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_0 \end{pmatrix}$$

with

$$c_i = a_0\alpha^0(b_i) + a_1\alpha^1(b_{i-1}) + \cdots + a_i\alpha^i(b_0), \quad 0 \le i \le n-1.$$

For simplicity, we denote the elements of $T_n(R,\alpha)$ as matrices. If α is the identity endomorphism, then $T_n(R,\alpha)$ is a subring of the ring of upper triangular matrices over R.

Theorem 3.2. *If $T_n(R,\alpha)$ is a QN-ring, then R is a QN-ring.*

Proof. Let

$$I = \left\{\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} : a_{ij} \in R\,(i \le j)\right\}.$$

Then, we can verify that $I^n = 0$ and $T_n(R,\alpha)/I \cong R$. Thus, $I \le J(T_n(R,\alpha))$, and according to Proposition 2.26, $T_n(R,\alpha)/I$ is a QN-ring. Therefore, R is a QN-ring by Proposition 2.4. □

Let α be an endomorphism of the ring R. Denote $R[x;\alpha]$ as the ring of polynomials in the variable x, with coefficients in R, under standard polynomial addition and multiplication defined via the action of α as follows:

$$xa = \alpha(a)x, \quad \text{for all } a \in R.$$

The ring homomorphism $\phi : R[x;\alpha]/(x^n) \to T_n(R,\alpha)$ defined by

$$\phi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n) = (a_0, a_1, \cdots, a_{n-1}),$$

where $a_i \in R$, $0 \le i \le n-1$, is an isomorphism of rings. Hence, $T_n(R,\alpha) \cong R[x;\alpha]/(x^n)$.

Corollary 3.3. *Let R be a ring. If $R[x;\alpha]/(x^n)$ is a QN-ring, then R is a QN-ring.*

Let R be a ring and M a bimodule over R. The trivial extension of R and M is defined as

$$T(R,M) = \{(r,m) : r \in R \text{ and } m \in M\},$$

with addition defined componentwise and multiplication defined by the equality

$$(r,m)(s,n) = (rs, rn + ms).$$

Observe that the trivial extension $T(R,M)$ is isomorphic to the subring

$$\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M\right\}$$

consisting of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$, and also $T(R,R) \cong R[x]/\left\langle x^2 \right\rangle$.

Theorem 3.4. *Let R be a ring and M a bimodule over R such that $rm = mr$ for all $m \in M$ and $r \in R$. Then, the trivial extension $T(R, M)$ is a QN-ring if and only if R is a QN-ring.*

Proof. Assume that $T(R, M)$ is a QN-ring, and we can consider R as a rationally closed subring of $T(R, M)$ (because, $R \cong T(R, 0)$). Therefore, by Theorem 2.7, R is a QN-ring. Now, conversely, assume that R is a QN-ring and $\begin{pmatrix} q & m \\ 0 & q \end{pmatrix} \in Q(T(R, M))$. From $rm = mr$ for all $m \in M$ and $r \in R$, it infers that $q \in Q(R)$. Since R is a QN-ring, we have $q \in N(R)$. Then, there exists $n \in \mathbb{N}$ such that $q^n = 0$. One can check that

$$\begin{pmatrix} q & m \\ 0 & q \end{pmatrix}^{n+1} = \begin{pmatrix} q^{n+1} & (n+1)q^n m \\ 0 & q^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is shown that $\begin{pmatrix} q & m \\ 0 & q \end{pmatrix} \in N(T(R, M))$. We deduce that $T(R, M)$ is a QN-ring. □

Corollary 3.5. *Let R be a commutative ring. Then, $R[x]/\left\langle x^2 \right\rangle$ is a QN-ring if and only if R is a QN-ring.*

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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