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Existence and approximation for nonlinear dynamic equations using monotone iteration method

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Abstract
This paper improves a generalized monotone iterative technique for solving dynamic initial value problems (IVPs) on time scales, using the method of coupled lower and upper solutions. We construct monotone sequences of iterates, each of which corresponds to a solution of the dynamic IVP on time scales. Under suitable conditions, we establish the uniform and monotonic convergence of sequences to the extremal (minimal and maximal) solutions of the problem. The results provide a unified framework for analyzing dynamic equations on time scales via monotone iteration.

Keywords: Comparison result, Monotone iterative technique, Coupled upper and lower solutions

INTRODUCTION
In recent decades, the theory of dynamic equations on time scales has played a pivotal role in unifying and extending the frameworks of differential and difference equations [2,4].

The monotone iterative technique, coupled with the method of lower and upper solutions, has proven to be a powerful and flexible tool for establishing existence results for nonlinear differential equations [3,5,6,7]. In 2002 and 2004, Bhaskar [1] and West [10] advanced this methodology by developing a generalized monotone iterative technique for initial value problems (IVPs). Their work demonstrated the existence of minimal and maximal solutions for differential equations where the nonlinear function decomposes into the sum of a monotone non-decreasing and a monotone non-increasing function. At this stage, Ramirez and Vatsala [8] extended these results to Caputo fractional differential equations with periodic boundary conditions via an initial value problem approach. Meanwhile, Wang and Tian [9] introduced an alternative method to derive unique solutions for boundary value problems.

In this study, we employ the method of coupled lower and upper solutions for dynamic initial value problems on a time scale $\mathbb{T}$ By constructing monotone iterative sequences from corresponding linear IVPs, we establish the uniform and monotone convergence of these sequences to the coupled minimal and maximal solutions of the nonlinear dynamic IVP.

Our results hold under the assumption that the nonlinear function f is rd-continuous and can be expressed as the difference of two monotone non-decreasing functions.

The paper is organized as follows. In Section 2, we review some basic concepts and preliminary results. The main results are presented in Section 3 followed by their proofs in Section 4. In Section 5 we present numerical examples to motivate the applicability and relevance of the main results. Concluding remarks are provided in Section 6.

PRELIMINARIES
In this section, we mention some basic concepts and theories used in subsequent references.
A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. The intervals with a subscript $\mathbb{T}$ are used to indicate the ordinary interval intersects with $\mathbb{T}$, i.e., $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ are defined respectively by

$$\sigma(t)=\inf\{s>t:s\in\mathbb{T}\} \text{ and } \rho(t)=\sup\{s<t:s\in\mathbb{T}\}$$

A point is called right-scattered, right-dense, left-scattered and left-dense if $\sigma(t)>t, \sigma(t)=t, \rho(t)<t$ and $\rho(t)=t$; respectively. Points that are both right- and left-scattered are isolated, while those that are both right- and left-dense are dense. The graininess function $\mu^*:\mathbb{T}\to[0,\infty)$is given by $\mu^*(t)=\sigma(t)-t$. The mapping $f:\mathbb{T}\to\mathbb{R}^n$ is said to be differentiable, if f has exactly one derivative $f^{\Delta}(t)$; that is, given any $\varepsilon>0$, there exists a neighborhood U_t of t such that f satisfies

$$\left|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)\right|\le\varepsilon|\sigma(t)-s| \text{ for } s\in U_t$$

The set $\mathbb{T}^{\kappa}$ is defined as

$$\mathbb{T}^{\kappa}:=\begin{cases}\mathbb{T}\setminus(\rho(\mathbb{T}),\sup\mathbb{T}], & \text{if }\sup\mathbb{T}<\infty;\\ \mathbb{T}, & \text{if }\sup\mathbb{T}=\infty.\end{cases}$$

A function $f:\mathbb{T}\to\mathbb{R}$ is regulated if it has finite left-sided limits at left-dense points. A function $f:\mathbb{T}\times\mathbb{R}^n\to\mathbb{R}^n$ is rd-continuous (denoted $f\in\mathscr{C}_{\mathrm{rd}}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^n]$if it is continuous at each right-dense $t\in\mathbb{T}$, and the left sided limit $f(t^-,x)$ exists in each left-dense. A function $F:\mathbb{T}^{\kappa}\to\mathbb{R}$ is an antiderivative of $f:\mathbb{T}^{\kappa}\to\mathbb{R}$ if $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^{\kappa}$. In this case, the delta integral of f is $\int_a^t f(s)\Delta s=F(t)-F(a)$, $t\in\mathbb{T}$.
Consider the dynamic initial value problem (IVP)

$$u^{\Delta}=f(t,u) \text{ for } t\in[t_0,\infty)_{\mathbb{T}} \tag{1}$$

with $u(t_0)=u_0$ As usual here the time scale $\mathbb{T}$ is assumed to have $t=t_0$ as the minimal element and $t=\tilde{t}$ as the maximal element which is not left scattered.

We need the following definitions before we proceed further.

Definition 1. A function $f:\mathscr{C}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^n]$ is quasi-monotone non-decreasing if $x\le y$ and $x_i=y_i$ for some $i\in\mathbb{N}$ implies $f_i(t,x)\le f_i(t,y)$.
Definition 2. A function $u\in\mathscr{C}[t_0,\infty)_{\mathbb{T}},\mathbb{R}^n]$ is a solution of (1) for some $s\in[t_0,\infty)_{\mathbb{T}}$ if it differentiable on $[s,\infty)_{\mathbb{T}}$ and satisfies (1) identically on $[s,\infty)_{\mathbb{T}}$ with $u(t_0)=u_0$.

Now, let us introduce comparison theorem.

Theorem 1 ([4]). Let $\mathbb{T}$ be a time scale with $t_0\ge0$ as its minimal element (no maximal element). Suppose
(i) $v,w\in\mathscr{C}\mathrm{rd}^1\left[\mathbb{T},\mathbb{R}^n\right]$ for each $t\in\mathbb{T}$ and satisfy the

following inequalities

$$\begin{aligned} v^{\Delta}(t) &\le f(t, v(t)), \quad v(t_0) \le u_0 \\ w^{\Delta}(t) &\ge f(t, w(t)), \quad w(t_0) \ge u_0 \end{aligned} \tag{2}$$

(ii) $f \in \mathscr{C}_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,u)$ is quasi-monotone non-decreasing in u and for each $i \in \mathbb{N}$, $1 \le i \le n$, $f_i(t,u)\mu^*(t) + u_i$, is non-decreasing in u_i for $t \in \mathbb{T}$.

Moreover, let $f(t,u)$ satisfies the following condition

$$f_i(t,x) - f_i(t,y) \le L \sum_{i=1}^{n} (x_i - y_i) \text{ for } x \ge y, L > 0. \tag{3}$$

Then $v(t_0) \le w(t_0)$ implies that $v(t) \le w(t)$ for $t \in \mathbb{T}$.

We conclude the following corollaries which will be useful in our main results.

Corollary 1. If $g \in \mathscr{C}_{\mathrm{rd}}{}^1[\mathbb{T}, \mathbb{R}^n]$ satisfies $g^{\Delta}(t) \le 0$ and $g(t_0) \le 0$, then $g(t) \le 0$ for all $t \in \mathbb{T}$.
Corollary 2. If $g \in \mathscr{C}_{\mathrm{rd}}{}^1[\mathbb{T}, \mathbb{R}^n]$ satisfies $g^{\Delta}(t) \ge 0$ and $g(t_0) \ge 0$, then $g(t) \ge 0$ for all $t \in \mathbb{T}$.

Now, consider the nonlinear dynamic initial value problem

$$u^{\Delta}(t) = f_1(t, u(t)) - f_2(t, u(t)), \quad u(0) = u_0, \tag{4}$$

where $f_1(t,u)$ and $f_2(t,u)$ may be non-decreasing functions.

Firstly, we introduce four types of coupled lower and upper solutions for (4).

Definition 3. Let $v_0, w_0 \in \mathscr{C}_{\mathrm{rd}}{}^1[\mathbb{T}, \mathbb{R}]$. Then v_0, w_0 are said to be for all $t \in \mathbb{T}$.

(i) natural lower and upper solutions of (4), if

$$\begin{aligned} v_0{}^{\Delta}(t) &\le f_1(t, v_0(t)) - f_2(t, v_0(t)),\ v_0(0) \le u_0 \\ w_0{}^{\Delta}(t) &\ge f_1(t, w_0(t)) - f_2(t, w_0(t)),\ w_0(0) \ge u_0 \end{aligned}$$

(ii) coupled lower and upper solutions of Type I of (4), if

$$\begin{aligned} v_0{}^{\Delta}(t) &\le f_1(t, v_0(t)) - f_2(t, w_0(t)),\ v_0(0) \le u_0 \\ w_0{}^{\Delta}(t) &\ge f_1(t, w_0(t)) - f_2(t, v_0(t)),\ w_0(0) \ge u_0 \end{aligned}$$

(iii) coupled lower and upper solutions of Type II of (4), if

$$\begin{aligned} v_0{}^{\Delta}(t) &\le f_1(t, w_0(t)) - f_2(t, v_0(t)),\ v_0(0) \le u_0 \\ w_0{}^{\Delta}(t) &\ge f_1(t, v_0(t)) - f_2(t, w_0(t)),\ w_0(0) \ge u_0 \end{aligned}$$

(iv) coupled lower and upper solutions of Type III of (4) if

$$\begin{aligned} v_0{}^{\Delta}(t) &\le f_1(t,,, w_0(t)) - f_2(t,,, w_0(t)), \quad v_0(0) \le u_0 \\ w_0{}^{\Delta}(t) &\ge f_1(t,,, v_0(t)) - f_2(t,,, v_0(t)), \quad w_0(0) \ge u_0 \end{aligned}$$

MAIN RESULTS

In this section, we now present our main results on the existence and convergence of solutions to the nonlinear dynamic equation (4). Our approach leverages coupled lower and upper solutions to construct monotone iterative sequences that converge uniformly to extremal solutions. Theorem 2 employs Type I coupled solutions, we derive natural monotone sequences that converge uniformly and monotonically to the coupled minimal and maximal solutions of Type I of (4). We can also give result using coupled lower and upper solutions of Type I and Type III, however it require an additional assumption to obtain intertwined monotone sequences which converge uniformly to coupled minimal and maximal solutions of (4).

Theorem 2. Assume the following conditions hold
(A1) the functions v_0, w_0 are coupled lower and upper solutions of Type I for (4) satisfying $v_0(t) \le u \le w_0(t)$ for all $t \in \mathbb{T}$,
(A2) the functions $f_1, f_2 \in C^{\mathrm{rd}}(\mathbb{T}^{\kappa} \times \mathbb{R}^n, \mathbb{R}^n)$, $f_1(t,u)$ and $f_2(t,u)$ are non-decreasing in u for each $t \in \mathbb{T}$.

Then, there exist monotone sequences $(v_j(t))$ and $(w_j(t))$ generated by the iterative scheme such that $v_j(t) \to \alpha(t)$ and $w_j(t) \to \beta(t)$ are uniformly convergent and monotonically, where α, β are coupled minimal and maximal solutions of (4), respectively. Moreover, they satisfy the coupled system

$$\begin{aligned} \alpha^{\Delta} &= f_1(t, \alpha) - f_2(t, \beta), \quad \alpha(0) = u_0 \\ \beta^{\Delta} &= f_1(t, \beta) - f_2(t, \alpha), \quad \beta(0) = u_0 \end{aligned}$$

for all $t \in \mathbb{T}$.

Theorem 3. Assume conditions (A1) and (A2) of Theorem 2 hold. Consider the iterative scheme

$$\begin{aligned} v_{j+1}{}^{\Delta} &= f_1(t, w_j) - f_2(t, v_j), \quad v_{j+1}(0) = u_0 = w_j(t_0), \\ w_{j+1}{}^{\Delta} &= f_1(t, v_j) - f_2(t, w_j), \quad w_{j+1}(0) = u_0 = v_j(t_0). \end{aligned} \tag{5}$$

Then, the iterates generate monotone sequences $\{v_{2j}\}$, $\{v_{2j+1}\}$, $\{w_{2j}\}$ and $\{w_{2j+1}\}$ satisfying

$$v_0 \le w_1 \le \ldots \le v_{2j} \le w_{2j+1} \le v_{2j+1} \le w_{2j} \le \ldots \le v_1 \le w_0$$

for all $j \in \mathbb{N}$ on $\mathbb{T}$, provided $v_0 \le u \le w_0$. Furthermore, the sequences converge $\{v_{2j}\}$ and $\{w_{2j+1}\}$ converge uniformly to α, and $\{v_{2j+1}\}$ and $\{w_{2j}\}$ converge uniformly to β, where α and β are coupled minimal and maximal solutions of Type I, respectively, of the system

$$\begin{aligned} \alpha^{\Delta} &= f_1(t, \alpha) - f_2(t, \beta), \quad \alpha(0) = u_0 \\ \beta^{\Delta} &= f_1(t, \beta) - f_2(t, \alpha), \quad \beta(0) = u_0 \end{aligned} \tag{6}$$

PROOF OF THE MAIN RESULTS

Here, we present proof of the our main results.
Proof of Theorem 2. For each $j \ge 0$, consider the system

$$\begin{aligned} v_{j+1}{}^{\Delta} &= f_1(t, v_m) - f_2(t, w_j), \quad v_{j+1}(0) = u_0, \\ w_{j+1}{}^{\Delta} &= f_1(t, w_j) - f_2(t, v_j), \quad w_{j+1}(0) = u_0. \end{aligned} \tag{7}$$

By assumption, $v_0(0) \le u_0 \le w_0(0)$. From (7), we have

$$\begin{aligned} v_1{}^{\Delta} &= f_1(t, v_0) - f_2(t, w_0), \quad v_1(0) = u_0, \\ w_1{}^{\Delta} &= f_1(t, w_0) - f_2(t, v_0), \quad w_1(0) = u_0. \end{aligned}$$

We now show that $v_0 \le v_1 \le u \le w_1 \le w_0$ on $\mathbb{T}$. Let $\psi = w_1 - w_0$. Then, $\psi(0) \le 0$, and

$$\begin{aligned} \psi^{\Delta} &= w_1{}^{\Delta} - w_0{}^{\Delta} \\ &\le f_1(t, w_0) - f_2(t, v_0) - f_1(t, w_0) + f_2(t, v_0) \\ &= 0. \end{aligned}$$

By Corollary 1, $\psi(t) \le 0$, so $w_1(t) \le w_0(t)$ on $\mathbb{T}$. Similarly, we can show that $v_1 \ge v_0$ on $\mathbb{T}$. Next, let $\phi = u - w_1$. Since $\phi(0) = 0$ and

$$\begin{aligned}&\phi^{\Delta} = u^{\Delta} - {w_1}^{\Delta}\\&= f_1(t,u) - f_2(t,u) - f_1(t,w_0) + f_2(t,v_0)\\&\le 0\end{aligned}$$

it follows that $u(t) \le w_1(t)$ on $\mathbb{T}$. Similarly, $u(t) \ge v_1(t)$. Thus, $v_0 \le v_1 \le u \le w_1 \le w_0$ holds for $k = 1$. Assume by induction that for some $k > 1$,

$$v_{k-1} \le v_k \le u \le w_k \le w_{k-1} \tag{8}$$

holds on $\mathbb{T}$. We now prove

$$v_k \le v_{k+1} \le u \le w_{k+1} \le w_k \quad \text{on } \mathbb{T}. \tag{9}$$

Let $\psi = w_{k+1} - w_k$. Then, $\psi(0) = 0$ and

$$\begin{aligned}&\psi^{\Delta} = {w^{\Delta}}_{k+1} - {w^{\Delta}}_k\\&= f\_1(t,w_k) - f\_2(t,v_k) - f\_1(t,w_{k-1}) + f\_2(t,v_{k-1})\\&\le 0.\end{aligned}$$

Thus, $w_{k+1}(t) \le w_k(t)$. Similarly, $v_{k+1}(t) \ge v_k(t)$. Now, let $\varphi = u - w_{k+1}$. Since $\varphi(0) = 0$ and

$$\begin{aligned}&\varphi^{\Delta} = u^{\Delta} - {w^{\Delta}}_{k+1}\\&= f_{\_1}(t,u) - f_{\_2}(t,u) - f_{\_1}(t,w_k) + f_{\_2}(t,v_k)\\&\le 0,\end{aligned}$$

we have $u(t) \le w_{k+1}(t)$. Similarly, $u(t) \ge v_{k+1}(t)$. By induction, we conclude that

$$v_0 \le v_1 \le \ldots \le v_j \le u \le w_j \le \ldots \le w_1 \le w_0 \text{ on } \mathbb{T}. \tag{10}$$

To show uniform convergence of $\{v_j\}$ and $\{w_j\}$, we prove they are uniformly bounded and equicontinuous. From (10) both sequences are uniformly bounded. For equicontinuity, let $0 \le t_1 \le t_2$ on $\mathbb{T}$. For $j > 0$,

$$\begin{aligned}|w_j(t_1) - w_j(t_2)| &= \left|\int_{t_1}^{t_2} [f_1(s,w_{j-1}(s)) - f_2(s,v_{j-1}(s))\Delta s\right|\\&\le \int_{t_2}^{t_1} |f_1(s,w_{j-1}(s)) - f_2(s,v_{j-1}(s))|\Delta s.\end{aligned}$$

Since $\{v_j\}$ and $\{w_j\}$ are uniformly bounded and f_1, f_2 are bounded on $\mathbb{T}$, there exists $\gamma > 0$ such that

$$|w_j(t_1) - w_j(t_2)| \le \gamma\,|t_1 - t_2|$$

Thus, for any $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{\gamma}$ ensures $|w_j(t_1) - w_j(t_2)| \le \varepsilon$ whenever $|t_1 - t_2| \le \delta$ Similarly, $\{v_j\}$ is equicontinuous. By the Arzelà-Ascoli theorem, there exist subsequences $\{v_{j_k}\}$ and $\{w_{j_k}\}$ converging uniformly to $\alpha(t)$ and $\beta(t)$, respectively. The monotonicity of $\{v_j\}$ and $\{w_j\}$ implies uniform convergence of the entire sequences. To show that α and β are coupled solutions of Type I for (4), integrate (7)

$$\begin{aligned}v_{j+1}\left(t\right) &= u_0 + \int_0^t [f_1(s,v_j(s)) - f_2(s,w_j(s))]\Delta s\\w_{j+1}\left(t\right) &= u_0 + \int_0^t [f_1(s,w_j(s)) - f_2(s,v_j\ (s))]\Delta s\end{aligned} \tag{11}$$

Taking $j \to \infty$ and using uniform convergence, we obtain

$$\begin{aligned}\alpha\left(t\right) &= u_0 + \int_0^t [f_1(s,\alpha(s)) - f_2(s,\beta(s))]\Delta s,\\\beta\left(t\right) &= u_0 + \int_0^t [f_1(s,\beta(s)) - f_2(s,\alpha(s))]\Delta s.\end{aligned} \tag{12}$$

Differentiating yields

$$\begin{aligned}\alpha^{\Delta} &= f_1(t,\alpha) - f_2(t,\beta),\ \alpha(0) = u_0,\\\beta^{\Delta} &= f_1(t,\beta) - f_2(t,\alpha),\ \beta(0) = u_0.\end{aligned}$$

Finally, since $v_0 \le v_j \le u \le w_j \le w_0$ for all j, taking $j \to \infty$ gives $v_0 \le \alpha \le u \le \beta \le w_0$ on $\mathbb{T}$. Thus, α and β are coupled minimal and maximal solutions of Type I for (4), respectively. This completes the proof.

Proof of Theorem 3. By assumption, $v_0 \le u \le w_0$ on . From the iterative scheme (5), we derive the first iterates

$$\begin{aligned}{v_1}^{\Delta} &= f_1(t,w_0) - f_2(t,v_0), \quad v_1(0) = u_0,\\{w_1}^{\Delta} &= f_1(t,v_0) - f_2(t,w_0), \quad w_1(0) = u_0.\end{aligned}$$

We first show $w_1 \le w_0$ on $\mathbb{T}$. Let $\phi = w_1 - w_0$. Then $\phi(0) \le 0$ and

$$\begin{aligned}&\phi^{\Delta} = {w_1}^{\Delta} - {w_0}^{\Delta}\\&\le f_1(t,v_0) - f_2(t,w_0) - f_1(t,w_0) + f_2(t,v_0)\\&= (f_1(t,v_0) - f_1(t,w_0)) + (f_2(t,v_0) - f_2(t,w_0))\\&\le 0,\end{aligned}$$

since \$$v_0 \le w_0$ and f_1, f_2 are nondecreasing. By Corollary 1, $\phi(t) \le 0$, proving $w_1 \le w_0$. Similarly, $v_0 \le v_1$. Next, we verify the ordering $v_0 \le w_1 \le v_1 \le w_0$. Let $\psi = v_1 - w_0$. Then, $\psi(0) \le 0$, and

$$\psi^{\Delta} \le f(t,,w_0) - f(t,,v_0) - f(t,,w_0) + f(t,,v_0) = 0$$

Thus, $v_1 \le w_0$. Let $\varphi = w_1 - v_1$. Then, $\varphi(0) = 0$, and

$$\varphi^{\Delta} = (f_1(t,v_0) - f_1(t,w_0)) + (f_2(t,v_0) - f_2(t,w_0)) \le 0$$

Hence, $w_1 \le v_1$. Assume for some $k \ge 1$,

$$v_{2k-2} \le w_{2k-1} \le v_{2k} \le w_{2k} \le v_{2k-1} \le w_{2k-2} \text{ on } \mathbb{T}.$$

We show the ordering extends to $k + 1$

$$v_{2k} \le w_{2k+1} \le v_{2k+2} \le w_{2k+2} \le v_{2k+1} \le w_{2k} \text{ on } \mathbb{T}.$$

Let $\psi = v_{2k} - w_{2k+1}$. Then,

$$\psi^{\Delta} \le f_1(t,w_{2k-1}) - f_1(t,v_{2k}) + f_2(t,w_{2k}) - f_2(t,v_{2k-1}) \le 0,$$

since $w_{2k-1} \le v_{2k}$ and $w_{2k} \le v_{2k-1}$.
Analogously, $\varphi = w_{2k+1} - v_{2k+1}$ yields $\varphi^{\Delta} \le 0$. By induction, we get

$$v_0 \le w_1 \le \cdots \le v_{2j} \le w_{2j+1} \le v_{2j+1} \le w_{2j} \le \cdots \le v_1 \le w_0 \text{ on } \mathbb{T}. \tag{13}$$

As in Theorem 2, the sequences $\{v_{2j}, w_{2j+1}\}$ and $\{v_{2j+1}, w_{2j}\}$ converge uniformly to α and β, respectively, satisfying the coupled system

$$\alpha^{\Delta} = f_1\left(t,\alpha\right) - f_2\left(t,\beta\right), \quad \alpha\left(0\right) = u_0,$$
$$\beta^{\Delta} = f_1\left(t,\beta\right) - f_2\left(t,\alpha\right), \quad \beta\left(0\right) = u_0$$

For all j, $v_{2j} \le w_{2j+1} \le u \le v_{2j+1} \le w_{2j}$. Taking limits as $j \to \infty$, we obtain

$$v_0 \le \alpha \le u \le \beta \le w_0 \text{ on } \mathbb{T}.$$

proving α and β are coupled minimal and maximal solutions of Type I for (4).

ILLUSTRATIVE EXAMPLES

We give some examples which are application of Theorem 2 and Theorem 3.

Example 1.
Consider the nonlinear dynamic equation

$$u^{\Delta}(t) = \tfrac{u^2}{2}(t) - \tfrac{u}{4}(t) \quad \text{for} t \in (0,1)\mathbb{R} \tag{14}$$

with the initial condition $u(0) = -1$. Define the functions $v_0, w_0 : [0,1]_{\mathbb{R}} \to \mathbb{R}$ by

$$v_0(t) = -1 \quad \text{and} \quad w_0(t) = t-1 \quad \text{for all } t \in [0,1]_{\mathbb{R}}.$$

Observe that $v_0(t) \le w_0(t)$ for all $[0,1]_{\mathbb{R}}$, and $v_0(0) = w_0(0) = -1 = u(0)$.

We now verify that v_0 and w_0 are coupled lower and upper solutions of Type I for (14). Then,

$${v_0}^{\Delta}(t) = 0 \le \tfrac{3}{4} - \tfrac{t}{4} \quad \text{for all} \quad t \in (0,1)_{\mathbb{R}}$$

and

$${w_0}^{\Delta}(t) = 1 \ge \tfrac{(t-1)^2}{2} + \tfrac{1}{4} \quad \text{for all} \quad t \in (0,1)_{\mathbb{R}}$$

Thus, v_0, w_0 satisfy the conditions for coupled lower and upper solutions of Type I for (14). Furthermore, assumptions (A1) and (A2) are readily verified. By Theorem 2, there exist monotone sequences that converge uniformly to the extremal solutions of (14) in the sector $[-1, t-1]$, and, by Theorem 3, alternating sequences also converge to the extremal solutions.

Example 2.
Let $\mathbb{T} = \mathbb{N}_0 = \{0,1,2,\ldots\}$, and consider the nonlinear dynamic equation

$$u^{\Delta}(t) = u(t) + t - te^{-u(t)} \quad \text{for} \quad t \in [0,,,\infty)_{\mathbb{T}} \tag{15}$$

with the initial condition $u(0) = 1$. Define the functions $v_0, w_0 : \mathbb{T} \to \mathbb{R}$ by

$$v_0(t) = 0 \quad \text{and} \quad w_0(t) = 2^t \quad \text{for all} \quad t \in \mathbb{T}.$$

Obviously, $v_0(t) \le w_0(t)$ for $t \in \mathbb{T}$. Moreover, we verify that v_0 and w_0 are coupled lower and upper solutions of Type I for (15). Indeed,

$${v_0}^{\Delta}(t) = 0 \le t\left(1 - e^{-2^t}\right) \quad \text{for all} \quad t \in \mathbb{T}$$

and

$${w_0}^{\Delta}(t) = 2^t \ge 2^t \quad \text{for all} \quad t \in \mathbb{T}$$

It is straightforward to check those conditions (A1) and (A2) are satisfied. By Theorem 2, we conclude the existence of monotone sequences that converge uniformly to the extremal solutions of (15) within the sector $\left[0, 2^t\right]$. Additionally, by Theorem 3, there exist alternating sequences that also converge to extremal solutions.

CONCLUSION

In this work, we have developed a comprehensive monotone iterative method for dynamic equations with initial value problems on time scales. Our approach, based on the fundamental concepts of coupled lower and upper solutions, successfully constructs natural and intertwined monotone sequences that converge uniformly to coupled extremal solutions. We make our final comments to conclude the paper.

Corollary 3. In addition to assumption of the Theorem 2, if for $u_1 \ge u_2$ and f_1, f_2 satisfy

$$\begin{aligned} f_1(t,u_1) - f_1(t,u_2) &\le L_1\,(u_1 - u_2), \\ f_2(t,u_1) - f_2(t,u_2) &\le L_2\,(u_1 - u_2), \end{aligned} \tag{14}$$

where L_1, L_2>0. Then $\alpha(t) = u(t) = \beta(t)$ is the unique solution.
Proof. We know that ; and so, we need to prove that . Let $p(t) = \beta(t) - \alpha(t)$ then $p(0) = 0$ and by using (14), it follows that

$$\begin{aligned} &p^{\Delta}\left(t\right) = \beta^{\Delta}\left(t\right) - \alpha^{\Delta}\left(t\right) \\ &= f_1(t,\beta(t)) - f_2(t,\alpha(t)) - f_1(t,\alpha(t)) + f_2(t,\beta(t)) \\ &\le L_1\,(\beta(t) - \alpha(t)) + L_2\,(\beta(t) - \alpha(t)) \\ &\le (L_1 + L_2)p(t) \end{aligned}$$

Thus, we get by Theorem 1 that $p(t) \le 0$, which proves that $\beta(t) \le \alpha(t)$ Hence, $\alpha(t) = u(t) = \beta(t)$.

The next theorem utilizes Type II coupled solutions, we obtain natural monotone sequences that converge uniformly and monotonically to the coupled minimal and maximal solutions of Type I of (4).

Theorem 4. Assume the following conditions hold
(C1) the functions v_0, w_0 are coupled lower and upper solutions of Type II for (4) satisfying $v_0 \le w_0$ for all $t \in \mathbb{T}$;
(C2) the functions $f1, f2 \in \mathscr{C}_{-\mathrm{rd}}\left(\mathbb{T}^{\kappa} \times \mathbb{R}^n, \mathbb{R}^n\right)$, $f_1(t,u)$ and $f_2(t,u)$ are non-decreasing in u for each $t \in \mathbb{T}$.

Then, there exist two monotone sequences $\{v_j(t)\}$ and $\{w_j(t)\}$ defined by

$${v_{j+1}}^{\Delta} = f_1\left(t,v_j\right) - f_2\left(t,w_j\right),\ v_{j+1}\left(0\right) = u_0$$
$${w_{j+1}}^{\Delta} = f_1\left(t,w_j\right) - f_2\left(t,v_j\right),\ w_{j+1}\left(0\right) = u_0,$$

satisfy the monotonicity property

$$v_0(t) \le v_1(t) \le \cdots \le v_j(t) \le w_j(t) \le \cdots \le w_1(t) \le w_0(t)$$
for all $t \in \mathbb{T}$

provided $v_0 \le v_1$ and $w_1 \le w_0$ on $\mathbb{T}$. Furthermore, the sequences $\{v_j(t)\}$ and $\{w_j(t)\}$ converge uniformly to $\alpha(t), \beta(t)$, respectively, where α, β are coupled minimal and maximal solutions of (4) such that they satisfy the coupled system,

$$\alpha^\Delta = f_1(t,\alpha) - f_2(t,\beta), \quad \alpha(0) = u_0$$
$$\beta^\Delta = f_1(t,\beta) - f_2(t,\alpha), \quad \beta(0) = u_0$$

for all $t \in \mathbb{T}$.

Remark 1. In Theorem 3,
(i) if $f_1(t,u) - f_2(t,u)$ is non-increasing on $\mathbb{T}$, then there exists a unique solution on $\mathbb{T}$.

(ii) if $f_2 \equiv 0$, then the conclusion of Theorem 3 is true.
Theorem 5. Assume that

(H1) v_0, w_0 are coupled lower and upper solutions of Type III for (4) with $v_0 \le u \le w_0$ on $t \in \mathbb{T}$;

(H2) $f_1, f_2 \in \mathscr{C}_{\mathrm{rd}}(\mathbb{T}^\kappa \times \mathbb{R}, \mathbb{R})$, $f_1(t,u)$ and $f_2(t,u)$ are nondecreasing in u for each $t \in \mathbb{T}$.

Then the sequences defined by

$$v_{j+1}{}^\Delta = f_1(t, w_j) - f_2(t, v_j),\ v_{j+1}(0) = u_0$$
$$w_{j+1}{}^\Delta = f_1(t, v_j) - f_2(t, w_j),\ w_{j+1}(0) = u_0.$$

give alternating monotone sequences $\{v_{2j}, w_{2j+1}\}$ and $\{w_{2j}, v_{2j+1}\}$ in $\mathscr{C}_{\mathrm{rd}}{}^1\left(\mathbb{T}, \mathbb{R}^n\right)$ where the sequences are given by

$$v_0 \le w_1 \le \cdots \le v_{2j} \le w_{2j+1} \le u \le v_{2j+1} \le w_{2j} \le \cdots \le v_1 \le w_0 \text{ on } \mathbb{T}$$

provided that $v_0 \le w_1 \le u \le v_1 \le w_0$ on $t \in \mathbb{T}$. Furthermore, the monotone sequences $\{v_{2j}, w_{2j+1}\}$ converge to α and $\{w_{2j}, v_{2j+1}\}$ converge to β, where α, β are coupled minimal and maximal solutions of (4) respectively, such that they satisfy the coupled system

$$\alpha^\Delta = f_1(t,\alpha) - f_2(t,\beta), \quad \alpha(0) = u_0$$
$$\beta^\Delta = f_1(t,\beta) - f_2(t,\alpha), \quad \beta(0) = u_0 \text{ on } \mathbb{T}.$$

Future research should extend this method to nonlinear functional dynamic equations. As this relatively new field continues to develop rapidly, significant opportunities exist for advancing both theoretical foundations and applications, particularly in hybrid dynamical systems.

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